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Improved treatment of bosonized QED around a large-Z nucleus

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In this paper we improve the semiclassical treatment of bosonized QED around a large-Z nucleus. In particular, the ground state of the system is approximated by means of a variational method based on a wider class of coherent trial states. These are defined in terms of excitations for a vacuum appropriate to a free boson theory with a space-dependent mass κ/r, where r is the distance from the nucleus and κ is a dimensionless constant. Our key issue consists in treating κ as a further variational parameter. The expectation value of the Hamiltonian is computed via a normal-ordering technique analogous to that used by Coleman in a similar context. The reliability of the improved treatment is suggested by our satisfactory prediction for $Z_{\rm cr}$, the critical value of the nuclear charge corresponding to the instability of the conventional neutral vacuum.

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I. INTRODUCTION

It is well known that the normal ground state of QED becomes unstable when the charge of a hypothetical nucleus exceeds the critical value $Z_{\rm cr} \sim 170$. Stability is achieved via the emission of positrons and the associated formation of the so-called charged vacuum. This phenomenon has been extensively discussed by Greiner, Müller, Rafelski, and collaborators in Refs. [1–3] which are based on the one-particle point of view, where the electromagnetic field is considered as an external classical object and the fermion-fermion interaction is therefore neglected. Some years ago, Hirata and Minakata [4–6] proposed a different approach which naturally takes into account the quantum field aspect of the problem. Their formalism relies on the bosonization technique [7,8] and it could represent a useful framework for the nonperturbative treatment of QED in very strong fields. The great advantage of the new approach is to render feasible a semiclassical analysis of the problem. Unfortunately, in addition to some difficulties of "technical" origin [9], the theory gives an unsatisfactory result for $Z_{\rm cr}$. As we shall discuss in Sec. III, the error in this fundamental quantity is rather large and it is then legitimate to doubt the validity of the semiclassical approximation in the boson theory. The purpose of our work is to elucidate this point by suggesting a simple but significant improvement of the bosonized formalism developed in Refs. [4–6]. In particular, we refine the variational method used in those papers by introducing a wider family of coherent trial states. Moreover, the expectation value of the bosonized Hamiltonian on these trial configurations is computed by fully exploiting a normal-ordering technique suggested by Coleman in [8,10]. As we shall see in Sec. III, this procedure will result in a more realistic value for $Z_{\rm cr}$ thereby providing strong evidence in favor of our improved semiclassical approximation.

For the sake of completeness, we close this introduction by recalling the main results of Ref. [4], which are the starting point for the following two sections. For details the reader should consult the original work.

The bosonized lowest-partial-wave QED is described by the Hamiltonian density

$$\mathcal{H} = N_\mu\left\{\sum_m \frac{1}{2}(\Pi_m^2 + P_m^2 + \Phi_m'^2 + Q_m'^2) + \sum_{m\delta} -\frac{1}{2\pi r^2}\cos\left[\sqrt{\pi}\left(\Phi_m + Q_m - \delta\int_r^\infty ds[\Pi_m(s) - P_m(s)]\right)\right]\right.$$
$$\left. - \sum_m \frac{c\mu m_0}{\pi}[\cos(2\sqrt{\pi}\Phi_m) + \cos(2\sqrt{\pi}Q_m)] - \frac{e^2}{4\pi\sqrt{\pi}r^2}\theta(r)\sum_m(\Phi_m + Q_m) + \frac{e^2}{8\pi^2 r^2}\left[\sum_m(\Phi_m + Q_m)\right]^2\right\} . \tag{1a}$$

The symbol N_μ means that the Hamiltonian density is normal ordered at the mass μ, which is the small but arbitrary mass of the Bose fields. The degrees of freedom Φ_m and Q_m are boson fields living in a (t,r) universe with $r \geq 0$ and obeying the boundary condition

$$\Phi_m(r=0,t) + Q_m(r=0,t) = 0 , \tag{1b}$$

while Π_m and P_m denote their canonical momenta. The index $m(=\pm 1/2)$ represents the z component of the angular momentum, $\delta(=\pm 1)$ corresponds to the chirality and $\theta(r)$ is the external charge contained in a sphere of radius r. In Eq. (1a) the second and third summations correspond to the centrifugal and mass terms, respectively, and c is a numerical constant (related to Euler's constant) whose value is irrelevant in what follows. Finally, the last two entries of Eq. (1a) describe the coupling with the external source $\theta(r)$ and the fermion-fermion interaction [9]. Since our aim is simply to ascertain the validity of the bosonized semiclassical approximation, we shall hereafter neglect this last interaction term. As stressed in [9], such a truncation will enable us to compare the predictions of the bosonized theory with those obtained by more conventional tools, i.e., the one-particle Dirac equation. It is also worthwhile to remark that we shall work in the symmetrical ansatz $\Phi_m(r,t) = Q_m(r,t)$ in strict analogy with Refs. [4–6].

II. THE RENORMAL ORDERING OF THE BOSONIZED HAMILTONIAN

In this section we shall introduce a different normal-ordering point for the bosonized Hamiltonian. This is an essential step in view of the forthcoming semiclassical treatment. As stressed in Refs. [4–6], at large radii compared with $1/m_e$ the theory reduces to two decoupled sine-Gordon theories, the centrifugal barrier and the Coulomb interaction being negligible; it is then quite natural to choose the renormal-ordering point in such a way that the theory reproduces the observed electron mass. By this prescription the coefficient of the mass term is renormalized as $c\mu m_0/\pi \to \pi/4m_e^2$ [5,6]. Things are more difficult for small radii, $r \ll 1/m_e$. In such a region the tree-level boson mass seems to be radius dependent and equal to κ/r, with $\kappa = \sqrt{2}$ as obtained by expanding the $r^{-2}\times$cosine term of Eq. (1a). This observation naturally leads [5] to consider the vacuum of a free bosons theory with a space-dependent mass given by $\sqrt{2}/r$. Starting from this vacuum $|0,\kappa=\sqrt{2}\rangle$, the authors of Refs. [4–6] constructed a family of coherent states $|\phi,\kappa=\sqrt{2}\rangle$ which were used as trial configuration in a variational estimate of the ground-state energy. The normal-ordering point was then fixed at the tree-level boson mass $\sqrt{2}/r$. This choice, without affecting the bozonized Hamiltonian $\mathcal{H}$ makes it easy to calculate the relevant expectation values. In the following we shall deviate from this approximated treatment by enlarging the family of coherent trial states $|\phi,\kappa\rangle$. The simplest way to generate a wider class of trial configurations is to release the condition $\kappa=\sqrt{2}$, by treating κ as a further variational parameter. Correspondingly, it turns out to be extremely convenient to use κ/r as a variable normal-ordering point. A similar method has been used by Coleman in [8,10] (see p. 237 of [10] in particular).

As a first step, let us now consider the effect of the normal-ordering operation on the kinetic part of a bosonized Hamiltonian density:

$$\mathcal{H}_{\text{kin}} = \tfrac{1}{2}(\pi^2 + \varphi'^2) \tag{2}$$

where, for the sake of simplicity, we have considered a single boson field φ and its canonical conjugate π. Obviously, the following relation holds:

$$\begin{aligned}\tfrac{1}{2}(\pi^2+\varphi'^2) &= N_{\kappa/r}\tfrac{1}{2}(\pi^2+\varphi'^2) + \tfrac{1}{2}\{[\pi^-,\pi^+] + [\varphi'^-,\varphi'^+]\} \\ &= N_{\kappa/r}\tfrac{1}{2}(\pi^2+\varphi'^2) \qquad (3a)\\ &\quad + \tfrac{1}{2}(\partial_t\partial_{t'} + \partial_r\partial_{r'})\Delta_\kappa(r,t;r',t')|_{(r,t)\to(r',t')}\ , \qquad (3b)\end{aligned}$$

where Δ_κ is the fundamental commutator:

$$\Delta_\kappa(r,t;r',t') = [\varphi^-(r,t),\varphi^+(r',t')]\ . \tag{4}$$

Different normal-orderings are then connected by

$$N_{\kappa'/r}\tfrac{1}{2}(\pi^2+\varphi'^2) = N_{\kappa/r}\tfrac{1}{2}(\pi^2+\varphi'^2) + \tfrac{1}{2}(\partial_t\partial_{t'} + \partial_r\partial_{r'})[\Delta_\kappa(r,t;r',t') - \Delta_{\kappa'}(r,t;r',t')]|_{(r,t)\to(r',t')}\ . \tag{5}$$

Since our renormal-ordering points are radius dependent, the fundamental commutators $\Delta_{\kappa,\kappa'}$ involve the Legendre function of second kind [6]:

$$[\varphi^-(r,t),\varphi^+(r',t')] = \frac{1}{2\pi}Q_\nu\left[1 + \frac{(r-r')^2-(t-t')^2}{2rr'}\right]\ , \tag{6}$$

with $\nu \equiv \sqrt{\kappa^2+\frac{1}{4}}-\frac{1}{2}$. The expansion of $Q_\nu(z)$ for $z\sim 1$ reads

$$Q_\nu(z) = -\frac{1}{2}\left[1+\frac{\nu(\nu+1)}{2}(z-1)\right]\ln(z-1) + R(z-1)\ , \tag{7}$$

where the remainder $R(z-1)$ does not contribute to the derivatives appearing in Eq. (5). From Eqs. (5)–(7), a simple algebra gives the fundamental relation

$$N_\mu\frac{1}{2}(\pi^2+\varphi'^2) = N_{\kappa/r}\frac{1}{2}(\pi^2+\varphi'^2) + \frac{1}{8\pi r^2}\nu(\nu+1) \tag{8}$$

[since μ is an arbitrary small parameter, it suffices to set $\kappa'=0$ in Eq. (5)]. As far as the centrifugal term is concerned, its renormal ordering has been discussed in Ref. [6]. The relevant equation is

$$N_\mu\cos(\sqrt{2\pi}\varphi) = \exp\left[\frac{\psi(\nu+1)+\gamma}{2}\right]N_{\kappa/r}\cos(\sqrt{2\pi}\varphi)\ , \tag{9}$$

where $\psi(\nu+1)$ is the digamma function. Using Eqs. (8) and (9) we are in a position to express the Hamiltonian density of the system in terms of operators normal ordered with respect to κ/r. We stress the essential point that we are not changing or correcting the operator $\mathcal{H}$ appearing in Eq. (1a). In fact, we are simply rewriting it by means of field functions with a different ordering point. Therefore, our prescription does not introduce any additional or spurious r dependence in the Hamiltonian density. Keeping this in mind, we can readily calculate the expectation value of the Hamiltonian on coherent states obtained from the vacuum $|0,\kappa\rangle$, i.e., the vacuum appropriate for a free bosons theory with a radius-dependent mass κ/r. In particular, recalling that $\Phi_m(r=0)=Q_m(r=0)=0$ [4], we obtain an expression for the energy density in the vicinity of the origin:

$$\langle \phi, \kappa | \mathcal{H} | \phi, \kappa \rangle \underset{r \to 0}{\sim} \frac{1}{r^2} \left[\frac{\nu(\nu+1)}{4\pi} - \frac{2}{\pi} \exp\left(\frac{\psi(\nu+1)+\gamma}{2} \right) \right] \equiv \frac{A(\kappa)}{r^2} \,, \tag{10}$$

where $|\phi, \kappa\rangle$ is a coherent state with vanishing momenta and specified by the expectation value $\phi(r)$ of the boson fields $[\Phi_m(r) + Q_m(r)]/\sqrt{2}$. The integration of the RHS in a small region around the origin gives a divergent contribution to the energy of the vacuum or, better, to its coherent state approximation. As a consequence, the best value of κ is that which minimizes the coefficient $A(\kappa)$ of the $1/r^2$ term. We call the reader's attention to the fact that no r^2 term is present in the measure of radial integration since a $1/r^2$ factor was introduced in the partial wave expansion of the original fermion fields, see Ref. [4] for details. In Fig. 1 $A(\kappa)$ is plotted as a function of κ. The unique minimum is found at $\kappa_{\min} \cong 1.84$, which is quite different from the value $\kappa = \sqrt{2}$ used in Refs. [5,6]. With this new value of κ, the centrifugal term is renormalized as

$$N_\mu \frac{1}{r^2} \cos(\sqrt{2\pi}\,\varphi) \to \frac{C(\kappa_{\min})}{r^2} \cos(\sqrt{2\pi}\,\phi) \tag{11a}$$

with

$$C(\kappa_{\min}) = \exp\left[\frac{\psi(\nu_{\min}+1)+\gamma}{2} \right] \cong 1.85 \,, \tag{11b}$$

which should be compared with the smaller value $C = \sqrt{e}(e = 2.718\ldots)$ obtained in Refs. [5,6]. From Fig. 1 we see that the coefficient $A(\kappa)$ is negative for all the reasonable values of its argument κ (the tree-level value $\kappa = \sqrt{2}$ included). As a consequence, the divergent piece of the energy tends to minus infinity. However, this divergence can be removed by subtracting a field independent energy density, as required for any sensible field theory. Within our variational approach, the most natural subtraction is $\mathcal{H} \to \mathcal{H}' = \mathcal{H} - A(\kappa_{\min})/r^2$. This regularization will be used in the numerical computations of the next section.

III. THE CRITICAL VALUE OF THE NUCLEAR CHARGE

In this section we show that the new coefficient $C(\kappa_{\min})$ of Eq. (11b) determines a more realistic value for Z_{cr} the critical value of the nuclear charge. In order to avoid misunderstanding about this crucial point, we find it appropriate to remark some relations between the Bose fields and the quantum numbers of the system we are considering. In Ref. [4] the following relations were obtained:

$$Q_{\rm em} = -\frac{e}{\sqrt{\pi}} \sum_m [\Phi_m + Q_m] \Big|_0^\infty \,, \tag{12a}$$

$$J_3 = \frac{1}{\sqrt{\pi}} \sum_m m[\Phi_m + Q_m] \Big|_0^\infty \,, \tag{12b}$$

where $Q_{\rm em}$ and J_3 are the electromagnetic charge and the third component of the angular momentum, respectively. Since, for technical reasons [4], we work in the symmetrical ansatz $\Phi_m(r) = Q_m(r)$, some constraints are imposed on the quantum numbers of the states we actually take into account. In particular, we cannot consider the state with $Q_{\rm em} = -2e(e > 0)$ and $J_3 = 0$ (corresponding to a $1S_{1/2}$ occupied level); as a consequence, we are only able to compare the energy of the neutral vacuum ($Q_{\rm em} = 0, J_3 = 0$) with the four positron emission state ($Q_{\rm em} = -4e, J_3 = 0$). The latter represents the charged vacuum which is realized when both the $1S_{1/2}$ level and the $2P_{1/2}$ level join the positron continuum. If the radius R of the nuclear charge is $R \approx 10$ fm, the transition to the vacuum with $Q_{\rm em} = -4e$ takes place at $Z \approx 185$ [1–3]. This value is obtained by solving the Dirac equation in the presence of an extended nucleus and looking for the condition $E_{2P_{1/2}}(Z) = -m_e$, $E_{2P_{1/2}}(Z)$ being the eigenvalue for the $2P_{1/2}$ state as a function of Z. As stressed in the Introduction, the bosonized framework of Refs. [5,6] seems to give a quite lower value $Z_{cr} \approx 170$, under the same assumptions about the nuclear charge distribution. Possible numerical errors are well below the effects we are dealing with. Actually, the above discrep-

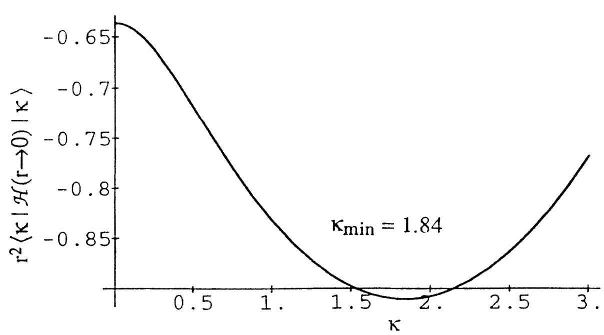

FIG. 1. The coefficient of the $1/r^2$ term in the energy density at the origin as a function of the variational parameter κ [see Eq. (10)].

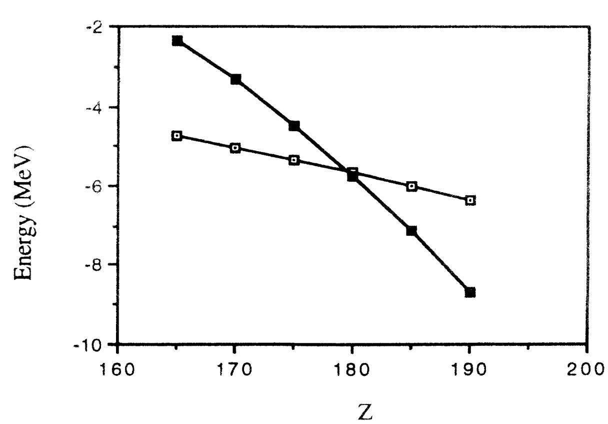

FIG. 2. The energies of the normal (open square) and the supercritical (solid square) vacua are plotted as functions of the nuclear charge Z. The external source is a uniformly charged sphere of radius $R = 10$ fm.

ancy is cured by the new criterion we have employed to identify the best normal-ordering point. In fact, we have already noted that our coefficient $C(\kappa_{\min})$ is larger with respect to that used in Refs. [5,6]; consequently the height of the centrifugal barrier is enhanced and a larger value of the nuclear charge is required in order to reach the critical conditions. The precise determination of $Z_{\rm cr}$ has been obtained in close analogy with Refs. [4–6], looking for the classical minimum of our effective Hamiltonian. As expected two local minima are found, corresponding to the neutral vacuum and to the charged supercritical one. The energies of these vacua are plotted in Fig. 2 as a function of the nuclear charge Z. The two curves intersect at $Z_{\rm cr} \approx 180$ which is rather close to $Z_{\rm cr} \approx 185$ as obtained by the more conventional treatment based on the one-particle Dirac equation.

IV. SUMMARY

In this paper we have improved the bosonized treatment of QED around a large-Z nucleus. Our procedure can be summarized as follows. From Ref. [4] we have taken the bosonized Hamiltonian describing the lowest partial wave of QED in the presence of a spherically symmetric charge distribution. The operators appearing in the Hamiltonian, which is normal ordered with respect to the vanishingly small mass μ, have been expressed in terms of operators normal ordered at the radius-dependent mass κ/r, with κ arbitrary. Then, we have considered the expectation value of the Hamiltonian on coherent states obtained from the vacuum of a free bosons theory with a radius-dependent mass κ/r. As a result we have obtained an equation for $\kappa_{\min}$, defined as the value of κ which minimizes the expectation value of the Hamiltonian. Finally, a numerical estimate of $\kappa_{\min}$ has been introduced in the classical equations of motion we have solved in order to find the best coherent state approximation to the vacuum. The calculation has been repeated for several values of Z (the nuclear charge) and, for $Z = Z_{\rm cr} \approx 180$, we have verified that the so-called charged vacuum with $Q_{\rm em} = -4e$ is favored with respect to the conventional neutral vacuum. The critical value obtained within the bosonized formalism of Refs. [4–6] was $Z_{\rm cr} \approx 170$, so that our prediction is much closer to the expected value $Z_{\rm cr} \approx 185$ [1–3]. Needless to say the agreement is not perfect. Nevertheless, we think that the improvement is significant and, above all, we believe that it has been obtained in a consistent and very natural way. So long as the ground state is concerned, our result suggests that the semiclassical treatment of bosonized QED should be trusted. However, we have to mention that its application to the excited states is not straightforward. In fact, we have already noted [9] that there is no serious justification for a small fluctuation approximation around the vacuum. We hope to return to this important point in the future.

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