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Relativistic description of heavy $q\bar{q}$ bound states

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We study the relativistic description of heavy $q\bar{q}$ bound states in the context of the relativistic wave equation. We used some attractive QCD based potentials where the vector part incorporates in the two loop perturbation QCD effects at short distances while the scalar part approaches the linear confining potential at large distances. We calculate the energy levels, leptonic and hadronic decay widths, as well as the $E1$ rate transition for $c\bar{c}$ and $b\bar{b}$. Results are compared with their experimental values.

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I. INTRODUCTION

Recently the relativistic description of the $q\bar{q}$ bound states has received much attention [1–6]. Gara and coworkers [1, 2] derived the relativistic wave equation by using the Bethe-Salpeter (BS) formalism [7], dropping the inhomogeneous interaction terms and following the standard approximations. They studied the relativistic description of $q\bar{q}$ in the context of the QCD potential with scalar confinement and an equal mixture of the scalar and vector confining potential. Later Lucha *et al.* [3–5] derived the same relativistic wave equation by constructing the T matrix and adopting the Born approximation. They studied $q\bar{q}$ and $q\bar{Q}$ in the context of some phenomenological potentials with somewhat flavor-dependent parameters.

In this work we investigate the relativistic description of $q\bar{q}$ and the meson couplings by solving the relativistic wave equation with some QCD base potentials. We solve the nonlocal relativistic wave equation using the method related to the orthogonal collocation method developed by Durand and Gara [8]. It approximates the action of all operators in the Hamiltonian on a set of basis functions evaluated at certain points. Hence the nonlocal relativistic wave equation is converted to a simple matrix eigenvalue problem for the wave function vector evaluated at these points, which are very suitable in the calculations of the meson couplings using a suitable quadrature with sufficient accuracy. The form of the vector kernel is given by the perturbative theory due to the asymptotic freedom of QCD at short distance. The scalar kernel is incorporated into the linear confining interaction as a long-range asymptotic term and a phenomenological intermediate term. We find that the phenomenological intermediate term fails to account for the intermediate interaction correctly. This flaw in the intermediate interaction term can be traced to the relativistic string behavior between interquarks [1, 2, 9–11].

The organization of this paper is as follows. In Sec. II we present the relativistic wave equation. In Sec. III we study the $q\bar{q}$ states using vector and scalar potentials proposed basically from the QCD theory. In Sec. IV we present the calculations and the results. Finally we give our conclusion in Sec. V.

II. THE RELATIVISTIC WAVE EQUATION

The relativistic wave equation is constructed by considering the kinetic energies of the constituents and the effective potential is constructed by considering the elastic scattering of the particles which build up the bound states [3, 4]. The effective potential reads [12] as

$$V^{\text{eff}} = -(2\pi)^3 \int d^3k e^{-i\mathbf{k}\cdot\mathbf{r}} T_{fi}^{\text{Born}}, \tag{1}$$

with the modulus $\mathbf{k} = \mathbf{p} - \mathbf{p}'$. So the Hamiltonian takes the form

$$H = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + V^{\text{eff}}(r). \tag{2}$$

The T-matrix elements for the elastic scattering process of fermion-antifermion interaction, namely,

$$f(\mathbf{p}_1, \tau_1) + \bar{f}(\mathbf{p}_2, \tau_2) \to f(\mathbf{p}_1', \tau_3) + \bar{f}(\mathbf{p}_2', \tau_4), \tag{3}$$

read

$$T = \frac{1}{(2\pi)^6} \frac{m_1 m_2}{\sqrt{E_1 E_2 E_1' E_2'}} \bar{\mathbf{u}}(\mathbf{p}_1', \tau_3)\Gamma_i \mathbf{u}(\mathbf{p}_1, \tau_1) \times \bar{\mathbf{v}}(\mathbf{p}_2, \tau_2)\Gamma_i \mathbf{v}(\mathbf{p}_2', \tau_4) K_{\text{int}}, \tag{4}$$

where Γ_i, $i = 1, 2, \ldots$ represent some Dirac matrices and K_{int} is an interaction kernel, while $\mathbf{u}(\mathbf{p}_i, \tau_i)$ and $\mathbf{v}(\mathbf{p}_i, \tau_i)$ are Dirac spinors. This relativistic wave equation which is derived from T matrix leads to the same equation that is obtained from the reduction of Bethe-Salpeter equation [1, 2]. The spin independent of the bound-state equation reads [1, 3] as

$$
\begin{aligned}
(M-E_1-E_2)\psi(r) = \frac{1}{4E_1E_2}\Big\{ & S_1S_2[V_S(r)+V_V(r)] + S_1\left[-\frac{dV_V(r)}{dr}\right]\frac{d}{dr}\frac{1}{S_2} \\
& + S_1\left[-V_V(r)\nabla_l^2\right]\frac{1}{S_2} + S_2\left[-\frac{dV_V(r)}{dr}\right]\frac{d}{dr}\frac{1}{S_1} + S_2\left[-V_V(r)\nabla_l^2\right]\frac{1}{S_1} \\
& + S_1\left[-\frac{dV_V(r)}{dr}\right]\frac{d}{dr}\frac{1}{S_1} + S_1\left[-V_V(r)\nabla_l^2\right]\frac{1}{S_1} + S_2\left[-\frac{dV_V(r)}{dr}\right]\frac{d}{dr}\frac{1}{S_2} + S_2\left[-V_V(r)\nabla_l^2\right]\frac{1}{S_2} \\
& + S_1\left[\frac{dV_S(r)}{dr}\right]\frac{d}{dr}\frac{1}{S_2} + S_1\left[V_S(r)\nabla_l^2\right]\frac{1}{S_2} + S_2\left[\frac{dV_S(r)}{dr}\right]\frac{d}{dr}\frac{1}{S_1} + S_2\left[V_S(r)\nabla_l^2\right]\frac{1}{S_1} \\
& - \nabla_l^2 V_V(r) + \left[\left(\frac{d^2V_S}{dr^2}+\frac{d^2V_V}{dr^2}\right)\frac{d^2}{dr^2}\right. \\
& \left. + \left(\frac{dV_S}{dr}+\frac{dV_V}{dr}\right)\left(\nabla_l^2\frac{d}{dr}+\frac{d}{dr}\nabla_l^2+\frac{l(l+1)}{r^3}\right) + (V_S+V_V)\nabla_l^2\nabla_l^2\right]\frac{1}{S_1S_2} \\
& - S_1S_2V_V(r)\nabla_l^2\frac{1}{S_1S_2}\Big\}\psi(r),
\end{aligned}
\tag{5}
$$

where $E_i = \sqrt{-\nabla_l^2 + m_i^2}$ is the nonlocal square root operator with $-\nabla_l^2 = -\frac{1}{r}\frac{d^2}{dr^2}r + \frac{l(l+1)}{r^2}$ and $S_i = E_i + m_i$, while V_V and V_S are the vector and scalar parts of the interaction, respectively.

III. SOME QCD BASE POTENTIALS

In the relativistic description case, the two-loop perturbative QCD formula is usually taken as the short-distance asymptotic form of the potential containing $\Lambda_{\overline{\rm MS}}$ as a parameter where $\overline{\rm MS}$ denotes the modified minimal subtraction scheme; furthermore, the nonperturbative part contributes the Lüscher [13] term as a transverse zero oscillation for large distance. A linear confining interaction between quarks appears naturally as a consequence of the fact that the color electric flux is quantized on the lattice and it will be taken as the large distance asymptotic form of the potential.

A. Effective potential

The one-gluon-exchange (OGE) interaction including vacuum polarization corrections is found as [14, 15]

$$
V_{\rm OGE}(r) = -C_F\frac{\alpha_s(\mu_{\overline{\rm MS}})}{r}\left[1+\frac{\alpha_s(\mu_{\overline{\rm MS}})}{\pi}\{b_0\ln(\mu_{\overline{\rm MS}}r)+A_1\} + O\left(\left(\frac{\alpha_s(\mu_{\overline{\rm MS}})}{\pi}\right)^2\right)\right],
\tag{6}
$$

where α_s is obtained by iterating [16]

$$
\frac{2\pi}{\alpha_s} + 2\frac{b_1}{b_0}\ln\left(\frac{\frac{b_0\alpha_s}{2\pi}}{1+\frac{b_1\alpha_s}{b_0\pi}}\right) = b_0\ln\left(\frac{\mu}{\Lambda_{\overline{\rm MS}}^{n_f}}\right)^2,
\tag{7}
$$

and

$$
A_1 = b_0\gamma_E + \frac{31}{37}C_A + \frac{2}{3}T_F\sum_{q=1}^{n_f}\Big[\gamma_E + \ln(m_q r) + E_1(m_q r e^{\frac{5}{6}})\Big],
\tag{8}
$$

with

$$
b_0 = \frac{11}{6}C_A - \frac{2}{3}n_fT_F,
\tag{9}
$$

$$
b_1 = \frac{17}{12}C_A^2 - \frac{5}{6}C_An_fT_F - \frac{1}{2}C_Fn_fT_F.
\tag{10}
$$

Here n_f is the number of effective flavors; $T_F=\frac{1}{2}$, $C_F=\frac{4}{3}$, and $C_A=3$ are color factors; γ_E is the Euler constant and E_1 is the exponential integral. The short-distance potential $V_{\rm OGE}$ can be written as [17]

$$
V_{\rm OGE} = -C_F\frac{\alpha_s(Q(r))}{r} + O(\alpha^3)
\tag{11}
$$

with scale $Q(r)\equiv\mu(r)=\frac{1}{r}e^{-A_1/b_0}$. We adopt the effective four- and five-flavor theories ($n_f = 4,5$ respectively) with $m_u = m_d = m_s = 0$, $m_c = 1.5$, and $m_b = 5$ GeV (fixed). Since, in the massless quark limit, the next-to-leading-order correction factor $A_1(r)$ reduces to

$$
A_1(r) \to b_0\gamma_E + \frac{31}{37}C_A - \frac{5}{9}n_lT_F,
\tag{12}
$$

then the complete next-to-leading-order correction factor $A_1(r)$ in Eq. (8) becomes

$$
A_1(r) = b_0\gamma_E + \frac{31}{37}C_A - \frac{5}{9}n_lT_F + \frac{2}{3}T_F \times \sum_{m_q=m_c,m_b}[\gamma_E + \ln(m_q r) + E_1(e^{5/6}m_q r)],
\tag{13}
$$

with $n_l = 3$ the number of massless quarks. The perturbative running coupling constant α_s has the well known Landau singularity ghost when $Q^2/\Lambda^2_{\overline{\rm MS}} = 1$. A simple regularization is obtained by replacing [1, 15]

$$\frac{Q^2}{\Lambda^2_{\overline{\rm MS}}} \longrightarrow \frac{Q^2}{\Lambda^2_{\overline{\rm MS}}} + \left(\frac{b_0^2}{2b_1}\right)^{2b_1/b_0^2} . \tag{14}$$

However this transformation removes the Landau singularity to infinite quark separation.

We have taken the interaction as a sum of scalar and vector terms with a fairly flexible parametrization of the potentials given by Gara *et al.* [1, 2], and with the only difference we have worked in the effective four- and five-flavor theories. We refer to the effective four-flavor potential with $\alpha^{nf=4}$ as potential I and the effective five-flavor potential with $\alpha^{nf=5}$ as potential II. Hence the potentials I and II read as

$$V_V(r) = -\frac{4}{3}\frac{\alpha_s^{nf=4,5}(r)}{r}e^{-\mu' r} +\delta\left(-\frac{\beta}{r}+kr\right)(1-e^{-\mu r}), \tag{15}$$

and

$$V_S(r) = (1-\delta)\left(-\frac{\beta}{r}+kr\right)(1-e^{-\mu r}) + V_0 +(c_0+c_1 r+c_2 r^2)(1-e^{-\mu r})e^{-\mu r}. \tag{16}$$

Herein, δ is used as a mixing factor between vector and scalar parts. Using the two-loop formula, we have found the derivative

$$\frac{d}{dr}\alpha_s(\mu) = \frac{\frac{\mu^2}{\Lambda^2_{\overline{\rm MS}}}}{\frac{\mu^2}{\Lambda^2_{\overline{\rm MS}}}+\left(\frac{b_0^2}{2b_1}\right)^{2b_1/b_0^2}}\frac{\alpha_s^2(\mu)}{\pi}\left[b_0+\frac{b_1\alpha_s}{\pi}\right] \times\left[\frac{1}{r}+\frac{1}{b_0}\frac{d}{dr}A_1(r)\right], \tag{17}$$

and [16]

$$\Lambda^{(4)}_{\overline{\rm MS}} \sim \Lambda^{(5)}_{\overline{\rm MS}}\left[\frac{m_b}{\Lambda^{(5)}_{\overline{\rm MS}}}\right]^{2/25}\left[2\ln\frac{m_b}{\Lambda^{(5)}_{\overline{\rm MS}}}\right]^{963/14375} . \tag{18}$$

Therefore if we take $\Lambda^{(4)}_{\overline{\rm MS}} = 0.430$ GeV in four-flavor theory, then it reduces to $\Lambda^{(5)}_{\overline{\rm MS}} \sim 0.310$ GeV in five-flavor theory.

We have used the fitting parameters [1, 2]

$$\begin{aligned}&\left[\Lambda^{n_f=4}_{\overline{\rm MS}},\Lambda^{n_f=5}_{\overline{\rm MS}},\beta,k\right] = \left[0.430\text{ GeV}, 0.310\text{ GeV}, \frac{\pi}{12}, 0.177\text{ GeV}^2\right],\\ &[V_0, C_0, C_1, C_2] = \left[-0.366\text{ GeV}, 2.45\text{ GeV}, -0.074\text{ GeV}^2, 0.343\text{ GeV}^3\right],\\ &[\mu,\mu'] = [0.933, 0.740]\text{ GeV}.\end{aligned} \tag{19}$$

B. Improved QCD-motivated potential

The above potentials have many parameters, so we consider another interesting potential (potential III) having two parameters μ and μ'. The potential approaches the two-loop perturbative formula at short distance and linear confining potential at large distance [18, 19]. The improved OGE interaction reads as [19]

$$V_{\rm OGE}(r) = -\frac{16\pi}{25}\frac{1}{rf(r)}\left[1+\frac{2\gamma_E+\frac{53}{75}}{f(r)}-\frac{462}{625}\frac{\ln f(r)}{f(r)}\right], \tag{20}$$

where

$$f(r) = \ln\left[\frac{1}{\Lambda_{\overline{\rm MS}}r}+4.62-\left[1-\frac{1}{4}\frac{\Lambda_{\overline{\rm MS}}}{\Lambda^I_{\overline{\rm MS}}}\right]\frac{1-\exp\left\{-\left[15\left(3\frac{\Lambda^I_{\overline{\rm MS}}}{\Lambda_{\overline{\rm MS}}}-1\right)\Lambda_{\overline{\rm MS}}r\right]^2\right\}}{\Lambda_{\overline{\rm MS}}r}\right]^2, \tag{21}$$

with $\Lambda^I_{\overline{\rm MS}} = 180$ MeV.

To have a smooth transition of $V_{\rm OGE}$ from the vector part to scalar part, we write the vector and scalar parts of potential III as

$$V_V = V_{\rm OGE}e^{-\mu r}, \tag{22}$$

and

$$V_S = V_C + V_{\rm OGE}(1-e^{-\mu' r}), \tag{23}$$

with

$$V_C = kr. \tag{24}$$

Since the exponential damping is not considered as an ideal partition function such as a Fermi-Dirac function, it is in general $\mu' \neq \mu$. The string tension k is related to the Regge slope α' by [18, 20]

$$k = \frac{1}{2\pi\alpha'}. \tag{25}$$

The value of α' is approximately 1 GeV^{-2}, so that $k \sim 0.16$ GeV^2. The value of $\Lambda_{\overline{\rm MS}}$ is $\Lambda_{\overline{\rm MS}} = 200^{+150}_{-80}$ GeV. We have used these values and the following fitting parameters in our calculations:

$$[\mu_c, \mu'_c, k, m_c]$$
$$= [0.75 \text{ GeV}, 1.20 \text{ GeV}, 0.16 \text{ GeV}^2, 1.516 \text{ GeV}],$$
$$[\mu_b, \mu'_b, k, m_b]$$
$$= [0.74 \text{ GeV}, 0.80 \text{ GeV}, 0.16 \text{ GeV}^2, 4.884 \text{ GeV}].$$

IV. MESON COUPLINGS

A. The wave function at the origin

We have found the wave function at the origin by adopting a smearing procedure based strongly on the nonrelativistic nature of the problem at large radii. From the Schrödinger equation with the usual boundary conditions at $r = 0$ and ∞, we get, for $l = 0$ [21],

$$\mid R_S(0) \mid^2 = m_q \left\langle \frac{dV}{dr} \right\rangle, \tag{26}$$

and, for $l = 1$,

$$R'_p(0) = \frac{m_q}{9} \left\langle \frac{1}{r^2}\frac{dV(r)}{dr} + \frac{4[E - V(r)]}{r^3} \right\rangle, \tag{27}$$

where $E = M(q\bar{q}) - 2m_q$, and $V = V_S + V_V$. We have estimated $\frac{v^2}{c^2}$ by using the virial theorem

$$\frac{v^2}{c^2} = \frac{1}{2m_q} \left\langle r \frac{dV}{dr} \right\rangle. \tag{28}$$

B. Leptonic decay widths

The leptonic decay widths are given for the $J^{PC} = 1^{--}$ states $n\,^3S_1$:

$$\Gamma^{(0)}_{ee} = \frac{4\alpha^2 e_Q^2}{M^2_{(Q\bar{Q})}} \mid \psi_S(0) \mid^2, \tag{29}$$

where α is the fine structure. In the QCD formalism, Eq. (29) is modified by the second-order radiative corrections to

$$\Gamma_{ee} = \Gamma^0(1 - \Delta^{\rm rad}), \tag{30}$$

where

$$\Delta^{\rm rad} = \frac{16}{3}\left[\frac{\alpha_s}{\pi}\right] - (24.26 - 0.11 n_f)\left[\frac{\alpha_s}{\pi}\right]^2. \tag{31}$$

In the spirit of Refs. [22, 23], the weight of the annihilation amplitude can be corrected relativistically by writing

$$\psi_l^{(l)} \longrightarrow S_l(\psi_i), \tag{32}$$

and

$$S_l(\psi_i) = \left| \frac{1}{(2\pi)^3} \int d^3p\phi_i(p) \left[\frac{m_i p}{E_i}\right]^l \frac{m_i}{E_i} \right|. \tag{33}$$

With a rough and reliable approximation Eq. (33) reads

$$S_0(\psi_i) \sim \left\langle \frac{m_i}{E_i} \right\rangle \left| \frac{1}{(2\pi)^3} \int d^3p\phi_i(p) \right| = \left\langle \frac{m_i}{E_i} \right\rangle |\psi(0)|, \tag{34}$$

and

$$S_1(\psi_i) \sim \left\langle \frac{m_i}{E_i} \right\rangle \left\langle \frac{m_i}{E_i} \right\rangle |\psi'(0)|, \tag{35}$$

for $l = 0, 1$ respectively. Hence the wave function at the origin can be modified to

$$|\psi_S(0)| \longrightarrow \left\langle \frac{m_i}{E_i} \right\rangle |\psi_S(0)|, \tag{36}$$

and

$$|\psi'_P(0)| \longrightarrow \left\langle \frac{m_i}{E_i} \right\rangle \left\langle \frac{m_i}{E_i} \right\rangle |\psi'_P(0)|. \tag{37}$$

Consequently Γ_{ee} can be corrected relativistically to

$$\Gamma_{ee} \longrightarrow \frac{m_i}{E_i}\Gamma_{ee}\frac{m_i}{E_i}. \tag{38}$$

C. Hadronic decay widths

The decay rates of heavy quark system via annihilation to the minimum possible number of gluons forbidden by the Okubo-Zweig-Iizuka can be written as [24]

$$\Gamma(n^1S_0 \longrightarrow \text{hadron}(2g)) = \frac{8}{3}\alpha_s^2 \frac{|\psi_{n0}(0)|^2}{M^2_{n0}}, \tag{39}$$

$$\Gamma(n^3P_0 \longrightarrow \text{hadron}(2g)) = 96\alpha_s^2 \frac{|\psi'_{n1}(0)|^2}{M^4_{n1}}, \tag{40}$$

and

$$\Gamma(n^3P_2 \longrightarrow \text{hadron}(2g)) = \frac{128}{5}\alpha_s^2 \frac{|\psi'_{n1}(0)|^2}{M^4_{n1}}. \tag{41}$$

However the relativistic corrections read

$$\Gamma(nS) \longrightarrow \frac{m_i}{E_i}\Gamma(nS)\frac{m_i}{E_i} \tag{42}$$

and

$$\Gamma(nP) \longrightarrow \left(\frac{m_i}{E_i}\right)^2 \Gamma(nP) \left(\frac{m_i}{E_i}\right)^2. \tag{43}$$

D. Radiative transition

The radiative transitions between heavy quark states are well defined even if the instantaneous approximation is adopted. The transition decay $E1$ is given by

$$\Gamma_{if} = \frac{4}{3}e_Q^2\alpha\omega^3 S_{if} E^2_{if}(2j_f + 1), \tag{44}$$

where $E_{if} = \langle i|r|f\rangle$, and S_{if} is given in terms of the Wigner 6-j symbol:

$$S_{if} = \begin{Bmatrix} j_i & 1 & j_f \\ l_f & s & l_i \end{Bmatrix} \max\{l_i, l_f\}. \quad (45)$$

Thus we have

$$\Gamma(n\,{}^3S_1 \longrightarrow {}^3P_j) = \frac{4}{9}\frac{(2j_f+1)}{3} e_Q^2 \alpha E_{if}^2 \omega^3, \quad (46)$$

and

$$\Gamma({}^3P_j \longrightarrow {}^3S_1) = \frac{4}{9} e_Q^2 \alpha E_{if}^2 \omega^3. \quad (47)$$

V. RESULTS AND CONCLUSIONS

Using some realistic interquark potentials, we studied the relativistic description of $q\bar{q}$ bound states within the relativistic wave equation. In the calculations, the orthogonal collocation method [8] was applied.

The masses of the lowest S, P, and D states of self-conjugate mesons were calculated with the potentials I and II in the case of scalar confinement (unmixing vector-scalar interaction) and an equal mixture of scalar and vector confining potential (equal mixing vector-scalar interaction) and potential III. Results are presented in Table I and compared with experimental values. Although the effective four-flavor theory can give a good description of the short distance for a wide range of interquark distances $r \geq \frac{1}{m_b}$, the effective five-flavor theory describes the short distance more accurately. Hence it is reasonable to expect that potential II should give more accurate results than potential I. Results obtained with potential I are in better agreement than the ones obtained with the potential II. The possible explanation of this is that the phenomenological potential of Eqs. (15) and (16) and that given in Refs. [1,2] may not reflect the real intermediate range interaction correctly. Although the unmixing vector-scalar interaction gives better results than the mixing one, the mixing vector-scalar interaction still gives acceptable results. These surprising results, if the long-range interaction including the confinement contributes in the vector part, contradict the QCD predictions. We have tried to avoid the flaw of the model by considering potential III. We have considered the improved QCD-motivated potential as a short range interaction and we have proposed a somewhat flavor-dependent intermediate range interaction. The intermediate interaction may be mass dependent, since a more light flavor becomes a more relativistic one and has more active range for the transition from the vector part to the scalar part. We have probed this proposed intermediate interaction as a soft transition from the vector part to the scalar part of the interaction. The results of the mass spectra of the potential III are listed in the last column of Table I. Our model does not account $1P$ for $c\bar{c}$ exactly and we trace this to the intermediate part of the interaction.

The leptonic decay widths are calculated in Table II using the results of Eqs. (26) and (27) and the second-order radiative corrections given by Eq. (30). We have smeared the wave functions at the origin to account for the relativistic corrections. Our scheme to account for the relativistic corrections is not exact, but sufficient for reliable corrections. We have estimated $\frac{v^2}{c^2} \sim 0.4$ and 0.1 for $c\bar{c}$ and $b\bar{b}$, respectively, using the virial theorem. Hence $m_i/E_i \sim (1 - \frac{v^2}{c^2})$ is equal to 0.6 and 0.9 for $c\bar{c}$ and $b\bar{b}$, respectively. Results of the leptonic decay widths for charmonium and bottomonium after considering the above corrections are found acceptable and sufficient to conclude that our short range interaction has a correct asymptotic behavior since the short range interaction is essential for the contribution of the wave func-

TABLE I. The quarkonium spectra for $c\bar{c}$ and $b\bar{b}$ by using potentials I and II with unmixing and equal mixing vector-scalar interaction potentials as well as potential III.

State	Expt. data[a]	I		II		III
		$\delta = 0$	$\delta = \frac{1}{2}$	$\delta = 0$	$\delta = \frac{1}{2}$	
$c\bar{c}\backslash m_c$ (GeV)		1.635	1.618	1.663	1.648	1.516
$1S$	3.068	3.079	3.064	3.072	3.060	3.067
$2S$	3.663	3.656	3.668	3.660	3.676	3.663
$3S$	4.022	4.019	4.078	4.039	4.099	4.026
$1P$	3.525	3.516	3.536	3.503	3.524	3.510
$1D$	3.770	3.805	3.876	3.803	3.876	3.776
$2D$	4.160	4.103	4.262	4.122	4.279	4.096
$b\bar{b}\backslash m_b$ (GeV)		4.966	4.965	4.991	4.990	4.883
$1S$	9.436	9.432	9.422	9.446	9.437	9.436
$2S$	10.017	10.022	10.020	10.020	10.018	10.020
$3S$	10.341	10.353	10.359	10.367	10.371	10.356
$4S$	10.576	10.607	10.616	10.622	10.629	10.636
$5S$	10.861	10.771	10.798	10.781	10.810	10.896
$6S$	11.016	10.996	11.032	11.040	11.075	10.993
$1P$	9.900	9.902	9.903	9.892	9.893	9.898
$2P$	10.261	10.260	10.268	10.259	10.267	10.259

[a]Here we cite Ref. [16].

TABLE II. Leptonic decay widths for $c\bar{c}$ and $b\bar{b}$ by using potentials I and II with unmixing and equal mixing vector-scalar interaction as well as potential III. Second-order radiative $\Gamma^0_{ee}(1-\Delta)$ (SOR) and relativistic corrections (RC) are included.

Expt. Data	I			II			III		
$c\bar{c}$ ($\alpha_s = 0.21$)	Γ^0_{ee}	SOR	RC	Γ^0_{ee}	SOR	RC	Γ^0_{ee}	SOR	RC
$\Gamma_{ee}(1S) = 5.36 \pm 0.35$ keV									
$\delta = 0$	12.54	9.40	5.64	12.27	9.20	5.52	9.05	6.79	4.07
$\delta = \frac{1}{2}$	13.40	10.05	6.03	13.07	9.80	5.88			
$\Gamma_{ee}(2S) = 2.14 \pm 0.21$ keV									
$\delta = 0$	4.66	3.50	2.10	4.93	3.70	2.22	3.85	2.88	1.73
$\delta = \frac{1}{2}$	4.95	3.71	2.23	5.21	3.91	2.35			
$b\bar{b}$ ($\alpha_s = 0.16$)									
$\Gamma_{ee}(1S) = 1.34 \pm 0.04$ keV									
$\delta = 0$	1.91	1.51	1.36	1.80	1.42	1.28	1.50	1.18	1.06
$\delta = \frac{1}{2}$	1.98	1.56	1.40	1.87	1.48	1.33			
$\Gamma_{ee}(2S) = 0.56 \pm 0.04$ keV									
$\delta = 0$	0.81	0.64	0.58	0.84	0.67	0.60	0.63	0.50	0.45
$\delta = \frac{1}{2}$	0.84	0.67	0.60	0.88	0.69	0.62			
$\Gamma_{ee}(3S) = 0.44 \pm 0.03$ keV									
$\delta = 0$	0.56	0.44	0.40	0.61	0.48	0.43	0.42	0.33	0.30
$\delta = \frac{1}{2}$	0.59	0.47	0.42	0.65	0.52	0.47			

tion at the origin. Moreover the results of the hadronic decay widths listed in Table III confirm the concluding remark above. The second-order radiative corrections and the hadronic decay widths depend essentially on the value of α_s. The well known scheme to find the value of $\alpha_s(\mu)$ is to take $\mu = 2m_q$ and the world-averaged value $\Lambda^4_{\overline{\mathrm{MS}}} = 200$ MeV. Therefore the values of the second order radiative corrections are estimated to be 0.75 and 0.79 for $\alpha_s(2m_c) = 0.21$ and $\alpha_s(2m_b) = 0.16$, respectively.

Results of the rates of the electromagnetic transitions are listed in Table IV and they are acceptable. However, the long range interaction is essential for the electromagnetic transition.

In summary, the short range interaction and the long range interaction seem to have a correct asymptotic behavior. The problem seems to be in the phenomenological intermediate interaction, since our intermediate interaction does not account for the relativistic string behavior between the interquarks. Although an immense number of fitting parameters is used in potentials I and

TABLE III. Hadronic total widths Γ for $c\bar{c}$ and $b\bar{b}$ by potentials I, II, and III. The relativistic corrections (RHC) of the hadronic total widths are considered.

Expt. Data	$\delta = 0$		$\delta = \frac{1}{2}$	
Potential I	Γ	RHC	Γ	RHC
$c\bar{c}$ ($\alpha_s = 0.21$)				
$\Gamma(1\,^1S_0 \longrightarrow 2g) = 10.3^{3.8}_{-3.4}$ MeV	22.2 MeV	13.2 MeV	22.0 MeV	13.2 MeV
$b\bar{b}$ ($\alpha_s = 0.16$)				
$\Gamma(1\,^3P_2 \longrightarrow 2g) = 144 \pm 35$ keV	218 keV	176 keV	245 keV	198 keV
$\Gamma(2\,^3P_2 \longrightarrow 2g) = 130^{130}_{-40}$ keV	192 keV	155 keV	214 keV	173 keV
Potential II				
$c\bar{c}$ ($\alpha_s = 0.21$)				
$\Gamma(1\,^1S_0 \longrightarrow 2g) = 10.3^{3.8}_{-3.4}$ MeV	20.0 MeV	12.0 MeV	21.6 MeV	13.0 MeV
$b\bar{b}$ ($\alpha = 0.16$)				
$\Gamma(1\,^3P_2 \longrightarrow 2g) = 144 \pm 35$ keV	229 keV	185 keV	248 keV	207 keV
$\Gamma(2\,^3P_2 \longrightarrow 2g) = 130^{130}_{-40}$ keV	146 keV	118 keV	160 keV	129 keV
Potential III				
$c\bar{c}$ ($\alpha_s = 0.21$)	Γ	RHC		
$\Gamma(1^1S_0 \longrightarrow 2g) = 10.3^{3.8}_{-3.4}$ MeV	14.9 MeV	8.9 MeV		
$b\bar{b}$ ($\alpha_s = 0.16$)				
$\Gamma(1\,^3P_2 \longrightarrow 2g) = 144 \pm 35$ keV	161 keV	130 keV		
$\Gamma(2\,^3P_2 \longrightarrow 2g) = 130^{130}_{-40}$ keV	167 keV	135 keV		

II and the flavor-dependent assumption is considered in potential III, the relativistic wave equation fails to fit the quarkonia exactly. Since the results of the meson couplings are in good agreement with the experimental ones, we can conclude that the model is not completely flawed. Nonetheless, our results confirm the ones given in Refs. [1, 5], that the relativistic wave equation is not an adequate starting point, at least to account for the inter-

TABLE IV. Radiative transition (all transitions are between triplet states) for $c\bar{c}$ and $b\bar{b}$ by using potentials I, II, and III.

Transition state	Expt. (keV)	$\langle f\|r\|i\rangle$ (keV)	Γ $\delta = 0$ (keV)	$\langle f\|r\|i\rangle$ (keV)	Γ $\delta = \frac{1}{2}$ (keV)
Potential I					
$c\bar{c}$					
$1P_2 \longrightarrow 1S$	270	1.418	230	1.396	223
$1P_1$	240		171		165
$1P_0$	92		81		78
$2S \longrightarrow 1P_2$	21.7	2.000	19.7	1.980	19.3
$1P_1$	24.2		28.8		28.3
$1P_0$	25.9		34.1		33.5
$b\bar{b}$					
$2S \longrightarrow 1P_2$	2.88	1.387	1.54	1.377	1.52
$1P_1$	2.84		1.56		1.54
$1P_0$	1.85		0.98		0.97
$3S \longrightarrow 2P_2$	2.77	2.312	2.11	2.336	2.16
$2P_1$	2.74		1.93		1.97
$2P_0$	1.31		1.20		1.22
Potential II					
$c\bar{c}$					
$1P_2 \longrightarrow 1S$	270	1.414	229	1.390	221
$1P_1$	240		170		164
$1P_0$	92		80		78
$2S \longrightarrow 1P_2$	21.7	1.891	17.6	1.871	17.2
$1P_1$	24.2		25.8		25.2
$1P_0$	25.9		30.6		29.9
$b\bar{b}$					
$2S \longrightarrow 1P_2$	2.88	1.360	1.48	1.351	1.46
$1P_1$	2.84		1.50		1.48
$1P_0$	1.85		0.94		0.93
$3S \longrightarrow 2P_2$	2.77	2.156	1.84	2.190	1.90
$2P_1$	2.74		1.68		1.73
$2P_0$	1.31		1.04		1.07

Potential III

Transition state	Expt. (keV)	$\langle f\|r\|i\rangle$ (keV)	Γ (keV)
$c\bar{c}$			
$1P_2 \longrightarrow 1S$	270	1.831	384
$1P_1$	240		285
$1P_0$	92		135
$2S \longrightarrow 1P_2$	21.7	2.402	28.4
$1P_1$	24.2		41.6
$1P_0$	25.9		49.3
$b\bar{b}$			
$2S \longrightarrow 1P_2$	2.88	1.536	1.89
$1P_1$	2.84		1.91
$1P_0$	1.85		1.20
$3S \longrightarrow 2P_2$	2.77	2.496	2.46
$2P_1$	2.74		2.24
$2P_0$	1.31		1.39

mediate interaction. The intermediate interaction might be explained by the relativistic string tube. Very recently Brambilla and Prosperi [25] formulated the semirelativistic potential to account for the retardation corrections, which agrees with the QCD predictions and Olsson and Williams [10] formulated a quantized relativistic tube model.

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