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Stage-two spin-correlation functions: Tests for non-CKM-type leptonic CP violation in $\tau \to \rho\nu$ decay

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There are two tests for leptonic CP violation in $\tau \to \rho\nu$ decay by inclusion of ρ polarimetry observables in the energy correlation function for Z^0 or $\gamma^* \to \tau^-\tau^+ \to (\rho^-\nu)\ (\rho^+\bar{\nu})$. By CP invariance the moduli ratio of, and phase difference between, the two helicity amplitudes for $\tau^- \to \rho^-\nu_\tau$ decay should equal those for $\tau^+ \to \rho^+\bar{\nu}_\tau$ decay. The full angular distribution for the above process, including the $\pi^\mp$ momentum direction versus that of the $\rho^\mp$ momentum, can be used to test for such a non-CKM-type leptonic CP violation in $\tau \to \rho\nu$ decay. Since this adds on spin-correlation information from the next stage of decays in the decay sequence, we call such an energy-angular distribution a stage-two spin-correlation (S2SC) function. Ideal statistical errors for $\tau \to \rho\nu$ decay are calculated for possible application at the Z^0, at a B factory, or at the τ-charm threshold. S2SC functions should be useful for testing for possible non-CKM-type CP violation in top quark and in W boson decay processes.

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I. INTRODUCTION

While in K^0 decay CP and T violations are phenomenologically well-described by the CKM matrix, the fundamental origin of these symmetry violations is still unknown [1]. For instance, the well-established and successful method to formulate fundamental theories with spontaneous symmetry violations is in terms of a relativistic Lagrangian operator in local quantum field theory. So rather paradoxically, although the "CKM paradigm" does naturally imply a remarkable experimental future in lepton physics in the long term, assuming quark-lepton symmetry and massive neutrinos, nevertheless the CKM matrix has developed in a quite Keplerian manner. Its formulation and parameters are indeed very important to test and measure, but the CKM matrix itself probably is not truly basic, mathematically or physically. Second, most astrophysics studies of electroweak baryogenesis conclude that additional sources of CP violation, beyond CKM, in elementary particle physics are necessary to explain the observed baryon-to-photon ratio. For these two reasons it is important to use new collider data to systematically search for possible experimental surprises such as for a non-CKM-type leptonic CP violation in τ lepton decays.

The idea in this paper [1] is to test for leptonic CP violation by generalizing the τ spin-correlation function $I(E_\rho, E_B)$ by including the ρ polarimetry [2–4] information that is available from the $\rho_{\mathrm{ch}} \to \pi_{\mathrm{ch}}\pi^0$ decay distribution [5–8].

Recall that since the ρ mode has the largest branching ratio [9], $B(\tau \to \rho\nu) \approx 25\%$, ongoing experiments with unpolarized e^-e^+ collisions do contain many events for the production-decay sequence

$$e^-e^+ \to Z^0\ ,\quad \gamma^* \to \tau^-\tau^+ \to (\rho^-\nu_\tau)(B^+X) \qquad (1.1)$$

and for the CP-conjugate sequence. Here γ^* denotes an off-mass shell photon, such as that produced in the Υ (10 GeV) resonance region at a B factory or near the τ-charm threshold at 4 GeV. The symbol $B = \rho, \pi, l$ and $X = \bar{\nu}_\tau$ or $\bar{\nu}_\tau\nu_l$, with $l = \mu$ or e. In the energy correlation function $I(E_\rho, E_B)$, the ρ's energy in the Z^0 or γ^* rest frame is E_ρ and the B's energy is E_B. The two-variable distribution $I(E_\rho, E_B)$ is useful as a probe for new physics because the empirical E_ρ and E_B energy correlation is a kinematic consequence of the τ^- and τ^+ spin correlation which depends in turn on the dynamics of the Z^0 or $\gamma^* \to \tau^-\tau^+$ amplitude, and of the $\tau^- \to \rho^-\nu$ and $\tau^+ \to B^+X$ decay amplitudes. It is found [4–7] that measurement of $I(E_A, E_B)$ determines independently the parameters $\sin^2\theta_W$, the τ Michel parameters for $\tau^- \to \bar{l}\nu_\tau\bar{\nu}_l$, and the chiral polarization parameter (chirality parameter)

$$\xi_A = \frac{|g_L|^2 - |g_R|^2}{|g_L|^2 + |g_R|^2} = \frac{2\,\mathrm{Re}(v_A a_A^*)}{|v_A|^2 + |a_A|^2} \qquad (1.2)$$

for $\tau^- \to A^-\nu_\tau$ ($\xi_A = \pm 1$ for $V \mp A$ coupling). ξ_A partially characterizes the Lorentz structure of the $\tau^- \to A^-\nu_\tau$ coupling [7]. In the special case of $m_\nu = 0$ and of only V and A couplings, it can be physically interpreted as (twice) the negative of the τ neutrino helicity $\xi_A = -\langle h_\nu \rangle$, see Eqs. (2.9) and (2.10) below.

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$I(E_\rho, E_B)$ has previously been [10,7] generalized, by including θ_e and ϕ_e to specify the initial e^- beam direction versus the final-state decay momenta. This permits complete measurement of the γ^* and Z^0 couplings to the $\tau^-\tau^+$ system, and enables four tests for CP and/or T in the production process, Z^0 or $\gamma^* \to \tau^-\tau^+$.

Both of these techniques are being applied in various ways in ongoing experiments in τ physics and elsewhere [11,12]. In this paper we study a different kinematic generalization by proceeding to the next stage in the decay sequence—we add ρ polarimetry information from $\rho_{\rm ch} \to \pi_{\rm ch}\pi^0$. This information is incorporated by generalizing $I(E_\rho, E_B)$ for process (1.1) to include the dependence of the $\rho^\mp$ decays on the $\pi^\mp$ polar and azimuthal angles $(\tilde\theta_1, \tilde\phi_1)$ and $(\tilde\theta_2, \tilde\phi_2)$ shown in Fig. 1. Since this adds on spin-correlation information from the next stage of decays in the decay sequence, we will call such a resulting energy-angular distribution $I(E_\rho, E_B; \tilde\theta_1, \tilde\phi_1, \tilde\theta_2, \tilde\phi_2)$ a stage-two spin-correlation (S2SC) function.

In analysis of S2SC functions, we often concentrate on the CP-symmetric decay sequence

$$Z^0 \text{ or } \gamma^* \to \tau^-\tau^+ \to (\rho^-\nu_\tau)(\rho^+\bar\nu_\tau) \tag{1.3}$$

followed by both $\rho^\mp \to \pi^\mp\pi^0$ because this sequence appears particularly promising experimentally to avoid a ρ^- versus ρ^+ bias [12]. As a simple consequence of adding in several more momenta variables, any prototype S2SC function depends on several additional variables and accordingly is more complex. The limited goal of this paper is to cast a big enough net to see what new physics information could be obtained, and how well, from applying such S2SC functions in τ physics [13].

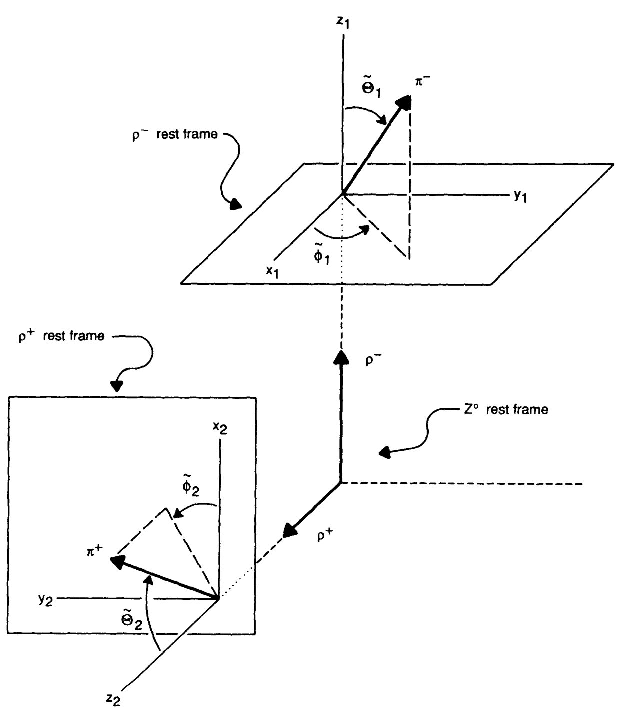

FIG. 1. The spherical angles $\tilde\theta_1, \tilde\phi_1$ specify the π^- momentum in $\rho_1{}^- \to \pi^-\pi^0$ decay in the $\rho_1{}^-$ rest frame when the boost is directly from the Z^0 or γ^* rest frame. Similarly, $\tilde\theta_2, \tilde\phi_2$ specify the π^+ momenta in the $\rho_2{}^{+-} \to \pi^+\pi^0$. The $\rho^-\rho^+$ production half-plane specifies the position x_1 and x_2 axes.

In a sentence, the principal conclusion of this paper is: By inclusion of ρ polarimetry observables, ongoing experiments with unpolarized e^-e^+ collisions enable two distinct tests for non-CKM-type leptonic CP violation in $\tau \to \rho\nu$ decay by generalization of the energy correlation function for Z^0, or $\gamma^* \to \tau^-\tau^+ \to (\rho^-\nu)(\rho^+\bar\nu)$.

It is simple to see why two CP tests are, in principle, possible kinematically: By Lorentz invariance, the decay $\tau^- \to \rho^-\nu_\tau$ depends on two independent helicity amplitudes, assuming a left-handed ν_τ, and the CP-conjugate decay $\tau^+ \to \rho^+\bar\nu_\tau$ also depends on two independent helicity amplitudes, assuming a right-handed $\bar\nu_\tau$.

(i) By CP invariance, the phase difference between the two amplitudes for $\tau^- \to \rho^-\nu_\tau$ decay should equal the phase difference between the two amplitudes for $\tau^+ \to \rho^+\bar\nu_\tau$ decay. To be precise, by CP invariance

$$\tilde\beta \equiv \beta_a - \beta_b = 0 \ , \tag{1.4}$$

where

$$\beta_a \equiv \phi^a_{-1} - \phi^a_0 \tag{1.5a}$$

since the ρ^- has helicity $\lambda_\rho = -1$ or 0, and

$$\beta_b \equiv \phi^b_1 - \phi^b_0 \tag{1.5b}$$

since ρ^+ has helicity $\bar\lambda_\rho = 1$ or 0. Rotational invariance forbids the other ρ^- and ρ^+ helicities; so that is why there are two, and not three, amplitudes for $\tau^- \to \rho^-\nu$.

(ii) By CP invariance, the ratio of the moduli of the two amplitudes for $\tau^- \to \rho^-\nu_\tau$ should equal that for $\tau^+ \to \rho^+\bar\nu_\tau$. That is, by CP invariance

$$r_a/r_b = 1 \ , \tag{1.6}$$

where for $\tau^- \to \rho^-\nu_\tau$ the moduli ratio is

$$r_a \equiv \frac{|A(-1,-1/2)|}{|A(0,-1/2)|} \tag{1.7a}$$

and for $\tau^+ \to \rho^+\bar\nu_\tau$ it is

$$r_b \equiv \frac{|B(1,1/2)|}{|B(0,1/2)|} \ . \tag{1.7b}$$

Notice that, respectively, the neutrino and antineutrino helicity is denoted by the second entry in the helicity amplitudes $A(\lambda_\rho, \lambda_\nu)$ and $B(\lambda_{\bar\rho}, \lambda_{\bar\nu})$. In the standard lepton model, $\beta_a = 0$ and

$$r_a = \frac{\sqrt{2}m_\rho}{E_\rho + q_\rho} \approx \frac{\sqrt{2}m_\rho}{m_\tau} \approx 0.613 \tag{1.8}$$

for $m_\nu \to 0$ and $m_\tau = 1.777$ GeV. (Ordinary CKM-type mixing in the lepton sector will not change β_a or r_a.)

Sections II–IV of this paper contain the derivations of the full S2SC function $I(E_1, E_2, \phi; \tilde\theta_a, \tilde\phi_a; \tilde\theta_b, \tilde\phi_b)$, and of the simpler $I(E_1, E_2, \tilde\theta_1, \tilde\theta_2)$, for the decay sequence

$$Z^0 \text{ or } \gamma^* \to \tau^-\tau^+ \to (\rho^-\nu_\tau)(B^+X) \ . \tag{1.9}$$

The full S2SC, indeed, depends on seven variables and the latter depends on four variables when applied to the $\{\rho^-\rho^+\}$ mode [three variables for the $\{\rho^-\pi^+\}$ and $\{\rho^- l^+\}$ modes].

Section V is for the reader who is only interested in the "analyzing power" of the two CP tests for the $\tau \to \rho\nu$ decay. There, the ideal statistical errors are tabulated in three tables. Neglecting completely the efficiency and/or acceptance depletion factors, we find that with 10^7 Z^0 events the $\tau \to \rho\nu$ decay amplitudes' moduli ratio r_a of Eq. (1.7) can be determined to about 1% by summing over the $B = \rho, \pi, l$ decay channels. We find that from only the $\{\rho^+\rho^-\}$ mode but by using the full seven-variable distribution, the amplitudes' phase difference $\tilde{\beta}$ of Eq. (1.4) can be measured to about 2° (i.e., two degrees). With 10^7 $\gamma^* \to \tau^-\tau^+$ events at either 10 or 4 GeV, we find that r_a can be determined to about 0.1% and $\tilde{\beta}$ to better than 1°.

Of course, the sensitivity of these two CP violation tests for $\tau \to \rho\nu$ should be compared with that of the partial width asymmetry of CP-conjugate reactions or processes

$$A_\Gamma \equiv \frac{\Gamma - \bar{\Gamma}}{\Gamma + \bar{\Gamma}} \tag{1.10}$$

which is a simple, but quite different, test for CP violation. For instance, in $\tau \to \rho\nu$ decay

$$\Gamma = \Gamma(\tau^- \to \rho^-\nu) = |A(-1,-1/2)|^2 + |A(0,-1/2)|^2 \ , \tag{1.11a}$$

$$\bar{\Gamma} = \bar{\Gamma}(\tau^+ \to \rho^+\bar{\nu}) = |B(1,1/2)|^2 + |B(0,1/2)|^2 \ . \tag{1.11b}$$

For τ two-body decay modes the denominator, $\Gamma + \bar{\Gamma}$, is known to about 1–4 % according to the Particle Data Group (1994). So A_Γ is considerably less sensitive than the two tests proposed in this paper. This is mainly because it is a test defined at the decay rate level.

The phenomenological significance of the signature $A_\Gamma \neq 0$ of Eq. (1.10) is also quite different: It tests for possible leptonic CKM-type CP violations, as well as for $r_a/r_b \neq 1$. It is not sensitive to $\tilde{\beta} \equiv \beta_a - \beta_b \neq 0$. On the other hand, the CP violation observables $\tilde{\beta} \equiv \beta_a - \beta_b$ and r_a/r_b for $\tau \to \rho\nu$ test for types of CP violation other than that due to a leptonic matrix, such as that due to multi-Higgs mechanisms. This is because any overall leptonic CKM-type phases will equally affect [14] the $A(-1,-\frac{1}{2})$ and $A(0,-\frac{1}{2})$ amplitudes, and so will cancel out in β_a and in r_a, and similarly in the $\tau^+ \to \rho^+\bar{\nu}_\tau$ amplitudes' β_b and r_b.

Measurement of a nonvanishing $\tilde{\beta} \equiv \beta_a - \beta_b \neq 0$, or of $r_a/r_b \neq 1$, would imply a violation of CP invariance. Measurement of $\beta_a \neq 0$ or of $\beta_b \neq 0$ implies a violation of T invariance when final-state interactions are absent. Such final-state effects are negligible for these S2SC functions in the case of process (1.9). This and a few other empirical matters are discussed in Sec. V.

It may prove fortunate empirically that there are tests for non-CKM-type leptonic CP violation in an age of active theoretical and/or experimental research on grand-unified theories and on supersymmetry theories which incorporate the known lepton-quark symmetry violations via multi-Higgs mechanisms [15]. In the context of other less empirically established theoretical formulations, kinematic tests for discrete symmetry violations in the lepton sector are probably even of greater interest, in that the tests are not dynamically dependent. Examples of latter theoretical formulations which come to mind are superstring theories which incorporate supersymmetry violation and $t\bar{t}$ condensate mechanisms, or even less precise compositeness ideas.

An advantage of working out the present tests in the helicity formalism is that the model independence and amplitude significance of the results is manifest. This is complementary to the greater dynamical information that can be obtained through other approaches, such as from studies of CP-violation effects based on higher-order diagrammatic calculations in multi-Higgs extensions of the standard model. Such increased dynamical information is obtained, of course, at the price of a greater model dependence which is one thing the tests in this paper are designed to avoid.

II. HELICITY AMPLITUDES FOR $\tau_{\rm ch} \to \rho_{\rm ch}\nu \to (\pi_{\rm ch}\pi^0)\nu$

In the $\tau_1{}^-$ rest frame, the matrix element for the decay $\tau_1{}^- \to \rho^-\nu_\tau$ is defined by [16]

$$\langle \theta_1{}^\tau, \phi_1{}^\tau, \lambda_\rho, \lambda_\nu | 1/2, \lambda_1 \rangle = D^{1/2*}_{\lambda_1,\mu}(\phi_1{}^\tau, \theta_1{}^\tau, 0) A(\lambda_\rho, \lambda_\nu) \ , \tag{2.1}$$

where the λ's denote the respective helicities, $\mu = \lambda_\rho - \lambda_\nu$. The final ρ^- momentum is in the $\theta_1{}^\tau$, $\phi_1{}^\tau$ direction, see Fig. 2(a). In Fig. 2(a), we have set $\phi_1{}^\tau = 0$ for ease in illustration.

An important, but elementary, technical point is that we have set the third Euler angle equal to zero in the

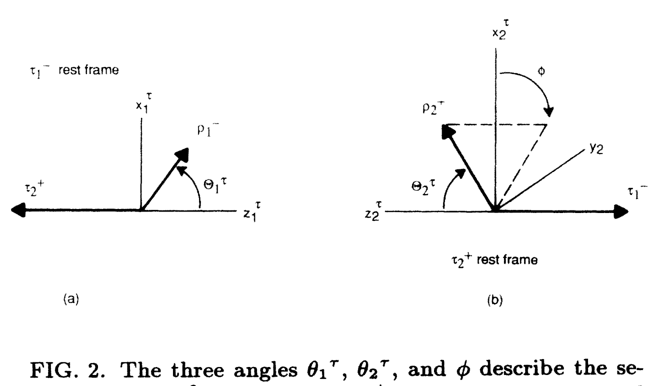

FIG. 2. The three angles $\theta_1{}^\tau$, $\theta_2{}^\tau$, and ϕ describe the sequential decay Z^0 or $\gamma^* \to \tau_1{}^-\tau_2{}^+$ with $\tau_1{}^- \to \rho_1{}^-\nu$ and $\tau_2{}^+ \to \rho_2{}^+\bar{\nu}$. From (a) a boost along the negative $z_1{}^\tau$ axis transforms the kinematics from the $\tau_1{}^-$ rest frame to the Z^0/γ^* rest frame and, if boosted further, to the $\tau_2{}^+$ rest frame shown in (b).

big D function in Eq. (2.1). We could have instead taken it to be a nonzero value such as the often used [17] value of $-\phi_1{}^\tau$. But a nonzero value of the third Euler angle implies an associated rotation about the final ρ^- momentum direction. Such an induced nonzero rotation would then have to be compensated for in defining the x_a coordinate axis in the ρ^- rest frame in Fig. 3 for $\rho^- \to \pi^-\pi^0$ decay (see [17]). This technical point is important in this paper because in the spin-correlation we exploit the azimuthal angular dependence of the second stage in the decay sequence $\tau_{\rm ch} \to \rho_{\rm ch}\nu \to (\pi_{\rm ch}\pi^0)\nu$.

Similarly in the ρ^- rest frame, the matrix element for $\rho^- \to \pi\pi^0$ is

$$\langle \tilde\theta_a, \tilde\phi_a | \lambda_\rho \rangle = D^1_{\lambda_\rho,0}{}^*(\tilde\phi_a, \tilde\theta_a, 0) c \ , \tag{2.2}$$

where the final π^- momentum is in the $\tilde\theta_a, \tilde\phi_a$ direction as shown in Fig. 3. Here c is a constant factor that is independent of λ_ρ. Note that these angles $\tilde\theta_a$ and $\tilde\phi_a$ specify the π^- in the ρ^- rest frame when the Lorentz boost (along the ρ^- momentum) is from the τ^- rest frame. These angles are for the (x_a, y_a, z_a) coordinate system in which the positive x_a axis lies in the τ^+ half-plane as shown in Fig. 3. In Eq. (2.2), the value of the third Euler angle can be chosen in any convention for it cancels out, at the observable's level, since we stop at the second stage in the decay sequence. We have set it equal to zero, and so are using the same convention here as we are using in Eq. (2.1).

For the CP-conjugate process $\tau_2 \to \rho^+\bar\nu_\tau \to (\pi^+\pi^0)\,\bar\nu_\tau$ the associated matrix element for $\tau_2{}^+ \to \rho^+\bar\nu_\tau$ is

$$\langle \theta_2{}^\tau, \phi_2{}^\tau, \lambda_{\bar\rho}, \lambda_{\bar\nu} | 1/2, \lambda_2 \rangle = D^{1/2*}_{\lambda_2,\bar\mu}(\phi_2{}^\tau, \theta_2{}^\tau, 0) B(\lambda_{\bar\rho}, \lambda_{\bar\nu}) \tag{2.3}$$

with $\bar\mu = \lambda_{\bar\rho} - \lambda_{\bar\nu}$. Figure 2 shows the relationship between the τ^- and τ^+ decay planes. The important azimuthal angle between these decay planes is defined by

$$\phi \equiv \phi_1{}^\tau + \phi_2{}^\tau \ . \tag{2.4}$$

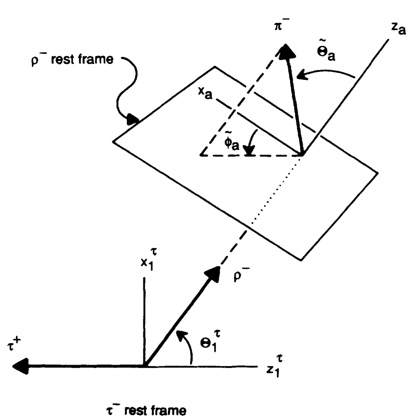

FIG. 3. The spherical angles $\tilde\theta_a, \tilde\phi_a$ specify the $\pi_1{}^-$ momentum in the $\rho_1{}^-$ rest frame when the boost is from the $\tau_1{}^-$ rest frame. The angular parameter ω_1, see text, specifies the necessary Wigner rotation about the implicit y_a axis to reach the ρ^- rest frame system $(x_1 y_1, z_1)$ of Fig. 1.

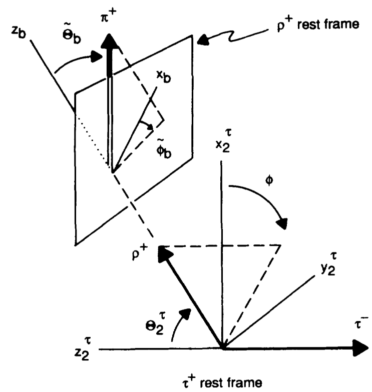

FIG. 4. Similar to Fig. 3, the spherical angles $\tilde\theta_b, \tilde\phi_b$ specify the $\pi_2{}^+$ momentum in the $\rho_2{}^+$ rest frame when the boost is from the $\tau_2{}^+$ rest frame. A Wigner rotation by ω_2 about the implicit y_b axis carries this $(x_b y_b z_b)$ coordinate system into the system $(x_2 y_2 z_2)$ of Fig. 1.

Since in Fig. 2(a) we set $\phi_1{}^\tau = 0$ for ease in illustration, in Fig. 2 we have $\phi = \phi_2{}^\tau$.

For $\rho^+ \to \pi^+\pi^0$, the matrix element is

$$\langle \tilde\theta_b, \tilde\phi_b | \lambda_{\bar\rho} \rangle = D^1_{\lambda_{\bar\rho},0}{}^*(\tilde\phi_b, \tilde\theta_b, 0)\bar c \ , \tag{2.5}$$

where the final π^+ momentum is in the $\tilde\theta_b$,$\tilde\phi_b$ direction as shown in Fig. 4. $\bar c$ is another constant factor. These angles $\tilde\theta_b, \tilde\phi_b$ specify the π^+ in the ρ^+ rest frame when the Lorentz boost is from the τ^+ rest frame. These angles are for the (x_b, y_b, z_b) coordinate system with the positive x_b axis in the τ^- half-plane.

Assuming a lefthanded ν_τ, $\tau^- \to \rho^-\nu_\tau$ depends only on

$$\begin{aligned} A(-1,-\tfrac{1}{2}) &= |A(-1,-\tfrac{1}{2})| e^{i\phi^a_{-1}} \ , \\ A(0,-\tfrac{1}{2}) &= |A(0,-\tfrac{1}{2})| e^{i\phi^a_0} \ . \end{aligned} \tag{2.6}$$

We neglect righthanded ν_τ amplitudes since for a pure V-A coupling [7] they are of order $A(1,\frac{1}{2})/A(-1,\frac{1}{2}) \approx m_\nu m_\tau/[(m_\tau)^2 - (m_\rho)^2]$ and $A(0,\frac{1}{2})/A(0,-\frac{1}{2}) \approx m_\nu (m_\rho)^2/\{m_\tau[(m_\tau)^2 - (m_\rho)^2]\}$. Likewise assuming a righthanded $\bar\nu_\tau$, $\tau^+ \to \rho^+\bar\nu_\tau$ depends on

$$\begin{aligned} B(1,\tfrac{1}{2}) &= |B(1,\tfrac{1}{2})| e^{i\phi^b_1} \ , \\ B(0,\tfrac{1}{2}) &= |B(0\tfrac{1}{2})| e^{i\phi^b_0} \ . \end{aligned} \tag{2.7}$$

By rotation invariance, $A(1,-\frac{1}{2}) = B(-1,\frac{1}{2}) = 0$. By CP invariance

$$B(\lambda_{\bar\rho}, \lambda_{\bar\nu}) = A(-\lambda_{\bar\rho}, -\lambda_{\bar\nu}) \ . \tag{2.8}$$

Although we neglect the right-handed ν_τ amplitudes in this paper, for completeness we note that in the case of both $(V \mp A)$ couplings and possibly $m_\nu \neq 0$, the $\tau^- \to \rho^-\nu$ amplitudes for $\lambda_\nu = -\frac{1}{2}$ are

$$\begin{aligned} A(0,-\tfrac{1}{2}) = g_L \left(\frac{E_\rho + q_\rho}{m_\rho}\right) \sqrt{m_\tau(E_\nu + q_\rho)} \\ - g_R \left(\frac{E_\rho - q_\rho}{m_\rho}\right) \sqrt{m_\tau(E_\nu - q_\rho)} \ , \end{aligned} \tag{2.9a}$$

$$A(-1,-\tfrac{1}{2}) = g_L\sqrt{2m_\tau(E_\nu + q_\rho)} - g_R\sqrt{2m_\tau(E_\nu - q_\rho)} \; . \tag{2.9b}$$

For $\lambda_\nu = \frac{1}{2}$ they are

$$A(-1,\tfrac{1}{2}) = 0 \; , \tag{2.10a}$$

$$A(0,\tfrac{1}{2}) = -g_L\left(\frac{E_\rho - q_\rho}{m_\rho}\right)\sqrt{m_\tau(E_\nu - q_\rho)} + g_R\left(\frac{E_\rho + q_\rho}{m_\rho}\right)\sqrt{m_\tau(E_\nu + q_\rho)} \; , \tag{2.10b}$$

$$A(1,\tfrac{1}{2}) = -g_L\sqrt{2m_\tau(E_\nu - q_\rho)} + g_R\sqrt{2m_\tau(E_\nu + q_\rho)} \; . \tag{2.10c}$$

Note that g_L and g_R, respectively, denote the chirality $(V \mp A)$ of the $\tau^- \to \rho^-\nu$ coupling whereas $\lambda_\nu = \mp\frac{1}{2}$ denotes the handedness of the (massive) τ neutrino. Thus, when $m_\nu = 0$, g_L (g_R), respectively, only appears in the $\lambda_\nu = -\frac{1}{2}$ $(+\frac{1}{2})$ amplitudes so the appropriately normalized, averaged neutrino helicity $\langle h_\nu\rangle = -\xi_p$. The $\tau^+ \to \rho^+\bar{\nu}_\tau$ amplitudes follow by CP invariance, Eq. (2.8).

III. JOINT DECAY DISTRIBUTION FUNCTIONS FOR $\tau_{\rm ch} \to \rho_{\rm ch}\nu \to (\pi_{\rm ch}\pi^0)\nu$

A. Composite decay density matrices

In Section II, the necessary helicity amplitudes have been defined. The associated composite decay density matrix for $\tau^- \to \rho^-\nu \to (\pi^-\pi^0)\nu$ is

$$R_{\lambda_1\lambda_1'}(\theta_1{}^\tau,\phi_1{}^\tau;\tilde{\theta}_a,\tilde{\phi}_a) = \sum_{\lambda_\rho,\lambda_\rho'} \rho_{\lambda_1\lambda_1';\lambda_\rho\lambda_\rho'}(\tau^- \to \rho^-\nu) \times \tilde{\rho}_{\lambda_\rho\lambda_\rho'}(\rho^- \to \pi^-\pi^0) \tag{3.1}$$

with

$$\rho_{\lambda_1\lambda_1';\lambda_\rho\lambda_\rho'}(\tau^- \to \rho^-\nu) = \sum_{\lambda_\nu=\mp 1/2} D^{1/2*}_{\lambda_1,\mu}(\phi_1{}^\tau,\theta_1{}^\tau,0) \times D^{1/2}_{\lambda_1',\mu'}(\phi_1{}^\tau,\theta_1{}^\tau,0)A(\lambda_\rho,\lambda_\nu) \times A^*(\lambda_\rho',\lambda_\nu) \; , \tag{3.2}$$

where $\mu = \lambda_\rho - \lambda_\nu$, $\mu' = \lambda_\rho' - \lambda_\nu$, and with

$$\tilde{\rho}_{\lambda_\rho\lambda_\rho'}(\rho^- \to \pi^-\pi^0) = D^{1*}_{\lambda_\rho,0}(\tilde{\phi}_a,\tilde{\theta}_a,0)D^1_{\lambda_\rho',0}(\tilde{\phi}_a,\tilde{\theta}_a,0) \; . \tag{3.3}$$

The overall $|c|^2$ factor has been omitted. Similarly, for $\tau^+ \to \rho^+\bar{\nu} \to (\pi^+\pi^0)\bar{\nu}$

$$\bar{R}_{\lambda_2\lambda_2'}(\theta_2{}^\tau,\phi_2{}^\tau;\tilde{\theta}_b,\tilde{\phi}_b) = \sum_{\lambda_{\bar{\rho}},\lambda_{\bar{\rho}}'} \rho_{\lambda_2\lambda_2';\lambda_{\bar{\rho}}\lambda_{\bar{\rho}}'}(\tau^+ \to \rho^+\bar{\nu}) \times \tilde{\rho}_{\lambda_{\bar{\rho}}\lambda_{\bar{\rho}}'}(\rho^+ \to \pi^+\pi^0) \; , \tag{3.4}$$

$$\rho_{\lambda_2\lambda_2';\lambda_{\bar{\rho}}\lambda_{\bar{\rho}}'}(\tau^+ \to \rho^+\bar{\nu}) = \sum_{\lambda_{\bar{\nu}}=\pm 1/2} D^{1/2*}_{\lambda_2,\bar{\mu}}(\phi_2{}^\tau,\theta_2{}^\tau,0) \times D^{1/2}_{\lambda_2',\bar{\mu}'}(\phi_2{}^\tau,\theta_2{}^\tau,0)B(\lambda_{\bar{\rho}},\lambda_{\bar{\nu}}) \times B^*(\lambda_{\bar{\rho}}',\lambda_{\bar{\nu}}) \; , \tag{3.5}$$

$$\tilde{\rho}_{\lambda_{\bar{\rho}}\lambda_{\bar{\rho}}'}(\rho^+ \to \pi^+\pi^0) = D^{1*}_{\lambda_{\bar{\rho}},0}(\tilde{\phi}_b,\tilde{\theta}_b,0)D^1_{\lambda_{\bar{\rho}}',0}(\tilde{\phi}_b,\tilde{\theta}_b,0) \; , \tag{3.6}$$

where in Eq. (3.5) $\bar{\mu} = \lambda_{\bar{\rho}} - \lambda_{\bar{\nu}}$, $\bar{\mu}' = \lambda_{\bar{\rho}}' - \lambda_{\bar{\nu}}$. In Eq. (3.6), the $|\bar{c}|^2$ factor is also omitted.

In this paper we assume a left-handed ν_τ in $\tau^- \to \rho^-\nu_\tau$ decay and a right-handed $\bar{\nu}_\tau$ in $\tau^+ \to \rho^+\bar{\nu}_\tau$. This assumption is discussed in Sec. II. It is straightforward to generalize [18] the following formulas to the case when pure ν_L and $\bar{\nu}_R$ couplings are not assumed, so as to include possible effects from $A(\lambda_\rho,\frac{1}{2})$ amplitudes, cf. Eq. (2.10).

It follows from Eq. (3.1) that the composite density matrix describing $\tau^- \to \rho^-\nu \to (\pi^-\pi^0)\nu$ for a left-handed ν is

$$R_{\lambda_1\lambda_1'} = \begin{pmatrix} R_{++} & e^{i\phi_1{}^\tau}r_{+-} \\ e^{-i\phi_1{}^\tau}r_{-+} & R_{--} \end{pmatrix} \; . \tag{3.7}$$

The diagonal elements are

$$R_{\pm\pm} = n_a[1 \pm f_a\cos\theta_1{}^\tau] \mp (1\sqrt{2})\sin\theta_1{}^\tau \sin 2\tilde{\theta}_a\cos(\tilde{\phi}_a - \beta_a)|A(0,-\tfrac{1}{2})||A(-1,-\tfrac{1}{2})| \tag{3.8}$$

and the off-diagonal elements depend on

$$\begin{aligned} r_{+-} &= (r_{-+})^* \\ &= n_a f_a\sin\theta_1{}^\tau + (1/\sqrt{2})\sin 2\tilde{\theta}_a[\cos\theta_1{}^\tau\cos(\tilde{\phi}_a - \beta_a) + i\,\sin(\tilde{\phi}_a - \beta_a)]|A(0,-\tfrac{1}{2})||A(-1,-\tfrac{1}{2})| \; . \end{aligned} \tag{3.9}$$

Note that only the moduli of the $\tau^- \to \rho^-\nu$ decay helicity amplitudes $A(0,-\frac{1}{2})$ and $A(-1,-\frac{1}{2})$ appear in the terms independent of the $\tilde{\phi}_a$ azimuthal angle since

$$\begin{pmatrix} n_a \\ n_a f_a \end{pmatrix} = \cos^2\tilde{\theta}_a|A(0,-\tfrac{1}{2})|^2 \pm \tfrac{1}{2}\sin^2\tilde{\theta}_a|A(-1,-\tfrac{1}{2})|^2 \; . \tag{3.10}$$

In $R_{\pm\pm}$ and $r_{\pm\mp}$ the terms sensitive to β_a do depend on $\tilde{\phi}_a$. In Eq. (3.18) below, this β_a dependence can be probed by the $\tilde{\theta}_1$ dependence of Fig. 1 because of the nonzero Wigner rotation by ω_1 [see following Eq. (3.20)]. The quantities in Eq. (3.10) depend on the longitudinal Γ_L and transverse Γ_T widths of $\tau \to \rho_{L,T}\nu$; whereas the β_a terms in Eq. (3.9) depend on the interference of the L and T amplitudes.

For the conjugate process $\tau^+ \to \rho^+\bar{\nu} \to (\pi^+,\pi^0)\bar{\nu}$, for a right-handed $\bar{\nu}$ it follows from Eq. (3.4) that

$$\bar{R}_{\lambda_2\lambda_2'} = \begin{pmatrix} \bar{R}_{++} & e^{i\phi_2{}^\tau}\bar{r}_{+-} \\ e^{-i\phi_2{}^\tau}\bar{r}_{-+} & \bar{R}_{--} \end{pmatrix} \tag{3.11}$$

with

$$\bar{R}_{\pm\pm} = n_b[1 \mp f_b \cos\theta_2{}^\tau] \pm (1/\sqrt{2}) \sin\theta_2{}^\tau \sin 2\tilde{\theta}_b \cos(\tilde{\phi}_b + \beta_b)|B(0,\tfrac{1}{2})||B(1,\tfrac{1}{2})| \;, \tag{3.12}$$

$$\begin{aligned} \bar{r}_{+-} &= (\bar{r}_{-+})^* \\ &= -n_b f_b \sin\theta_2{}^\tau - (1/\sqrt{2}) \sin 2\tilde{\theta}_b[\cos\theta_2{}^\tau \cos(\tilde{\phi}_b + \beta_b) + i\,\sin(\tilde{\phi}_b + \beta_b)]|B(0,\tfrac{1}{2})||B(1,\tfrac{1}{2})| \;, \end{aligned} \tag{3.13}$$

and

$$\begin{pmatrix} n_b \\ n_b f_b \end{pmatrix} = \cos^2\tilde{\theta}_b|B(0,\tfrac{1}{2})|^2 \pm \tfrac{1}{2}\sin^2\tilde{\theta}_b|B(1,\tfrac{1}{2})|^2 \;. \tag{3.14}$$

The diagonal elements in $R_{\lambda_1\lambda_1'}$ ($\bar{R}_{\lambda_2\lambda_2'}$) give the angular distributions for polarized $\tau_1{}^- \to \rho^-\nu \to (\pi^-\pi^0)\nu$ decay [$\tau_2{}^+ \to \rho^+\bar{\nu} \to (\pi^+\pi^0)\bar{\nu}$]. Specifically, for $\tau_1{}^-$ with positive helicity the two-stage, or joint, angular distribution is

$$\frac{dN}{d\cos\theta_1{}^\tau d\tilde{\Omega}_\rho} = R_{++}(\theta_1{}^\tau;\tilde{\theta}_a,\tilde{\phi}_a) \;, \tag{3.15}$$

where $d\tilde{\Omega}_\rho = d(\cos\tilde{\theta}_a)d\tilde{\phi}_a$. For $\tau_1{}^-$ with negative helicity the two-stage angular distribution is

$$\frac{dN}{d\cos\theta_1{}^\tau d\tilde{\Omega}_\rho} = R_{--}(\theta_1{}^\tau;\tilde{\theta}_a,\tilde{\phi}_a) \;. \tag{3.16}$$

[To rewrite these formulas for $\tau_2{}^+$ decay, the "barred" diagonal elements of Eq. (3.12) appear on the right-hand side and $\theta_1{}^\tau \to \theta_2{}^\tau$, $\tilde{\theta}_a \to \tilde{\theta}_b$, $\tilde{\phi}_a \to \tilde{\phi}_b$ on both sides of Eqs. (3.15) and (3.16).]

B. $dN/d(\cos\theta_1{}^\tau)d(\cos\tilde{\theta}_1)$ joint distribution for $\tau^- \to \rho^-\nu \to (\pi^-\pi^0)\nu$

Another two-stage angular distribution is particularly important for it has many possible polarimetry applications. We first list the distribution and then give its derivation which introduces the Wigner rotation angular parameter ω_1. It is the joint distribution for $\tau^- \to \rho^-\nu \to (\pi^-\pi^0)\nu$ in terms of the variables $\cos\theta_1{}^\tau$ and $\cos\tilde{\theta}_1$. Recall that $\theta_1{}^\tau$ is the polar angle of $\rho_1{}^-$ in the $\tau_1{}^-$ rest frame, see Fig. 2(a). Similarly $\tilde{\theta}_1$ is the polar angle of the $\pi_1{}^-$ in the $\rho_1{}^-$ rest frame when the boost is directly from the Z^0 rest frame (or γ^* rest frame), see Fig. 1. This joint distribution for $\tau_1{}^-$ with helicity $\lambda_1 = h/2$ is given by

$$\frac{dN}{d(\cos\theta_1{}^\tau)d(\cos\tilde{\theta}_1)} = \rho_{hh}(\theta_1{}^\tau,\tilde{\theta}_1) \;, \tag{3.17}$$

where the composite decay density matrix elements is given by [for off-diagonal counterparts see Eqs. (4.16)–(4.18) below]

$$\begin{aligned} \rho_{hh} = {} & (1 + h\cos\theta_1{}^\tau)[\cos^2\omega_1\cos^2\tilde{\theta}_1 + \tfrac{1}{2}\sin^2\omega_1\sin^2\tilde{\theta}_1] + (r_a^2/2)(1 - h\cos\theta_1{}^\tau)[\sin^2\omega_1\cos^2\tilde{\theta}_1 + \tfrac{1}{2}(1+\cos^2\omega_1)\sin^2\tilde{\theta}_1] \\ & + h(r_a/\sqrt{2})\cos\beta_a\sin\theta_1{}^\tau\sin 2\omega_1[\cos^2\tilde{\theta}_1 - \tfrac{1}{2}\sin^2\tilde{\theta}_1] \;. \end{aligned} \tag{3.18}$$

Here ω_1 is the Wigner rotation angular parameter [see Eq. (3.20) below]. Alternatively, ρ_{hh} can be written in the factorized form

$$\rho_{hh}(\theta_1{}^\tau,\tilde{\theta}_1) = \cos^2\tilde{\theta}_1 F(\theta_1{}^\tau) + \tfrac{1}{2}\sin^2\tilde{\theta}_1 G(\theta_1{}^\tau) \;, \tag{3.19a}$$

where

$$\begin{aligned} F(\theta_1{}^\tau) = {} & \cos^2\omega_1(1 + h\cos\theta_1{}^\tau) \\ & + (r_a^2/2)\sin^2\omega_1(1 - h\cos\theta_1{}^\tau) \\ & + h(r_a/\sqrt{2})\cos\beta_a\sin 2\omega_1\sin\theta_1{}^\tau \;, \end{aligned} \tag{3.19b}$$

$$\begin{aligned} G(\theta_1{}^\tau) = {} & \sin^2\omega_1(1 + h\cos\theta_1{}^\tau) \\ & + (r_a^2/2)(1+\cos^2\omega_1)(1 - h\cos\theta_1{}^\tau) \\ & - h(r_a/\sqrt{2})\cos\beta_a\sin 2\omega_1\sin\theta_1{}^\tau \;. \end{aligned} \tag{3.19c}$$

Equation (3.19a) is a simple generalization of $W^\pm(\cos\theta_1{}^\tau,\cos\tilde{\theta}_1)$ given in Rouge's Orsay paper [3,19].

C. Derivation of $dN/d(\cos\theta_1{}^\tau)d(\cos\tilde{\theta}_1)$ joint distribution

Equation (3.18) is very easily derived. The (x_a,y_a,z_a) coordinate system of Fig. 3 is transformed into the

(x_1, y_2, z_2) coordinate system of Fig. 1 upon [19] a Wigner rotation by ω_1 about the implicit y_a axis of Fig. 3. The angular parameter ω_1 is given by

$$\sin\omega_1 = m_\rho \beta\gamma \sin\theta_1{}^\tau / p_1 \;,$$
$$\cos\omega_1 = \frac{M(m^2 - m_\rho{}^2 + [m^2 + m_\rho{}^2]\beta\cos\theta_1{}^\tau)}{4m^2 p_1} \;, \tag{3.20}$$

where $M = E_{\mathrm{c.m.}} =$ the Z^0/γ^* mass, $m =$ the $\tau_1{}^-$ mass, $m_\rho = \rho^-$ mass, $p_1 =$ the magnitude of the ρ^- momentum in the Z^0/γ^* rest frame, and γ, β describe the relativistic boost to the $\tau_1{}^-$ rest frame [$\gamma = M/(2m)$]. Note that the ρ^- variables p_1, E_{ρ^-}, and $\theta_1{}^\tau$ are equivalent variables, see Eqs. (3.24)–(3.26) below.

The Wigner-rotation equations for transforming

$$\begin{pmatrix} \tilde\theta_a \\ \tilde\phi_a \end{pmatrix} \to \begin{pmatrix} \tilde\theta_1 \\ \tilde\phi_1{}^{(i)} \end{pmatrix} \tag{3.21}$$

are explicitly

$$\cos\tilde\theta_a = \cos\omega_1 \cos\tilde\theta_1 + \sin\omega_1 \sin\tilde\theta_1 \cos\tilde\phi_1{}^{(i)} \;, \tag{3.22a}$$

$$\sin\tilde\theta_a \cos\tilde\phi_a = -\sin\omega_1 \cos\tilde\theta_1 + \cos\omega_1 \sin\tilde\theta_1 \cos\tilde\phi_1{}^{(i)} \;, \tag{3.22b}$$

$$\sin\tilde\theta_a \sin\tilde\phi_a = \sin\tilde\theta_1 \sin\tilde\phi_1{}^{(i)} \;, \tag{3.22c}$$

where the i superscript denotes that this Wigner rotation is "initialized" versus the τ^- momentum direction in the Z^0 (or γ^*) rest frame. [The form of Eqs. (3.22), and of Eqs. (3.31) below, does not depend on whether it is the $i = B$ axis or the $i = A$ axis that the τ^- travels along. See Fig. 5, its caption, and the discussion in the next paragraph for the definition of these two axes. Also, the final result of the present derivation, i.e., Eq. (3.18), does not depend on which axis it is since $\tilde\phi_1{}^{(i)}$ is integrated out. However, the physical significance of the Wigner rotation versus the $\rho^-\rho^+$ half-plane does depend on which axis it is. For example, in Sec. IV in the discussion following the full S2SC function of Eq. (4.9), the experimentalist's observable event variables are expressed in terms of the $\rho^-\rho^+$ half-plane. Consequently, we introduce this terminology here.]

The first paper in Ref. [10] explains in detail the kinematics of the process

$$e^- e^+ \to \tau^- \tau^+ \to (\rho^- \nu)(\rho^+ \bar\nu) \tag{3.23}$$

in the present paper's notation. So, here, we only list the expressions needed to determine the Z^0, or γ^*, rest frame angles θ_1, θ_2, and $\cos\phi$. The angle between the two τ decay planes is ϕ, see Fig. 2. The direction of the $\tau_1{}^-$ momentum can be determined [20] up to an ambiguity as to whether to use a B or an A axis. This twofold ambiguity (due to the missing ν and $\bar\nu$ momenta) is illustrated in the present paper in Fig. 5. The three variables $E_\rho = E_1$, $p_\rho = p_1$, and $\theta_1{}^\tau$ are equivalent:

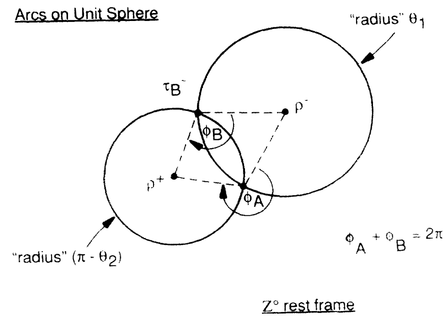

FIG. 5. Note that the remaining series of figures are all in the Z^0 or γ^* rest frame. Arcs on unit sphere about the ρ^- and ρ^+ momenta specify the τ^- momentum direction up to a twofold "A-axis" versus "B-axis" ambiguity. Note that $\phi_A + \phi_B = 2\pi$, and so $\cos\phi_A = \cos\phi_B$, but $\sin\phi_A = -\sin\phi_B$. Therefore, $\cos\phi$ is measurable but the sign of $\sin\phi$ is not, because of the missing ν and $\bar\nu$ momentum.

$$\theta_1{}^\tau = \arccos\left(\frac{-M(m^2 + m_\rho{}^2) + 4E_1 m^2}{(m^2 - m_\rho{}^2)\sqrt{M^2 - 4m^2}}\right) \;, \quad 0 \le \theta_1{}^\tau \le \pi \;, \tag{3.24}$$

where E_1 is the ρ^- energy in the Z^0 rest frame. So the angle θ_1 of $\rho_1{}^-$ in the Z^0 rest frame is determined uniquely from $\cos\theta_1$ and $\sin\theta_1$ of

$$p_1 \cos\theta_1 = \gamma(p_1{}^\tau \cos\theta_1{}^\tau + \beta E_1{}^\tau) \;, \tag{3.25a}$$

$$p_1 \sin\theta_1 = p_1{}^\tau \sin\theta_1{}^\tau \;, \tag{3.25b}$$

with p_1 the $\rho_1{}^-$ momentum in the Z^0 rest frame. Here $p_1{}^\tau$ and $E_1{}^\tau$ are, respectively, the momentum and energy of $\rho_1{}^-$ in the $\tau_1{}^-$ rest frame:

$$p_1{}^\tau = \frac{m^2 - m_\rho{}^2}{2m} \;, \quad E_1{}^\tau = [m_\rho{}^2 + (p_1{}^\tau)^2]^{1/2} \;. \tag{3.26}$$

Since θ_1 and θ_2 are known, $\cos\phi$ can be expressed explicitly in terms of the cosine of the opening angle ψ between the $\rho_1{}^-$ and $\rho_2{}^+$ momenta in the Z^0 rest frame:

$$\sin\theta_1{}^\tau \sin\theta_2{}^\tau \cos\phi = \frac{4m^2}{(m^2 - m_\rho{}^2)^2}\left(p_1 p_2 \cos\psi + \frac{(ME_1 - m^2 - m_\rho{}^2)(ME_2 - m^2 - m_\rho{}^2)}{M^2 - 4m^2}\right) \;. \tag{3.27}$$

To complete the derivation of Eq. (3.18), the azimuthal angle $\tilde\phi_1{}^{(i)}$ is integrated over (from zero to 2π).

D. $dN/d(\cos\theta_2{}^\tau)d(\cos\tilde\theta_2)$ joint distribution for $\tau^+ \to \rho^+\bar\nu \to (\pi^+\pi^0)\bar\nu$

Similarly for $\tau_2{}^+$ with helicity $\lambda_2 = h/2$ the joint distribution is given by

$$\frac{dN}{d(\cos\theta_2{}^\tau)d(\cos\tilde\theta_2)} = \bar\rho_{hh}(\theta_2{}^\tau, \tilde\theta_2) \;, \tag{3.28}$$

where

$$\bar\rho_{h,h} = \rho_{-h-h} \text{ (subscripts } 1 \to 2,\ a \to b) \;. \tag{3.29}$$

Equation (3.29) follows since $\tau^+ \to \rho^+\bar\nu \to (\pi^+\pi^0)\bar\nu$ is the CP-conjugate process and ρ_{hh} is even in β_a. This is a special case of the CP substitution rule shown below in Eq. (4.18).

The Wigner-rotation equations for transforming

$$\begin{pmatrix} \tilde{\theta}_b \\ \tilde{\phi}_b \end{pmatrix} \to \begin{pmatrix} \tilde{\theta}_2 \\ \tilde{\phi}_2{}^{(i)} \end{pmatrix} \tag{3.30}$$

are

$$\cos\tilde{\theta}_b = \cos\omega_2\cos\tilde{\theta}_2 + \sin\omega_2\sin\tilde{\theta}_2\cos\tilde{\phi}_2{}^{(i)} \ , \tag{3.31a}$$

$$\sin\tilde{\theta}_b\cos\tilde{\phi}_b = -\sin\omega_2\cos\tilde{\theta}_2 + \cos\omega_2\sin\tilde{\theta}_2\cos\tilde{\phi}_2{}^{(i)} \ , \tag{3.31b}$$

$$\sin\tilde{\theta}_b\sin\tilde{\phi}_b = \sin\tilde{\theta}_2\sin\tilde{\phi}_2{}^{(i)} \ , \tag{3.31c}$$

where i denotes that this Wigner rotation is initialized versus the τ^+ momentum direction in the Z^0 (or γ^*) rest frame. The angular parameter ω_2 is obtained as in Eq. (3.20) where now $\omega_1 \to \omega_2$ with the right-hand-side subscripts $1 \to 2$. This Wigner rotation by ω_2 is about the implicit y_b axis of Fig. 4.

The Wigner rotations in Eq. (3.22) [(3.31)] depend, respectively, on the τ^- (τ^+) momentum direction. So when a spin correlation between τ^- and τ^+ is being analyzed, e.g., in $I(E_1, E_2, \ldots)$ of Eq. (4.10) in the next section, one must choose the same i value in Eqs. (3.22) and (3.31).

IV. DERIVATION OF STAGE-TWO τ SPIN-CORRELATION FUNCTIONS

A. The full S2SC function

In this section we derive the full stage-two spin-correlation function for the decay sequence

$$\begin{aligned} Z^0 \ \text{or}\ \gamma^* &\to \tau_1{}^-\tau_2{}^+ \to (\rho_1{}^-\nu)(\rho_2{}^+\bar{\nu}) \\ &\to (\pi_1{}^-\pi_1{}^0\nu)(\pi_2{}^+\pi_2{}^0\bar{\nu}) \ . \end{aligned} \tag{4.1}$$

We start as for a beam-referenced spin-correlation function (see Sec. IV of first paper in Ref. [10]) and consider the production-decay sequence

$$e^-e^+ \to Z^0 \ \text{or}\ \gamma^* \to \tau_1{}^-\tau_2{}^+ \to \cdots \ , \tag{4.2}$$

where the ellipsis indicates the remaining decays of Eq. (4.1). The general angular distribution is

$$\begin{aligned} I(\Theta_B,\Phi_B;\theta_1{}^\tau,\phi_1{}^\tau;\tilde{\theta}_a,\tilde{\phi}_a;\theta_2{}^\tau,\phi_2{}^\tau;\tilde{\theta}_b,\tilde{\phi}_b) = \sum_{\lambda_1\lambda_2\lambda_1'\lambda_2'} & \rho^{\text{prod}}_{\lambda_1\lambda_2;\lambda_1';\lambda_2'}(e^-e^+ \to \tau_1{}^-\tau_2{}^+) \\ & \times R_{\lambda_1\lambda_1'}(\tau_1^- \to \rho^-\nu \to \cdots)\bar{R}_{\lambda_2\lambda_2'}(\tau_2{}^+ \to \rho^+\bar{\nu} \to \cdots) \ , \end{aligned} \tag{4.3}$$

where the composite decay density matrix for $\tau^- \to \rho^-\nu \to \cdots$ is given by Eq. (3.1), and that for $\tau^+ \to \rho^+\bar{\nu} \to \cdots$ is given by Eq. (3.4). For initially unpolarized particles in the e^-e^+ collision, the $\tau_1{}^-\tau_2{}^+$ production density matrix [10] is

$$\rho^{\text{prod}}_{\lambda_1\lambda_2;\lambda_1';\lambda_2'}(\Theta_B,\Phi_B) = \frac{T(\lambda_1,\lambda_2)T^*(\lambda_1',\lambda_2')}{|D_{Z/\gamma}|^2} e^{i(\lambda'-\lambda)\Phi_B} \left(1/4 \sum_{s_1,s_2} |\tilde{T}(s_1,s_2)|^2 d^1_{\lambda s}(\Theta_B) d^1_{\lambda' s}(\Theta_B) \right) \ , \tag{4.4}$$

where $\lambda = \lambda_1 - \lambda_2$, $\lambda' = \lambda_1' - \lambda_2'$, and $s = s_1 - s_2$. Equation (4.4) is for the center-of-mass frame of the e^-e^+ collision. The final τ^- direction specifies the z axis of the Z^0, or γ^*, polarization. In this reference system, Θ_B and Φ_B specify the e^- (beam) momentum's direction. In Eq. (4.4), the $e^-e^+ \to Z^0$, or γ^*, production amplitude is $\tilde{T}(s_1, s_2)$ and $D_{Z/\gamma}$ is the Z^0, or γ^*, propagator factor. With Eq. (4.3) there is an associated differential counting rate

$$\begin{aligned} dN = I(\Theta_B,\Phi_b,\ldots)&d(\cos\Theta_B)d\Phi_B d(\cos\theta_1{}^\tau)d\phi_1{}^\tau \\ &\times d(\cos\tilde{\theta}_a)d\tilde{\phi}_a d(\cos\theta_2{}^\tau)d\phi_2{}^\tau d(\cos\tilde{\theta}_b)d\tilde{\phi}_b \ , \end{aligned} \tag{4.5}$$

where, for full phase space, the cosine of each polar angle ranges from -1 to 1, and each azimuthal angle ranges from 0 to 2π.

From the general expression in Eq. (4.3), the full S2SC function follows directly: Neglecting $O(m_e/E_{\text{c.m.}})$ corrections due to the finite electron mass in the production density matrix, i.e., in the $\tilde{T}(s_1,s_2)$ amplitudes for $e^-e^+ \to Z^0$ or γ^*, in Eq. (4.4) the bracket factor

$$\begin{aligned} \{\,\} \to 1/4[&|\tilde{T}(+-)|^2 d^1_{\lambda 1}(\Theta_B) d^1_{\lambda' 1}(\Theta_B) \\ &+ |\tilde{T}(-+)|^2 d^1_{\lambda,-1}(\Theta_B) d^1_{\lambda',-1}(\Theta_B)] \ . \end{aligned} \tag{4.6}$$

Each term in Eq. (4.3) can depend on the angle between the two τ decay planes

$$\phi = \phi_1{}^\tau + \phi_2{}^\tau \tag{4.7}$$

and on the angular difference

$$\Phi_R = \Phi_B - \phi_1{}^\tau \ . \tag{4.8}$$

Holding ϕ and Φ_R fixed, the angle $\phi_1{}^\tau$ can be integrated out [21]. Next we integrate out the Φ_R and Θ_B angles. Consequently, the Z^0/γ^* production intensity factors are common for each term, i.e., there is an overall factor $[|\tilde{T}(+-)|^2 + |\tilde{T}(-+)|^2]$ times $|D_{Z/\gamma}|^{-2}$. We suppress it. The resulting full S2SC function is relatively simple:

$$I(E_1,E_2,\phi;\tilde{\theta}_a,\tilde{\phi}_a;\tilde{\theta}_b,\tilde{\phi}_b) = \sum_{h_1,h_2} |T(h_1,h_2)|^2 R_{h_1,h_1}\bar{R}_{h_2,h_2} + e^{i\phi}T(++)T^*(--)r_{+-}\bar{r}_{+-} + e^{-i\phi}T(--)T^*(++)r_{-+}\bar{r}_{-+} \ , \tag{4.9}$$

where the composite decay density matrix elements are given in Eqs. (3.7) and (3.11). The Z^0, or $\gamma^* \to \tau^-\tau^+$ amplitudes $T(h_1,h_2)$ are given in Ref. [10]. Note that on the left-hand side of Eq. (4.9), we have replaced the ${\theta_1}^\tau$ angular dependence by the center-of-mass system ρ^- energy E_1 [per Eq. (3.24)] and similarly ${\theta_2}^\tau$ by the ρ^+ energy E_2.

B. Wigner rotations and summation over $\tau^-\tau^+$ production axis

Due to the missing ν and $\bar{\nu}$ momenta, it is necessary to sum the Eq. (4.9) distribution over the B-axis–A-axis ambiguity to obtain the distribution to be compared with experimental data. It is straightforward to do this while also performing the associated Wigner rotations. We parametrize the difference between the A axis and the B axis by the two internal angles α_1 and α_2 shown in Fig. 6 for the case $0 \le \alpha_1 \le \pi$, and shown in Fig. 7 for the case $\pi \le \alpha_1 \le 2\pi$. α_1 and α_2 can be expressed [22] in terms of the observable variables $\cos\phi$, θ_1, and θ_2:

$$\tan(1/4[\alpha_1-\alpha_2]) = \cot(\phi/2)\frac{\cos(1/2[\theta_1+\theta_2])}{\cos(1/2[\theta_1-\theta_2])} \ , \tag{4.10a}$$

$$\tan(1/4[\alpha_1+\alpha_2]) = -\cot(\phi/2)\frac{\sin(1/2[\theta_1+\theta_2])}{\sin(1/2[\theta_1-\theta_2])} \ , \tag{4.10b}$$

where $\cot(\phi/2) = \sqrt{(1+\cos\phi)/(1-\cos\phi)}$.

We choose an experimental convention that the observable angle ϕ between τ^- and τ^+ decay planes which describes the "empirical event" lies in the range $[0,\pi]$. The corresponding full S2SC distribution is

$$dN = I(E_1,E_2,\ldots)d\tilde{\Omega}|_B + I(E_1,E_2,\ldots)d\tilde{\Omega}|_A \ , \tag{4.11a}$$

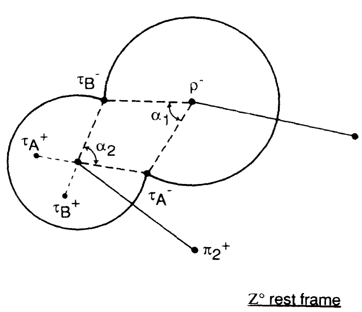

FIG. 6. Since $\cos\phi$, θ_1, and θ_2 are measurable, the angles α_1 and α_2 are known from elementary spherical trigonometry (see text). So, one can calculate the effect on the *full* stage-two spin-correlation function, Eq. (4.9), of not knowing the A axis versus the B axis. This figure is for the case $0 \le \alpha_1 \le \pi$.

where

$$d\tilde{\Omega}|_{(i)} \equiv d\phi\, d(\cos{\theta_1}^\tau)d(\cos{\theta_2}^\tau)d\tilde{\phi}_1\, d(\cos\tilde{\theta}_1) \times d\tilde{\phi}_2 d(\cos\tilde{\theta}_2) \ , \quad i=A,B. \tag{4.11b}$$

In Eq. (4.11a), $I(E_1,E_2,\ldots)$ is given by Eq. (4.9).

The first term in Eq. (4.11a) is the B-axis term. We express it in terms of the observable (experimentalist's) event variables. The B transformation equations are

$$\tilde{\phi}_1{}^B = \tilde{\phi}_1 + \alpha_1/2 \pm \pi \ , \qquad \tilde{\phi}_2{}^B = \tilde{\phi}_2 - \alpha_2/2 \ . \tag{4.12}$$

The upper (lower) sign in the first equation is for the case $0 \le \alpha_1 \le \pi$ ($\pi \le \alpha_1 \le 2\pi$). Thus, in $I(E_1,E_2,\ldots)$ of Eq. (4.9), to make the B transformation we use (upper signs for B-axis case)

$$\cos\tilde{\phi}_1{}^{B,A} = -\cos\tilde{\phi}_1\cos\alpha_1/2 \pm \sin\tilde{\phi}_1\sin\alpha_1/2 \ , \qquad \sin\tilde{\phi}_1{}^{B,A} = -\sin\tilde{\phi}_1\cos\alpha_1/2 \mp \cos\tilde{\phi}_1\sin\alpha_1/2 \ , \tag{4.13a}$$

and

$$\cos\tilde{\phi}_2{}^{B,A} = \cos\tilde{\phi}_2\cos\alpha_2/2 \pm \sin\tilde{\phi}_2\sin\alpha_2/2 \ , \qquad \sin\tilde{\phi}_2{}^{B,A} = \sin\tilde{\phi}_2\cos\alpha_2/2 \mp \cos\tilde{\phi}_2\sin\alpha_2/2 \ , \tag{4.13b}$$

in Eqs. (3.22) and (3.31) to replace explicitly $\tilde{\theta}_a,\tilde{\phi}_a \to \tilde{\theta}_1,\tilde{\phi}_1$ and $\tilde{\theta}_b,\tilde{\phi}_b \to \tilde{\theta}_2,\tilde{\phi}_2$.

The second term in Eq. (4.11a) is the A-axis term. It has $\phi_A = \phi + \pi$ in the analytic range $\phi_A \in [\pi,2\pi]$. The A-transformation equations, compare Fig. 7, are

$$\tilde{\phi}_1{}^A = \tilde{\phi}_1 - \alpha_1/2 + \pi \ , \qquad \tilde{\phi}_2{}^A = \tilde{\phi}_2 + \alpha_2/2 \ . \tag{4.14}$$

Thus, in $I(E_1,E_2,\ldots)$ to make the A transformation we

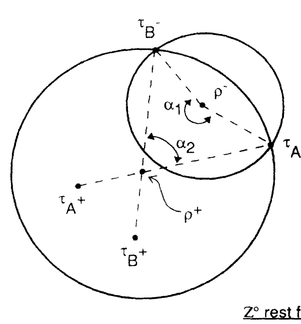

FIG. 7. Same as Fig. 6 except for case $\pi \le \alpha_1 \le 2\pi$.

use Eqs. (4.13a) and (4.13b) (lower signs for A-axis case) in Eqs. (3.22) and (3.31). [Since the τ^- momentum direction has been chosen in Fig. 5, Eqs. (4.12) and (4.14) are not symmetric.]

Equation (4.9) is therefore a parametric distribution, i.e., it is described explicitly in terms of $\tilde{\theta}_a$ and $\tilde{\phi}_a$ and parametrically in terms of $\tilde{\theta}_1$ and $\tilde{\phi}_1$, etc. This parametric formulation has occurred because of the necessary Wigner rotations [Eqs. (3.22)] and (3.31)] and because of the A-axis–B-axis summation [Eqs. (4.12) plus $\phi_B = \phi$ and Eqs. (4.14) plus $\phi_A = \phi + \pi$]. For many purposes, a simple parametric description is as useful as an explicit analytic distribution. In this manner, we obtain the results listed in Table III in the next section. [If the τ^- momentum direction were known, e.g., via a silicon vertex detector, one could use Eq. (4.9) directly, with greater sensitivity, and without these additional parametric transformations [18].]

C. Two simpler S2SC functions

The idea in this subsection is to integrate out some of the variables to obtain simple nonparametric distributions which still contain significant stage-two spin correlations.

In Eq. (4.9), the ρ^- composite density matrix elements $(R_{h_1,h_1}, r_{+-}, r_{-+})$ depend on $\tilde{\theta}_1{}^\tau, \tilde{\theta}_a, \tilde{\phi}_a$. By the Wigner rotation about the τ^- direction, Eq. (3.22), they depend on $\theta_1{}^\tau, \tilde{\theta}_1, \tilde{\phi}_1{}^{(i)}$. Similarly, the ρ^+ composite density matrix elements $(\bar{R}_{h_2,h_2}, \bar{r}_{+-}, \bar{r}_{-+})$ depend on $\theta_2{}^\tau, \tilde{\theta}_b, \tilde{\phi}_b$. So by the Wigner rotation about the τ^+ direction, Eq. (3.31), they depend on $\theta_2{}^\tau, \tilde{\theta}_2, \tilde{\phi}_2{}^{(i)}$. Since the ν and $\bar{\nu}$ momenta directions are unknown, we integrate out the two aximuthal angles $\tilde{\phi}_{1,2}{}^{(i)}$ and sum over the twofold A-axis–B-axis ambiguity. This gives a five-variable S2SC function

$$I(E_1, E_2, \phi; \tilde{\theta}_1, \tilde{\theta}_2) = \sum_{h_1 h_2} |T(h_1, h_2)|^2 \rho_{h_1 h_2} \bar{\rho}_{h_2 h_2} + 2\cos\phi \,\mathrm{Re}\{T(++)T^*(--)\rho_{+-}\bar{\rho}_{+-}\} \; . \tag{4.15}$$

If the A-axis–B-axis summation is not done because the τ^- momentum direction is known, then there is an extra term $(-2\sin\phi\,\mathrm{Im}\{\cdots\})$ where $\{\cdots\}$ is as in Eq. (4.15). This extra term would only vanish if both CP invariance holds in $(\tau^-\tau^+)$ production and $\beta_a = \beta_b = 0$ in $\tau^\mp$ decays.

The integrated, diagonal composite density matrix elements

$$\begin{aligned} \rho_{h_1 h_1} &\equiv (1/2\pi) \int_0^{2\pi} d\tilde{\phi}_1^{(i)} R_{h_1 h_1} / |A(0, -\tfrac{1}{2})|^2 \; , \\ \bar{\rho}_{h_2 h_2} &\equiv (1/2\pi) \int_0^{2\pi} d\tilde{\phi}_2^{(i)} \bar{R}_{h_2 h_2} / |B(0, \tfrac{1}{2})|^2 \; , \end{aligned} \tag{4.16}$$

were already listed in Eqs. (3.18) and (3.29). The off-diagonal counterparts are

$$\begin{aligned} \rho_{+-} &\equiv (1/2\pi) \int_0^{2\pi} d\tilde{\phi}_i^{(i)} r_{+-} / |A(0, -1/2)|^2 \\ &= (\rho_{-+})^* \equiv [(1/2\pi) \int_0^{2\pi} d\tilde{\phi}_1^{(i)} r_{-+} / |A(0, -1/2)|^2]^* \\ &= \sin\theta_1{}^\tau [\cos^2\omega_1 \cos^2\tilde{\theta}_1 + 1/2 \sin^2\omega_1 \sin^2\tilde{\theta}_1] - (r_a{}^2/2) \sin\theta_1{}^\tau [\sin^2\omega_1 \cos^2\tilde{\theta}_1 + 1/2(1 + \cos^2\omega_1)\sin^2\tilde{\theta}_1] \\ &\quad - \{r_a/(\sqrt{2})\}(\cos\beta_a \cos\theta_1{}^\tau - i\sin\beta_a)\sin 2\omega_1 [\cos^2\tilde{\theta}_1 - 1/2\sin^2\tilde{\theta}_1] \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \bar{\rho}_{+-} &\equiv (1/2\pi) \int_0^{2\pi} d\tilde{\phi}_1^{(i)} r_{+-} / |B(0, \tfrac{1}{2})|^2 \\ &= (\bar{\rho}_{-+})^* \equiv [(1/2\pi) \int_0^{2\pi} d\tilde{\phi}_2^{(i)} \bar{r}_{-+} / |B(0, \tfrac{1}{2})|^2]^* \\ &= -\rho_{+-}(\text{subscripts } 1 \to 2,\ a \to b,\ \beta_a \to -\beta_b) \; , \end{aligned} \tag{4.18}$$

which shows a useful CP substitution rule. These can also be written in the factorized forms

$$\begin{aligned} \rho_{+-}(\theta_1{}^\tau, \tilde{\theta}_1) &= \cos^2\tilde{\theta}_1 F_{+-}(\theta_1{}^\tau) + \tfrac{1}{2}\sin^2\tilde{\theta}_1 G_{+-}(\theta_1{}^\tau) \; , \\ \bar{\rho}_{+-}(\theta_2{}^\tau, \tilde{\theta}_2) &= \cos^2\tilde{\theta}_2 \bar{F}_{+-}(\theta_2{}^\tau) + \tfrac{1}{2}\sin^2\tilde{\theta}_2 \bar{G}_{+-}(\theta_2{}^\tau) \; , \end{aligned} \tag{4.19a}$$

where

$$F_{+-}(\theta_1{}^\tau) = [\cos^2\omega_1 - (r_a{}^2/2)\sin^2\omega_1]\sin\theta_1{}^\tau - \{r_a/(\sqrt{2})\}(\cos\beta_a \cos\theta_1{}^\tau - i\,\sin\beta_a)\sin 2\omega_1 \; , \tag{4.19b}$$

$$G_{+-}(\theta_1{}^\tau) = [\sin^2\omega_1 - (r_a{}^2/2)(1+\cos^2\omega_1)]\sin\theta_1{}^\tau + \{r_a/(\sqrt{2})\}(\cos\beta_a\cos\theta_1{}^\tau - i\,\sin\beta_a)\sin 2\omega_1\ . \tag{4.19c}$$

Here, $\bar{F}_{+-}(\theta_2{}^\tau)$ and $\bar{G}_{+-}(\theta_2{}^\tau)$ follow respectively from F_{+-} and G_{+-} by the transformation of Eq. (4.18), including of course the overall minus sign.

By integrating out the angle between the τ^- and τ^+ decay planes ($\int_0^\pi d\phi$), a simple four-variable S2SC function is obtained

$$\begin{aligned} I(E_1,E_2,\tilde\theta_1,\tilde\theta_2) &= \sum_{h_1h_2}|T(h_1h_2)|^2\rho_{h_1h_1}\bar\rho_{h_2h_2} \\ &= |T(+-)|^2\rho_{++}\bar\rho_{--} + |T(-+)|^2\rho_{--}\bar\rho_{++} + |T(++)|^2(\rho_{++}\bar\rho_{++}+\rho_{--}\bar\rho_{--})\ . \end{aligned} \tag{4.20}$$

In the second line, we have assumed $T(++) = T(--)$ which follows if one assumes CP invariance for the production process Z^0 (or γ^*) $\rightarrow \tau^-\tau^+$.

Note that the composite density matrices ρ_{hh} and $\bar\rho_{hh}$ of Eqs. (3.18) and (3.29) depend respectively on both r_a, β_a and r_b, β_b so the simpler S2SC given in Eq. (4.20) can be used to test for CP violation in $\tau \rightarrow \rho\nu$. In the next section, the ideal statistical errors for $\sigma(r_a)$ and $\sigma(\beta_a{}^2)$ associated with $I(E_{\rho^-}, E_{\rho^+}, \tilde\theta_1, \tilde\theta_2)$ are listed for the $\{\rho^-, \rho^+\}$ sequential decay mode.

V. IDEAL STATISTICAL ERRORS

Using both the simpler four variable and the full S2SC functions of Sec. IV, we have calculated the associated "ideal statistical errors" for a least-squares measurement [23] of the CP violation parameters $\tilde\beta = \beta_a - \beta_b$ and r_a/r_b. Our results are shown in Tables I–III. For simplicity in this paper we do not investigate correlations among the errors.

In Tables I and II, for the $\{\rho^-, \rho^+\}$ sequential decay mode the four-variable S2SC function $I(E_{\rho^-}, E_{\rho^+}; \tilde\theta_{\rho^-}, \tilde\theta_{\rho^+})$ of Eq. (4.20) was used. In order to sum over the modes, we consider usage of this S2SC from a $\{\rho^-\rho^+\}$ data set to measure r_a and β_a with r_b and β_b at standard model values, Eq. (1.8). Then it could be used from the same data set to measure r_b and β_b, with r_a and β_a with SM values. If CP invariance holds, the values should agree within errors. Since ρ_{hh} of Eq. (3.18) depends only on $\cos\beta_a$ and $\beta_a = 0$ in the standard lepton model, the error is listed in the square of β_a. To calculate the error in measurement of r_a and β_a, the SM values were assumed. In the case of the $\{\rho^-\pi^+\}$ and $\{\rho^- l^+\}$ decay sequences, the analogous three-variable S2SC functions $I(E_{\rho^-}, E_B; \tilde\theta_{\rho^-})$ were used ($B = \pi, l$). The well-known decay density matrices [7,11] for $\tau \rightarrow B\nu$ just need to be inserted into Eq. (4.20). Note that by inclusion of the $\tau^+ \rightarrow \pi^+\bar\nu_\tau$ and $\tau^+ \rightarrow l^+\nu_l\bar\nu_\tau$ modes, there is an improvement by a factor of 2 in $\sigma(r_a)$, but only slight improvement in $\sigma(\beta_a{}^2)$.

For Table III, the full S2SC function $I(E_1, E_2, \phi; \tilde\theta_a, \tilde\phi_a; \tilde\theta_b, \tilde\phi_b)$, see Eq. (4.9), for the $\{\rho^-, \rho^+\}$ sequential decay mode was used [see Eqs. (4.10)–(4.14)]. There is an improvement by a factor of 6 to 13 in $\sigma(\beta_a)$ from the full S2SC versus the simpler four-variable $I(E_{\rho^-}, E_{\rho^+}, \tilde\theta_1, \tilde\theta_2)$ of Eq. (4.20). For the full S2SC function we do not list the values for $\sigma(r_a)$ because there is only a 10–15 % improvement in the errors in the measurement of the moduli ratio r_a for $\tau^- \rightarrow \rho^-\nu$.

In the tables notice that measurement of the phase differences $\beta_a, \ldots$ at γ^* energies rather than at the Z^0 does not improve as much as expected, ~ 6, through the increase of statistics. This happens because in using ρ polarimetry a Wigner rotation connects the center-of-mass-frame's ρ observables to the respective τ-rest-frame's ρ observables. For example, the β_a dependence of ρ_{hh} in Eq. (3.18) disappears right at the $\tau\bar\tau$ threshold. For this reason a measurement of $\beta_a, \ldots$ is more powerful at 10 GeV than at 4 GeV.

In an effective Hamiltonian framework, measurement of $\tilde\beta \neq 0$ or $\beta' \neq 0$ implies a violation of T invariance when a first-order perturbation in a Hermitian Hamiltonian is reliable. Final state electroweak interactions can [10] simulate T violation effects for the production-decay sequence of Eq. (1.1) but such effects are negligible for these S2SC functions: In the case of these S2SC functions, the direction of the initial e^- beam has been integrated out so there is no contribution from the $\mathrm{Im}[T(+-)T^*(-+)]$ term which is affected by $\gamma^* - Z^0$ interference, by the one-loop $Z^0 \rightarrow \tau^-\tau^+$ vertex correction, and by the interference of the π^0's from the ρ^- and ρ^+. Unlike in K_{l3} decays, since ν_τ is only weakly interacting there is no simple electromagnetic rescattering of the ν_τ and ρ^-.

The experimental background to the $\tau^- \rightarrow \rho^-\nu_\tau$ mode from the $\tau^- \rightarrow a_1{}^-\nu_\tau$ mode where $a_1{}^- \rightarrow \pi^- 2\pi^0$ could be included in a combined S2SC based on ρ and a_1 polarimetry as has been recently done for other τ spin-correlation tests [4]. This does require a smearing over the S2SC distributions. Smearing is also required to include the finite ρ width, see Thurn and Kolanoski in [7], and the usual e^-e^+ QED radiative corrections. How-

TABLE I. At $E_{\rm c.m.} = M_Z$, ideal statistical errors for two tests for CP violation in $\tau \rightarrow \rho\nu$ by the simpler S2SC function $I(E_1, E_2, \tilde\theta_1, \tilde\theta_2)$, see Eq. (4.20), for the sequential decay $Z^0 \rightarrow \tau^-\tau^+$ with $\tau^- \rightarrow \rho^-\nu \rightarrow (\pi^-\pi^0)\nu$ and $\tau^+ \rightarrow \rho^+\bar\nu$, $\pi^+\bar\nu$, or $l^+\nu_l\bar\nu_\tau$. We use 10^7 Z^0 events.

$E_{\rm c.m.} = M_z$ Mode	Number of events	Ideal statistical errors $\sigma(r_a)$	$\sigma(\beta_a{}^2)$
$\{\rho^-\rho^+\}$	20 302	0.006 5	$(12°)^2$
$\{\rho^-\pi^+\}$	9 847	0.009 1	$(12°)^2$
$\{\rho^- l^+\}$	29 074	0.005 6	$(15°)^2$
Sum of above modes	59 223	0.0039 [0.6%]	$(10°)^2$

TABLE II. At $E_{c.m.} = 10$ and 4 GeV, respectively, ideal statistical errors for two tests for CP violation in $\tau \to \rho\nu$ by the simpler S2SC function, Eq. (4.20), for the decay of an off-mass-shell photon $\gamma^* \to \tau^-\tau^+$ with $\tau^- \to \rho^-\nu \to (\pi^-\pi^0)\nu$, and $\tau^+ \to \rho^+\bar{\nu}$, $\pi^+\bar{\nu}$, or $l^+\nu_l\bar{\nu}_\tau$. We use 10^7 $\gamma^* \to \tau^-\tau^+$ events.

Mode	Number of events	$E_{c.m.} = 10$ GeV $\sigma(r_a)$	$\sigma(\beta_a{}^2)$	$E_{c.m.} = 4$ GeV $\sigma(r_a)$	$\sigma(\beta_a{}^2)$
$\{\rho^-\rho^+\}$	605 127	0.001 2	$(5.5°)^2$	0.001 1	$(8.8°)^2$
$\{\rho^-\pi^+\}$	293 527	0.001 7	$(5.9°)^2$	0.001 6	$(9.1°)^2$
$\{\rho^-l^+\}$	866 658	0.001 0	$(7.5°)^2$	0.001 0	$(11.5°)^2$
Sum of above modes	1 765 312	0.000 7 [0.1%]	$(4.7°)^2$	0.000 7 [0.1%]	$(7.3°)^2$

ever, such improvements of the analysis also do not give rescattering between the ν_τ and $(\pi^-\pi^0)$.

VI. CONCLUSIONS

In the context of spin-correlation data analyses in tau τ-physics, the purpose of this paper is to point out that ρ polarimetry gives a fundamental measurement of the non-CKM-type CP violation observables $\tilde{\beta} = \beta_a - \beta_b$ and r_a/r_b for $\tau \to \rho\nu$ decay. Notice in regard to measurement of β_a that at γ^* energies it is necessary to include τ spin correlations because if one integrates out E_{ρ^+} and $\tilde{\theta}_2$ in Eq. (4.20), there remains no dependence on β_a since $T(+-) = T(-+)$ for $\gamma^* \to \tau^-\tau^+$. At γ^* energies, β_a cannot be measured by only analyzing the decay of an unpolarized τ^- via $\tau^- \to \rho^-\nu \to (\pi^-\pi^0)\nu$. Analogously at the Z^0, without τ spin correlations the dependence goes as $\alpha_H \cos\beta_a$ where

$$\alpha_H \approx -2a_\tau\nu_\tau/(\nu_\tau{}^2 + a_\tau^2) \approx \langle P_\tau\rangle \approx -0.138\ .$$

Notice that the measurement of β_a by a simpler four-variable S2SC function is possible *only* because of the existence of the Wigner rotation. In ρ_{hh}, see Eq. (3.18), $\cos\beta_a$ appear as "$\cos\beta_a \sin 2\omega_1$." This observation is consistent with the listings in Tables I–III which show smaller statistical errors $\sigma(\beta_a)$ at higher energies, whereas in the tables the statistical errors $\sigma(r_a)$ go as the square root of the total number of events.

Assuming a $V \mp A$ coupling for $\tau^- \to \rho^-\bar{\nu}$ ($\tau^+ \to \rho^+\bar{\nu}$), it is important to ask: Since by CP invariance $B(\lambda_{\bar{\rho}}, \lambda_{\bar{\nu}}) = A(-\lambda_{\bar{\rho}}, -\lambda_{\bar{\nu}})$, can we measure more amplitude ratios or phase differences from $e^-e^+ \to \gamma^*$, $Z^0 \to \tau^-\tau^+ \to (\rho^-\nu)\ (\rho^+\bar{\nu}) \to \cdots$ by some other angular distribution? The answer is no. One cannot measure $|A(0,-\frac{1}{2})|$ versus $|B(0,\frac{1}{2})|$ because they each appear squared in the overall normalization factor of any $I(\cdots)$. One similarly cannot measure $\phi_0{}^a$ versus $\phi_0{}^b$ because A^*A appears in any $I(\cdots)$ and so the net $\tau^- \to \rho^-\nu$ phase is $\pm\beta_a$, or zero, likewise from B^*B.

TABLE III. Ideal statistical errors for CP and/or T violation tests based on the full S2SC function of Eq. (4.9) for the $\{\rho^-\rho^+\}$ sequential decay mode. Note that $\tilde{\beta} \equiv \beta_a - \beta_b$ and $\beta' \equiv \beta_a + \beta_b$.

$E_{c.m.}$	Number of $\{\rho^-\rho^+\}$ events	Ideal statistical errors $\sigma(\tilde{\beta})$ (deg)	$\sigma(\beta')$ (deg)	$\sigma(\beta_a)$ (deg)
M_Z	20 302	1.88	3.15	1.84
10 GeV	605 127	0.43	0.74	0.42
4 GeV	605 127	0.86	1.13	0.71

It is very important to notice that the kinematic complications (see Sec. IV B) in the measurement of β_a and r_a arise because of the two-fold ambiguity as to the τ^- momentum direction. For a large sample of events, this ambiguity may in practice be absent when silicon vertex detectors are operational in γ^*, $Z^0 \to \tau^-\tau^+ \to \cdots$ experiments. However, in the present paper, we have assumed that the τ^- momentum direction is unknown.

The prototype S2SC functions in this paper need to be simplified and optimized. Kinematic symmetry methods could be investigated which describe the same production-decay sequence, but in variables more convenient in practice than those which naturally occur in the helicity formalism. Clearly generalizations of triple-product correlations, and of the related final-state momentum tensor techniques of Bernreuther and Nachtmann, and others, might be useful [8,12] if generalized to S2SC. It probably would be helpful to incorporate the optimal variable(s) and techniques for τ polarization, which are being developed by Rouge and others associated with the ALEPH collaboration [3].

As in the case of ordinary spin correlations, S2SC functions are independent of the polarization state of the decaying particle [24] here the Z^0 or off-mass-shell photon γ^*. Consequently, there exists good factorization [7] from initial-state effects such as from initial-state QED radiation.

This spin-correlation technique for searching for CP violation is of course also relevant to other production-decay sequences. For example, (a) S2SC functions can be derived [18] for $\tau \to a_1\nu$ decay, (b) W polarimetry information from $W \to l\bar{\nu}$ can be used to test for possible CP violation in top quark decay $t \to Wb$ arising from a $t\bar{t}$ production process, and (c) τ polarimetry information in a S2SC could be used to test for possible non-CKM-type CP violation in $W \to \tau\nu$ decay arising from a W^+W^- production process.

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[9] A. Schwarz, in *Lepton and Photon Interactions*, Proceedings of the 16th International Conference, Ithaca, New York, 1993, edited by P. Drell and D. Rubin, AIP Conf. Proc. No. 302 (AIP, New York, 1994); W. Hollik *ibid.*; M. Davier, *Proceedings of the Second Workshop on Tau Lepton Physics* [1]; R. S. Galik, *ibid.* τ spin correlations at the standard model level are incorporated in the Monte Carlo simulation KORALB for γ^* energies, see S. Jadach and Z. Was, Comput. Phys. Commun. **64**, 267 (1991); and report, 1993 (unpublished). For numerical purposes we use $m_\tau = 1.777$ GeV, $M_Z = 91.187$ GeV, $B(Z \to \tau^-\tau^+) = 0.033\,55$, $B(\tau \to \rho\nu) = 0.2460$, $B(\tau \to \{\mu\nu + e\nu\}) = 0.3523$, and $B(\tau \to \pi\nu) = 0.1193$.

[10] C. A. Nelson, Phys. Rev. D **43**, 1465 (1991); S. Goozovat and C. A. Nelson, Phys. Lett. B **267**, 128 (1991); Phys. Rev. D **44**, 2818 (1991); in *The Vancouver Meeting—Particles and Fields '91*, Proceedings of the Joint Meeting of the Division of Particles and Fields of the American Physical Society and the Particle Physics Division of the Canadian Association of Physicists, Vancouver, 1991, edited by D. Axen, D. Bryman, and M. Comyn (World Scientific, Singapore, 1992).

[11] In the simple $X \to \Phi\Phi \to (K^+K^-)(K^+K^-)$, etc., spin-correlation tests, the evidence for CP violation at the X vertex is a term(s) odd in the angle ϕ between the two (K^+K^-) decay planes. For instance, $\sin\phi$ or $\sin 2\phi$. This was first shown in C. A. Nelson, Phys. Rev. D **30**, 1937 (1984), and then generalized in J. R. Dell'Aquila and C. A. Nelson, *ibid.* **33**, 80 (1986); **33**, 101 (1986). For examples of useful analogies between τ lepton polarimetry tests and t quark polarimetry tests, see C. A. Nelson, *ibid.* **41**, 2805 (1990).

[12] A very important practical question regarding tests for possible CP violation in τ lepton processes and in the very analogous t quark processes is: How homogeneous, and charge symmetric, are the magnetic field and other detector-trigger components? At the Top Quark Collider Workshop, Madison, Wisconsin, 1992, G. L. Kane emphasized this and argued that in the case of t quark processes CP violation effects smaller than $\sim \frac{1}{2}\%$ are probably undetectable [see C. J. C. Im, G. L. Kane, and P. J. Malde, Phys. Lett. B **317**, 454 (1993). However, M. E. Peskin claimed that well-chosen observables could be less sensitive to such matters [C. R. Schmidt and M. E. Peskin, Phys. Rev. Lett. **69**, 410 (1992), the $(-\Delta N_{LR}) = \epsilon$ parameter of the last paper of [11]]. Similar challenges arise in the (i) design and quantification of CP-symmetric detector systematic errors and from (ii) standard model backgrounds in kaon, in b quark, and in hyperon CP violation experiments. The literature on methods to test for CP violation in top quark processes includes the last paper of Ref. [11]; J. F. Donoghue and G. Valencia, Phys. Rev. Lett. **58**, 451 (1987); M. B. Gavela, F. Iddir, A. Le Yaouanc, L. Oliver, O. Péne, and J. C. Raynal, Phys. Rev. D **39**, 1870 (1989); J. Kuhn, W. Bernreuther, O. Nachtmann, P. Overmann, and T. Schröder, in *e^+e^- Collisions at 500 GeV: The Physics Potential*, Proceedings of the Workshop, Munich-Annecy-Hamburg, 1991, edited by P. Zerwas (DESY Report No. 92-123A, Hamburg, 1992); G. L. Kane, G. A. Ladinsky, and C. P. Yuan, Phys. Rev. D **45**, 124 (1991); D. Atwood and A. Soni, *ibid.* **45**, 2405 (1992); J. P. Ma and A. Brandenburg, Phys. Lett. B **298**, 211 (1993); T. Arens and L. M. Sehgal, *ibid.* **302**, 501 (1993); A. Pilaftsis, Z. Phys. C **47**,

95 (1990); M. Nowakowski and A. Pilaftsis, Mod. Phys. Lett. A **6**, 1933 (1991); G. Eilam, J. R. Hewett, and A. Soni, Phys. Rev. Lett. **67**, 1979 (1991); J. M. Soares, *ibid.* **68**, 2102 (1992); C. R. Schmidt, Phys. Lett. B **293**, 111 (1992); D. Atwood, G. Eilam, A. Soni, R. Mendel, and R. Migneron, Phys. Rev. Lett. **70**, 1364 (1993); R. Cruz, B. Grzadkowski, and J. F. Gunion, Phys. Lett. B **289**, 440 (1992); Jiang Liu, University of Pennsylvania Report No. UPR-0525T, 1992 (unpublished); D. Atwood, G. Eilam, and A. Soni, Phys. Rev. Lett. **71**, 492 (1993); N. G. Deshpande, B. Margolis, and H. D. Trottier, Phys. Rev. D **45**, 178 (1992); W. Bernreuther, O. Nachtmann, P. Overmann, and T. Schröder, Nucl. Phys. **B388**, 53 (1992); **B406**, 516(E); W. Bernreuther, J. P. Ma, and T. Schröder, Phys. Lett. B **297**, 318 (1992); D. Chang and W.-Y. Keung, Phys. Lett. B **305**, 261 (1993); W. Bernreuther and A. Brandenburg, Phys. Rev. D **49**, 4481 (1994). See also R. H. Dalitz and G. R. Goldstein, *ibid.* **45**, 1531 (1992); Int. J. Phys. **9**, 635 (1994).

[13] An improved, simplified formulation of this papers' two tests would provide further checks of the CP conserving and/or violating parts of the angular distributions given in Eq. (4.20), and in Eq. (4.9) with its associated parametric transformations. We thank J. F. Donoghue for emphasizing this. The distributions in Eq. (4.20), and Eq. (4.9), are consistent with CP transformations at the helicity variable level which interchange $\tau^- \to \rho^- \nu \to (\pi^- \pi^0)\nu$ variables with those for the CP conjugate decay sequence.

[14] We thank S. Pakvasa for emphasizing this. The argument in the text is for a pure V and A leptonic CKM coupling. For instance, a CKM phase in an S and/or P coupling contributes at the tree level only to the $A(0,-\frac{1}{2})$ amplitude, but not to the $A(-1,-\frac{1}{2})$ amplitude and so is measurable by a S2SC function. In general, T and T_5 contribute unequally at the tree level to both amplitudes. Multi-Higgs-boson models, as well as multiloop induced leptonic CKM effects, can produce CP-violating amplitudes which are observable by S2SC's.

[15] S. Weinberg, Phys. Rev. Lett. **37**, 657 (1976); Phys. Rev. D **42**, 860 (1990); see also T. D. Lee, *ibid.* **8**, 1226 (1973); Phys. Rep. C **9**, 148 (1974). For more recent references and a review of multi-Higgs-boson models, see I. I. Bigi, A. I. Sanda, and N. G. Uraltsev, in *Perspectives on Higgs Physics*, edited by G. L. Kane (World Scientific, Singapore, 1993). In theories with *both* CKM and non-CKM sources of CP violation, the CKM phase might be measurable by S2SC's. For such a theory, see P. H. Frampton and T. W. Kephart, Phys. Rev. D **47**, 3655 (1993).

[16] M. Jacob and G. Wick, Ann. Phys. (N.Y.) **7**, 209 (1959). Very readable introductions to the helicity formalism are in H. Pilkuhn, *Interactions to Hadrons* (North-Holland, Amsterdam, 1967); the books by Perl and by Martin and Spearman in Ref. [19] below; and J. D. Richman, Caltech Report No. CALT-68-1231 (unpublished); Caltech Report No. CALT-68-1148 (unpublished). Useful symmetry transformations are in Appendix E of M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964); J. Werle, *Relativisitic Theory of Reactions* (North-Holland, Amsterdam, 1966).

[17] The third Euler angle locates the x_a axis which is used to reference the second stage $\rho^- \to \pi^- \pi^0$ decay's momenta directions relative to the first stage $\tau^- \to \rho^- \nu$ decay's momenta directions. The third Euler angle γ is set equal to zero by S. M. Berman and M. Jacob, SLAC Report No. 43, 1965 (unpublished); compare Phys. Rev. **139**, B1023 (1965); and by S. U. Chung, *ibid.* **169**, 1342 (1968). However, in the often used Jacob-Wick paper γ is set equal to $-\phi_1{}^\tau$. In the present paper, a bit awkwardly we could choose the x_a coordinate axis to follow the Jacob-Wick (JW) convention as we will explain in detail. Note that with the JW choice, $\phi_1{}^\tau = -\gamma$, the production azimuthal angle is also being used to specify the value of the third Euler rotation which is about the z_a axis which lies along the ρ^- momentum. Essentially, the $\gamma = -\phi_1{}^\tau$ Euler angle choice induces a rotation of the x_a referencing axis backward in the ρ^- rest frame by the angle $\phi_1{}^\tau$: In Eq. (2.1), in $D^{1/2\,*}_{\lambda_1,\mu}(\alpha,\beta,\gamma)$ in the JW convention we would first set $\gamma = -\phi_1{}^\tau$ where still $\alpha = \phi_1{}^\tau$, $\beta = \theta_1{}^\tau$. Since $\alpha = \phi_1{}^\tau$, in the τ^- rest frame the $\tau^- \to \rho^- \nu$ production plane is at azimuthal angle $\phi_1{}^\tau$. A boost along the ρ^- momentum then gives a ρ^- rest frame. But in this ρ^- rest frame, the JW choice that $\phi_1{}^\tau = -\gamma$ means that the x_a referencing axis is to be chosen so that the $\tau^- \to \rho^- \nu$ production plane is also oriented with an azimuthal angle $\phi_1{}^\tau$. For the $\rho^- \to \pi^- \pi^0$ decay, this JW choice x_a referencing axis is at $\tilde{\phi}_a = 0$ where the (new) $\tilde{\phi}_a$ and (same) $\tilde{\theta}_a$ specify the π^- momentum.

[18] C. A. Nelson, M. Kim, and H.-C. Yang, Report No. SUNY BING 5/27/94, 1994 (unpublished).

[19] See also S. Orteu, Report No. UAB-LFAE 89-04, 1989 (unpublished). Wigner rotations are treated clearly in Secs. 9-8 and 9-9 of M.L. Perl, *High Energy Hadron Physics* (Wiley, New York, 1974). See also A. D. Martin and T. D. Spearman, *Elementary Particle Physics* (North-Holland, Amsterdam, 1970).

[20] J. R. Dell'Aquila and C. A. Nelson, Nucl. Phys. **B320**, 61 (1989); **B320**, 86 (1989).

[21] At this point in the derivation one has a generalization of Eq. (4.9). This generalization is the full beam-referenced state-two spin-correlation function. It can also be expressed in terms of observable variables by the three steps in Sec. IV of the first paper of Ref. [10] in combination with the Wigner rotation procedure explained in the present paper in the subsection following Eq. (4.9).

[22] *CRC Handbook of Mathematical Sciences* (CRC, Boca Raton, FL, 1987).

[23] We calculate ideal statistical errors as in C. A. Nelson, Phys. Rev. D **40**, 123 (1989). For instance, for a two-variable correlation we would distribute N events ideally over a two-dimensional ij grid according to the theoretical result, $I(x,y) = Z_0(x,y) + aZ_1(x,y)$; the ideal error in bin ij is $\sigma_{ij} = \sqrt{I(x_i,y_j)}$. By χ^2 minimization, the "ideal statistical error" in the measurement of "a" is $\sigma_a = \{\sum_{ij}[Z_1(x_i,y_j)/\sigma_{ij}]^2\}^{-1/2}$.

[24] T. L. Trueman, Phys. Rev. D **18**, 3423 (1978); N. P. Chang and C. A. Nelson, Phys. Rev. Lett. **40**, 1617 (1978).