Natural parity states of SU(6)×O(3) in baryon spectra: Evidence for (56,odd$^-$) via Δ states

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The possibility of some natural odd parity Δ states in the latest PDG tables belonging to (**56**, odd$^-$) members of SU(6)×O(3) which start with ($N=3$) is explored dynamically, by employing a new complex harmonic oscillator (HO) representation which provides a compact S_3-symmetric description and is also realizable as a solution of a 3D dynamical equation (BSE) based on a vectorlike harmonic confinement. The complex representation permits the identification of an additional quantum number (N_a) over and above those already realizable in the standard (real) HO basis, and also facilitates the solution of the above BSE in terms of the Casimir terms of several distinct SO(2,1) algebras deducible from the former. The model is precalibrated to a fairly representative cross section of the known (N,Δ) spectra so as to provide the needed mandate for extrapolation. Fair agreement with expected mass locations for several new Δ^- resonances is found in terms of appropriate (**56**, odd$^-$) quantum numbers which are not only unique in cases such as G_{39} and $I_{3,13}$, but also compete very favorably with parallel (**70**$^-$) assignments in several others. More vigorous searches are therefore recommended for such Δ^- states which are much less contaminated by mixing than are the corresponding N^- states.

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I. INTRODUCTION

One of the most successful classifications in baryon spectroscopy is based on SU(6)×O(3) whose continued confirmation by Particle Data Group (PDG) tables [1] since Berkeley XIII [2] is generally regarded as a bedrock for the harmonic-oscillator (HO) framework that underlies this classification. The full HO classification consists of several types of towers labeled by the total quantum number N, the orbital quantum number L (where $\mathbf{L}=\mathbf{l}_{12}+\mathbf{l}_3$ in obvious notation), the quark spin S ($=\frac{1}{2}$ or $\frac{3}{2}$), the J^P and the S_3 symmetry types ($\mathbf{s},\mathbf{m},\mathbf{a}$) going with **56**, **70**, **20** states, respectively [3]. The more important among these are the *natural parity* states [4]

$$(\mathbf{56},\text{even}^+),\quad (\mathbf{70},\text{odd}^-),\quad (\mathbf{70},\text{even}^+),\quad (\mathbf{56},\text{odd}^-)\ . \tag{1.1}$$

To these may be added a second set of towers consisting of *unnatural parity* states:

$$(\mathbf{20},\text{odd}^+),\quad (\mathbf{70},\text{even}^-),\quad (\mathbf{70},\text{odd}^+),\quad (\mathbf{20},\text{even}^-)\ . \tag{1.2}$$

The internal structure of any one of these towers may be augmented by including *radial* excitations within each species, as well as certain other (specialized) types of *scalar* excitations. Note that **56** and **20** go only with natural- and unnatural-parity states, respectively, while **70** goes with both.

The first two members of (1.1), which may be regarded as the "main sequence" terms since they start with $N=0$ and 1, respectively, received the earliest confirmation [2], and for several years were thought to be the only states realized in nature [4]. The third member of the series which starts with $N=2$ found some early signatures in $P_{11}(1710)$ and satisfied a theoretical demand for strong SU(6) breaking [5] of the nucleon wave function through mixing with (**70**,0^+). A search for the last member of the natural-parity series (1.1), which starts with $N=3$, was initiated by Cutkosky *et al.* [6] in the context of $D_{35}(1930)$ as a member of (**56**,1^-), but they found its mass too low (by 200 MeV) for such an $N=3$ assignment [7].

In contrast, a direct experimental signature for the unnatural-parity series (1.2), whose prototype is (**20**,1^+) with $N=2$, continues to be practically nil, after almost three decades of the quark model. A possible reason [8] could be that its **a**-type symmetry deviates so strongly from the standard **s** symmetry of the nucleon that its production mechanism through the normal π-N channel [9] is severely strained due to the need for a big "symmetry transfer" ($\Delta S_3=2$). This means that while the production of a **70** (**m** type) of natural parity requires a standard one unit of symmetry transfer ($\Delta S_3=1$) from **s** type to **m** type, in accordance with the spirit of the additive quark model [10], the production of a **20** (**a** type) through a similar mechanism requires *two successive stages* of symmetry transfers with $\Delta S_3=1$ each, giving rise to a total of $\Delta S_3=2$. This results in a considerable suppression of its production compared with $\Delta S_3=1$ processes through the standard production channels (πN) and (γN) (see Sec. V for a further discussion). Similar remarks apply to other members of the series (1.2) each of which involves the generic antisymmetric vector $\boldsymbol{\zeta}=\boldsymbol{\xi}\times\boldsymbol{\eta}$ ($L^P=1^+$), where ($\boldsymbol{\xi},\boldsymbol{\eta}$) form a two-component m representation of S_3 symmetry [3].

No such inhibiting selection rule operates against the production of (**56**, odd$^-$) states, the only outstanding *natural-parity* member of the series (1.1), so that it should be of considerable interest to look afresh for their signatures in the context of the 1990s [1], many years after they were first suggested [6,7]. The latest PDG tables [1] do seem to indicate several new $P=-1$ candidates in the 2–2.5 GeV regime, and the search for $(\mathbf{56}^-)$ members among them can be considerably narrowed down by concentrating attention on the Δ^- states $(S=\frac{3}{2})$ of the highest seniority $(N=L)$ and the highest J value $(J=L+\frac{3}{2})$, since these are much less contaminated by the background of neighboring $\mathbf{70}^-$-type Δ's than are their nucleonic $(S=\frac{1}{2})$ counterparts. For example, the simplest assignments for $G_{39}(2400)$ and $I_{3,13}(2750)$ are $L^P=3^-$ and 5^-, and respectively, each with $S=\frac{3}{2}$, thus naturally qualifying them as Δ members $(J=L+\frac{3}{2})$ of $(\mathbf{56},3^-)$ and $(\mathbf{56},5^-)$, respectively. A few other Δ candidates of lower seniority, and hence, not free from mixing with $\mathbf{70}^-$ states, are $G_{37}(2200)$ and of course $D_{35}(1930)$ [6]. The interested reader is also referred to some recent reviews on the subject [11].

In this paper we shall examine the status of these Δ^- states with a view to exploring their $(\mathbf{56}, \text{odd}^-)$ assignments on *dynamical grounds of their energy levels.* While we are aware of the low-star status of these resonances, it is precisely this fact that probably makes such an exercise worthwhile, since a dynamical input could provide a valuable complementary tool to the general methods [9] of partial-wave analysis (with high inelasticity factor) which may not be discriminatory enough. This is apart from the general problem of "mass shifts" associated with strong couplings to inelastic channels [9] (see Sec. V again). In this respect the locations of well-established (four-star) resonances should prove particularly useful for calibration of the dynamical method itself through a sufficiently illustrative comparison.

For greater confidence in the spin-parity assignments of these resonances it is also necessary to probe the structure of their respective wave functions whose simplest manifestations would come about in terms of their principal channels of production and decay $(\pi N, \gamma N, \Delta\pi)$. This had indeed been the practice [12,8] since the Feynman-Kislinger-Ravndal (FKR) days [13] of the early 1970s and was resumed with renewed vigor in the 1980s [14], within the respective dynamical models of the authors concerned. In this respect, ideally speaking, a proper test of a full-fledged dynamical model lies in a *simultaneous* comparison of both mass spectra and decay width predictions with data, but it is good to remember that the latter in general are more difficult to evaluate than the former, and the gap increases rapidly with the degree of sophistication envisaged for the model. For example, in an effectively three-dimensional (3D) approach, the overlap integrals for a two-body decay are relatively simple [12–14] and come under the general description of, say, the quark-pair-creation model [15] which uses the overlap of the corresponding 3D wave functions of the particles participating in the decay process together with such unifying principles such as partial symmetry [16,8]; and indeed the width calculations noted above [12–14] are by and large based on a similar philosophy. If, on the other hand, the dynamical model is more involved, such as a 4D Bethe-Salpeter equation (BSE), the calculational gap between spectroscopy and decay widths can be quite substantial. For while a 3D reduction of the BSE usually proves adequate for the spectroscopy [17], a fully reconstructed 4D wave function must be employed for the evaluation of the transition amplitudes via quark loops [18] (involving 4D overlap integrals), in order that the detailed implications of such a dynamical approach may be properly addressed. This is already hard enough for mesonic transitions [19], but the complexity increases quite rapidly for baryonic transitions [20].

In this paper, which is based on a 4D model, we shall address only the baryon spectroscopy sector (through a 3D reduction of the BSE), and relegate the evaluation of the transition amplitudes to a later communication (so as to keep the length of this paper within reasonable bounds). Nevertheless, we shall offer a general perspective, based on semianalytical considerations, as to which channels (production as well as decay) may be profitably checked for possible identification of $\mathbf{56}^-$ states by analogy with the more familiar $\mathbf{70}^-$ states (see Sec. V). For a further check on our conclusions, we shall use a prior calibration of the BS model through a comparison with a good cross section of baryonic data for the well-established states, so as to provide a meaningful mandate for extrapolation to less familiar cases.

The paper is organized as follows. Section II gives a panoramic view of the main steps leading to the 3D form, Eq. (2.18), of the BSE for a qqq system, for a logically self-contained presentation of the basic dynamics, but omitting inessential details which may be found in the published literature. In Sec. III, we outline a practical method of reduction of Eq. (2.18), in terms of the two independent 3D variables (ξ,η) [3], leading to Eq. (3.16), so as to make it amenable to further analytical treatment. To that end, Sec. IV outlines a new classification scheme in a *complex HO basis* (model independent) which not only provides a transparent S_3-symmetric description of the various qqq states involved (including the $\mathbf{56}^-$ states under study), through the identification of an additional quantum number N_a (over and above those accessible through the standard (ξ,η) basis [3]), but also greatly facilitates the solution of Eq. (3.15) in terms of the Casimirs of several distinct SO(2,1) algebras deducible from the complex basis. Section V presents first the numerical results on a good cross section of the *known* baryonic states, including the effect of mixing among them caused by the different pieces of the one-gluon-exchange (OGE) term, Eq. (3.13), evaluated in the complex HO basis. The prior calibration thus provided on the working of the model is then followed by listing the actual predictions of the model in respect of the Δ states of 56 odd$^-$, together with a brief discussion of the patterns to be expected on their couplings to standard (πN) channels vis-à-vis those already known from the corresponding couplings of $\mathbf{70}^-$ states. The method of normalization for the spatial wave functions in the complex HO basis is summarized in Appendix A, while some essential calculational details including the specification of the basic constants, as well as the treatment of the OGE term, Eq. (3.13), in terms of the wave functions in the complex basis, are outlined in Appendix B.

II. DYNAMICAL BASIS FOR qqq SYSTEM: 3D REDUCTION OF THE BSE

Our starting point for qqq dynamics is a Bethe-Salpeter equation whose pairwise interaction kernel K has a 3D but Lorentz-covariant support [21]:

$$(2\pi)^4\Psi(p_1p_2p_3)=i\sum_{123}S_{F_1}(p_1)S_{F_2}(p_2)\int d^4q'_{12}K(\hat{q}_{12},\hat{q}'_{12})\Psi(p'_1p'_2p_3)\ , \tag{2.1}$$

$$K(\hat{q},\hat{q}')=i\gamma_\mu^{(1)}i\gamma_\mu^{(2)}V(\hat{q},\hat{q}')\tfrac{1}{2}\lambda_1\cdot\tfrac{1}{2}\lambda_2\ . \tag{2.2}$$

The kernel (2.2) is vector-exchange-like, with the scalar function V being a sum of one-gluon exchange $V_{\rm OGE}$ and a confining term $V_{\rm con}$ [17,22]. This is closely analogous to contemporary studies [23] which generally seek to interpret such structures as infrared modifications to the gluon propagator, but has one important point of difference, viz., the function V is effectively *3D in content,* ulike the 4D structures envisaged in the latter [23]. The 3D support to the BS kernel has had a long history since the 1960s [24], a logical perspective on which, vis-à-vis the spectroscopic data [1] is summarized elsewhere [25]. A 3D support strongly demanded by the data [1] which are O(3)-like is an essential ingredient of the present approach and has been aimed at an explicitly Lorentz-covariant formulation. This last was achieved in stages, beginning with the instantaneous approximation [17,18] which got upgraded to the null-plane ansatz (NPA) obeying limited covariance [19,22,26]. The latter NPA was to serve as a springboard for a full-fledged Lorentz covariance symbolized by the covariant-instantaneity ansatz (CIA) [21] which represents the content of Eq. (2.1). Now, for a two-body $q\bar{q}$ system, the spectroscopic predictions of the 3D BSE based on CIA [21] are identical with those of NPA [22,27] and therefore did not have to be repeated. For qqq systems, on the other hand, one expects some small variations from the NPA predictions [26] because of two-body off-shell effects, and therefore the CIA predictions for such systems requires a fresh calibration to the well-established (N,Δ) states before the same can be taken seriously enough for the less known (56^-) ones under the present study.

The other aspect of our ansatz is a "Gordon reduced form" of the 4D BSE which , in the notation of Ref. [21], reads as

$$i(2\pi)^4\Phi(p_1p_2p_3)=\sum_{123}\frac{\frac{1}{2}\lambda_1\cdot\frac{1}{2}\lambda_2}{\Delta_1\Delta_2}\int d^4q'_{12}V_\mu^{(1)}V_\mu^{(2)}V(\hat{q}_{12},\hat{q}'_{12})\Phi(p'_1p'_2p_3)\ , \tag{2.3}$$

where [17,21]

$$\Delta_i=m_q^2+p_i^2 \tag{2.4}$$

and the intermediate quantity Φ is related to the actual wave function Ψ by [21]

$$\Psi(p_i)=\prod_1^3 S_F^{-1}(-p_i)\Phi(p_i)\ , \tag{2.5}$$

while the four-vectors $V_\mu^{(i)}$ are [17,21]

$$V_\mu^{(i)}=(p_{i\mu}+p'_{i\mu})+i\sigma_{\mu\nu}^{(i)}(p_i-p'_i)_\nu,\quad i=1,2,3\ . \tag{2.6}$$

The transverse components $\hat{p}_{i\mu}$ of the four-momenta $p_{i\mu}$ are defined as [21]

$$\hat{p}_{i\mu}=p_{i\mu}-\nu_iP_\mu,\quad \nu_i=P\cdot p_i/P^2\ , \tag{2.7}$$

where P_μ is the four-momentum of the baryon (mass M) and ν_i is the longitudinal component of $p_{i\mu}$:

$$\nu_1+\nu_2+\nu_3=1,\quad \hat{p}_{1\mu}+\hat{p}_{2\mu}+\hat{p}_{3\mu}=0 \tag{2.8}$$

so that only two of each set are independent. In terms of these quantities, the relative momenta $\hat{q}_{12}$ for (12) pair in (2.3) are merely $2\hat{q}_{12\mu}=\hat{p}_{1\mu}-\hat{p}_{2\mu}$ and are effectively 3D quantities, while $V_\mu^{(i)}$ is expressible as [21]

$$V_\mu^{(i)}=2\nu_iP_\mu+(\hat{p}_{i\mu}+\hat{p}'_{i\mu})+i\sigma_{\mu\nu}^{(i)}(\hat{p}_i-\hat{p}'_i)_\nu\ ,\quad i=1,2,3\ . \tag{2.9}$$

The requirement of 3D support for the entire kernel in (2.3) is now made explicit by demanding for each pair $V^{(1)}\cdot V^{(2)}$ its "on-shell" value and this amounts to the replacement $\nu_i\to\omega_i/M$ in (2.9), where

$$\omega_i^2=m_q^2+\hat{p}_i^2\ . \tag{2.10}$$

At this stage a comment is in order regarding the BS status of (2.3): Because of the Gordon reduction on the γ_μ matrices in (2.2), the pairwise kernels in (2.3) agree with the corresponding ones in (2.1) only on the respective mass shells of the quarks involved, but since (2.3) is now intended for the off-shell extensions of these quarks, this form of BS dynamics [17–22,26] represents a conscious departure [21] from the standard BSE (2.1). Such modifications of the BSE are, however, not unknown in the literature [28], where the motivating factor has generally been a desire to achieve better control on its algebraic structure. Indeed the NPA formulation [26] of the

BSE, with emphasis on Gordon reduction from the outset [17], has all along been motivated by a similar concern, in view of the semiempirical status of the BS kernel (2.2). The CIA formalism [21] presently used is merely a reiteration of the same theme in a Lorentz covariant manner. Next we make a 3D reduction of Eq. (2.3) in order to make contact with baryon spectroscopy. This has been worked out fully in Ref. [21], but the main steps are summarized for ready reference. Define the 3D wave function ψ as

$$\psi(\hat{p}_1\hat{p}_2\hat{p}_3)=\int ds_i dt_i \Phi(p_1p_2p_3) , \tag{2.11}$$

where

$$\sqrt{3}s_3=\nu_1-\nu_2, \quad 3t_3=-2\nu_3+\nu_1+\nu_2 \tag{2.12}$$

together with two similar sets (s_1t_1) and (s_2t_2) obtained from (2.11) by cyclic permutation. Note that ds_idt_i is S_3 invariant, so that the definition (2.10) can be taken over for all three terms on the right-hand side (RHS) of (2.3) with appropriate indexing. The rest of the procedure is now straightforward and follows closely the pattern already laid out for the original formulations [17,26]. One integrates both sides of (2.3) with respect to ds_3dt_3, making use of (2.11) as well as the measure [21]

$$d^4q'_{12}=d^3\hat{q}'_{12}Mds'_3\sqrt{3}/2 \tag{2.13}$$

to give, on its RHS,

$$\int ds'_3dt_3\Phi(p'_1p'_2p_3)=\psi(\hat{p}'_1\hat{p}'_2\hat{p}_3)$$

while the ds_3 integration on the RHS is expressed by the result [21]

$$(\sqrt{3}/2)\int Mds_3\Delta_1^{-1}\Delta_2^{-1}=2\pi i D_{12}^{-1} . \tag{2.14}$$

D_{12} is the 3D denominator function for the (12) pair and is given by the triangle function (λ):

$$D_{12}=(-\tfrac{1}{2})\Omega_{12}\lambda(\omega_1^2,\omega_2^2,M^2(1-\nu_3)^2)/M^2(1-\nu_3)^2 , \tag{2.15}$$

$$2\Omega_{12}^{-1}=\hat{\mu}_{12}\omega_1^{-1}+\hat{\mu}_{21}\omega_2^{-1} , \tag{2.16}$$

$$\hat{\mu}_{12'21}=(1-\nu_3)/2\pm(\omega_1^2-\omega_2^2)/2M^2(1-\nu_3) , \tag{2.17}$$

where ν_3 has its on-shell value ω_3/M in Eqs. (2.14)–(2.17), befitting the spectator status of quark No. 3 in the first term of (2.3). Identical results hold for the other two terms of (2.3), so that its resultant 3D reduction takes the form (cf. Refs. [17,26])

$$(2\pi)^3\psi(\hat{p}_1\hat{p}_2\hat{p}_3)=\sum_{123}\frac{\frac{1}{2}\lambda_1\cdot\frac{1}{2}\lambda_2}{D_{12}}\int d^3\hat{q}'_{12}V^{(1)}\cdot V^{(2)}V(\hat{q}_{12},\hat{q}'_{12})\psi(\hat{p}'_1\hat{p}'_2\hat{p}_3) . \tag{2.18}$$

III. REDUCTION OF EQ. (2.18)

We now outline the main steps for the reduction of (2.18) in a manner closely analogous to a previous NPA treatment [26], but with several refinements in the procedure so as (i) to incorporate the effect of the relativistic reduced mass of the two interacting quarks [27] in the presence of a spectator and (ii) to avoid "correction terms" with "bad" analytic behavior (such as the term $D_s^{-1}\Delta H$ in Eq. (3.16) of Ref. [26]) irrespective of their (expected) smallness. Our new procedure is based on the expected smallness of the S_3-invariant quantity

$$-\delta\equiv\omega_1+\omega_2+\omega_3-M \tag{3.1}$$

compared to ω_i and/or M. This gives from (2.15)–(2.17), the crucial result

$$D_{12}=-4\delta\omega_1\omega_2+O(\delta^2) \tag{3.2}$$

which ensures from (2.18) that all the three terms on its RHS have a *common denominator* δ^{-1}, so that the quantity δ, when transferred to the LHS, serves as a *natural* energy denominator function for the entire qqq system while avoiding any "wrong" analytic behavior in the resulting 3D equation. [Since the terms of $O(\delta^2)$ in (3.2) are fully calculable, any effect of their omission can be estimated perturbatively if necessary.] Next we recall the structure of $V(\hat{q},\hat{q}')$ whose confining part for *light* (ud) quarks may be considered harmonic, cf. [26,27] with $A_0=0$:

$$V_{\text{con}}(\hat{q},\hat{q}')=(\tfrac{3}{4})(2\pi)^3\omega_{qq}^2[\nabla_q^2+C_0/\omega_0^2]\delta^3(\hat{q}-\hat{q}') . \tag{3.3}$$

The spring constant ω_{qq}^2 for the pairwise interaction has the form

$$\omega_{q_1q_2}^2=4M_{12}\hat{\mu}_{12}\hat{\mu}_{21}\omega_0^2\alpha_s(M_{12}^2) . \tag{3.4}$$

To explain this formula, we have used the definition [27] of a "relativistic reduced mass" $\mu_{12}=M\hat{m}_1\hat{m}_2$ of the two constituents of fractional four-momenta $\hat{m}_{1,2}$ (in the manner of Wightman and Garding) forming a composite of mass M, but generalized it to include the presence of a spectator of fractional four-momentum ν_3. Here the two fractional four-momenta are $\hat{m}_{1,2}=\hat{\mu}_{12'}\hat{\mu}_{21'}$ of (2.17), while the composite mass of the pair is $M=M_{12}$. For equal mass kinematics, the approximation $\hat{\mu}_{12}=\hat{\mu}_{21}\approx(1-\nu_3)/2$ is adequate, while $M_{12}\approx M-\omega_3$ since δ, in (3.1), is small.

The rest of the procedure is closely analogous to Ref. [26] and amounts to collecting the different pieces contained in (3.1)–(3.4) and (2.9) into the master equation (2.18) and simplifying. This is achieved in terms of a basis set $\boldsymbol{\xi},\boldsymbol{\eta}$ defined by [26]

$$\sqrt{3}\xi_\mu=\hat{p}_{1\mu}-\hat{p}_{2\mu}, \quad 3\eta_\mu=-2\hat{p}_{3\mu}+\hat{p}_{1\mu}+\hat{p}_{2\mu} , \tag{3.5}$$

in a notation which emphasizes their 3D character and the corresponding angular momenta

$$\mathbf{L}_\xi=-i\boldsymbol{\xi}\times\nabla_\xi, \quad \mathbf{L}_\eta=-i\boldsymbol{\eta}\times\nabla_\eta \tag{3.6}$$

with a resultant $\mathbf{L}=\mathbf{L}_\xi+\mathbf{L}_\eta$ representing the total orbital momentum of the qqq system. To take full advantage of the HO form (3.3), it is useful to recast the (small) energy denominator δ in the alternative form

$$-2M\delta \approx (\omega_1+\omega_2+\omega_3)^2-M^2 \le 3(\omega_1^2+\omega_2^2+\omega_3^2)-M^2 \equiv \Delta \ , \quad (3.7)$$

where the (Schwartz) inequality is, in fact, quite close to "equality" for equal mass kinematics ($m_1=m_2=m_3 \equiv m_q$) and Δ becomes

$$\Delta = 9m_q^2 - M^2 + \tfrac{9}{2}(\xi^2+\eta^2) \ . \quad (3.8)$$

The resultant Master equation (2.18) may now be written in obvious pairwise notation as

$$\Delta\psi = W_{\rm con}\psi + W_{\rm OGE}\psi \ , \quad (3.9)$$

$$W_{\rm con} = (2M)(\omega_0^2/2)\sum_{123}(1-\nu_3)^2\alpha_{12}^s M_{12} \times \left[(\nabla_{12}^2 + C_0/\omega_0^2) + \frac{1}{\omega_1\omega_2}\left[\hat{Q}_{12}/4 - \frac{C_0}{\omega_0^2} - \mathbf{L}_{12}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) + \frac{i}{2}\mathbf{p}_3\times\nabla_{12}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2) - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \right]\right] \ , \quad (3.10)$$

$$\mathbf{L}_{12} = \mathbf{q}_{12}\times\nabla_{12}, \quad \nabla_{12} = \nabla_{p_1} - \nabla_{p_2}, \quad 2\mathbf{q}_{12} \equiv 2\hat{q}_{12}^\mu = \hat{p}_{1\mu} - \hat{p}_{2\mu} \ , \quad (3.11)$$

$$\hat{Q}_{12} = 4\mathbf{q}_{12}^2\nabla_{12}^2 + 8\mathbf{q}_{12}\cdot\nabla_{12} + 6 \ , \quad (3.12)$$

and the OGE term is expressed in a mixed $(\mathbf{r},\mathbf{p})$ representation as

$$W_{\rm OGE} = (4M/3)\sum_{123}\alpha_{12}^s \left[\frac{1}{r_{12}} + \frac{1}{\omega_1\omega_2}\hat{q}_{12\mu}\frac{1}{r_{12}}\hat{q}_{12}^\mu + \frac{1}{\omega_1\omega_2}\pi\delta^3(r_{12})1 - \tfrac{2}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) + \text{noncentral effects} \right] \ . \quad (3.13)$$

The OGE term will be calculated perturbatively in a 6D HO basis provided by the main (confining) term (3.10). This is analytically facilitated with sufficient accuracy through an (iterative) averaging procedure which amounts to the replacements

$$\nu_3 \to \langle \nu \rangle, \quad \alpha_{12}^s \to \langle \alpha_s \rangle, \quad M_{12} \to M - \langle \omega \rangle, \quad \omega_1\omega_2 \to \langle \omega \rangle^2 \ . \quad (3.14)$$

The resulting structure of (3.9) without the OGE term becomes

$$\begin{aligned}\Delta\psi &= W_{\rm con}\psi \\ &= M\omega_0^2(1-\langle\nu\rangle)^2\langle\alpha_s\rangle(M-\langle\omega\rangle)\left[2(\nabla_\xi^2+\nabla_\eta^2) + \frac{1}{\langle\omega\rangle^2}\frac{C_0}{2\omega_0^2}(M^2-3m_q^2+\Delta) + \frac{1}{\langle\omega\rangle^2}(\hat{Q}_B - 8\mathbf{J}\cdot\mathbf{S} + 18)/4 \right]\psi \\ &\quad + O(\delta^2, (\omega_i - \langle\omega\rangle)^2) \ , \end{aligned} \quad (3.15)$$

where the operators $\hat{Q}_B$ and $\mathbf{J}\cdot\mathbf{S}$ are defined as in Eqs. (3.8)–(3.11) of Ref. [26].

IV. COMPLEX HO BASIS FOR qqq STATES

Our aim is first to express the confining part (3.15) in a complex HO basis characterized by a scale parameter which may be read from this equation as

$$\beta^4 = 4M\omega_0^2\bar{\alpha}_s(1-\langle\nu\rangle)^2(M-\langle\omega\rangle)/9 \ , \quad (4.1)$$

$$\bar{\alpha}_s = 1/[1/\alpha_s - 2C_0 M(1-\langle\nu\rangle)^2/(M-\langle\omega\rangle)] \ .$$

To that end we define the three-vector complex quantities (dimensionless) [29]

$$\sqrt{2}\beta Z_i = \xi_i + i\eta_i, \quad \sqrt{2}\beta Z_i^* = \xi_i - i\eta_i \quad (4.2)$$

and their derivative forms

$$\sqrt{2}\beta^{-1}\partial_{Z_i} = \partial_{\xi_i} - i\partial_{\eta_i}, \quad \sqrt{2}\beta^{-1}\partial_{Z_i^*} = \partial_{\xi_i} + i\partial_{\eta_i} \ . \quad (4.3)$$

The angular momentum operators in the complex basis are

$$\mathbf{L}_Z = -i\mathbf{Z}\times\nabla_Z, \quad \mathbf{L}_Z^* = i\mathbf{Z}^*\times\nabla_Z^* \ . \quad (4.4)$$

Then the total angular momentum is

$$\mathbf{L} = \mathbf{L}_Z + \mathbf{L}_Z^* = \mathbf{L}_\xi + \mathbf{L}_\eta \ , \quad (4.5)$$

which is adequate for handling the $\mathbf{J}\cdot\mathbf{S}$ term in (3.15). However, for the OGE term, (3.13), we shall also need the "mixed" angular momentum operators [30]

$$\mathbf{L}_c = -i\mathbf{Z}^*\times\nabla_Z, \quad \mathbf{L}_c^* = i\mathbf{Z}\times\nabla_Z^* \ . \quad (4.6)$$

To handle the main HO terms as well as the operator $\hat{Q}_B$ in (3.15), define two sets of complex operators (tensor notation):

$$\sqrt{2}a_i = Z_i + \partial_{Z_i^*}, \quad \sqrt{2}a_i^* = Z_i^* + \partial_{Z_i},$$
$$\sqrt{2}a_i^\dagger = Z_i^* - \partial_{Z_i}, \quad \sqrt{2}a_i^{*\dagger} = Z_i - \partial_{Z_i}^*, \tag{4.7}$$

which satisfy the commutation relations

$$[a_i, a_j^\dagger] = [a_i^*, a_j^{*\dagger}] = \delta_{ij} \tag{4.8}$$

with all other pairs commuting. Next define

$$N_c = a_i^\dagger a_i, \quad N_c^* = a_i^{*\dagger} a_i^*, \tag{4.9}$$

which can be simultaneously diagonalized and now play the roles of operators N_ξ, N_η in the (ξ,η) basis [26]. On the other hand, the pair (N_c, N_c^*) exhibits better S_3-symmetry properties than (N_ξ, N_η) which shows up through the appearance of an *additional* quantum number $N_a = N_c - N_c^*$ over and above the total quantum number $N = N_c + N_c^* = N_\xi + N_\eta$. Indeed while N is diagonal in both real and complex bases, N_a exists only in the complex basis and has no counterpart in the (ξ,η) basis. [The situation is analogous to the diagonality of the charge operator for a scalar field when expressed in the complex (ϕ,ϕ^*) basis but not in the real (ϕ_1,ϕ_2) basis, while the energy remains diagonal in both.] Thus, in the complex basis we have identified a *new* quantum number N_a which being fully rooted in S_3 symmetry should hopefully provide a sharper classification of baryonic states for purposes of "mixing" wherever possible than in the standard (ξ,η) picture (see Sec. V for discussion). Next we define the nondiagonal number operators in association with the diagonal ones:

$$N_m = a_i a_i^{*\dagger}, \quad N_m^\dagger = N_m^* = a_i^* a_i^\dagger, \tag{4.10}$$

$$N = N_c + N_c^*, \quad N_a = N_c - N_c^*. \tag{4.11}$$

It is also necessary to define three sets of *two-step* (both up and down) ladders:

$$A = 2a_i a_i^*, \quad A^\dagger = 2a_i^\dagger a_i^{*\dagger}, \tag{4.12}$$

$$C = a_i a_i, \quad C^\dagger = a_i^\dagger a_i^\dagger, \tag{4.13}$$

$$C^* = a_i^* a_i^*, \quad C^{*\dagger} = a_i^{*\dagger} a_i^{*\dagger}, \tag{4.14}$$

in which $(A, A^\dagger)$ are S_3 symmetric and $(C, C^\dagger)$, $(C^*, C^{*\dagger})$ transform according to a [2,1] representation of S_3. It has been shown elsewhere [30] that the three sets

$$(A, A^\dagger, N+3), \quad (C, C^\dagger, N_c + \tfrac{3}{2}), \quad (C^*, C^{*\dagger}, N_c^* + \tfrac{3}{2}) \tag{4.15}$$

satisfy as many SO(2,1) algebras, each with spectra bounded from below [31], while the trio $(N_m, N_m^\dagger, N_a/2)$ satisfy a regular SO(3) algebra with $-N \le N_a \le N$. These algebras in the complex basis are similar to those already encountered in the real basis [26], but now they have better S_3-symmetry properties which show up among other things through the appearance of an additional SO(3) algebra, that of $(N_m, N_m^\dagger, N_a/2)$. The SO(2,1) Casimir terms of (4.15) may be expressed in terms of two parameters $u, u_c = u_c^*$ as $u(u+1)$; $u_c(u_c+1)$ [31] where [22,26,30]:

$$u = -\tfrac{3}{2}(N \text{ even}), \quad -2(N \text{ odd}),$$
$$u_c = (u_c^*) = -\tfrac{3}{4}(N_c \text{ even}), \quad -\tfrac{5}{4}(N_c \text{ odd}), \tag{4.16}$$

while the Casimiar term of $(N_m, N_m^\dagger, N_a/2)$ is simply $(N/2)(N/2+1)$.

In terms of operators (4.7)–(4.9), the principal HO terms in (3.16) may be checked to be proportional to $(N+3)$, after taking account of the scale factor β of (4.1). Further, the operator $\hat{Q}_B$ defined by Eqs. (3.8)–(3.11) of Ref. [26] is now expressible in terms of the above Casimir terms as

$$\hat{Q}_B = -8u_c(u_c+1) - 8u_c^*(u_c^*+1) - 4u(u+1)$$
$$-2N(N+2) + 2(N_c + \tfrac{3}{2})^2 + 2(N_c^* + \tfrac{3}{2})^2$$
$$+2N_a^2 - (N+3)^2 - 18 \tag{4.17}$$

after leaving out some nondiagonal terms which connect states differing by $\Delta N = 4$ [26]. This already suffices for the solution of (3.15) [except for some small correction terms of $O(\delta^2)$, etc.] which comprises the HO part of the interaction. To obtain the full solution, one must add the OGE term, Eq. (3.13), calculated perturbatively and inserted in (3.9). The final result may then be expressed as (cf. Ref. [26])

$$F(M,N) \equiv F_{\rm con}(M,N) + F_{\rm OGE}(M,N) = N + 3, \tag{4.18}$$

where

$$9\beta^2 F_{\rm con} = M^2 - 9m_q^2 + \frac{M\omega_0^2 \bar{\alpha}_s (1 - \langle \nu \rangle)^2}{M - \langle \omega \rangle} \times \left[\hat{Q}_B - 8\mathbf{J}\cdot\mathbf{S} + 18 + \frac{C_0}{2\omega_0^2}(M^2 - 3m_q^2) \right], \tag{4.19}$$

$$F_{\rm OGE} = \langle \overline{W}_{\rm OGE} \rangle / 9\beta^2. \tag{4.20}$$

The relation of $\overline{W}_{\rm OGE}$ to $W_{\rm OGE}$ is explained in Appendix B 2.

The calculation of $F_{\rm OGE}$ requires the knowledge of the 3D wave function ψ as solutions of the HO equation (3.15) for the different states under study. To this end we first need the entire wave function in all the degrees of freedom (DF's) in the complex basis, after factoring out the color-singlet part of the wave function $\varepsilon_{\alpha\beta\gamma}/\sqrt{6}$ [20] which is already antisymmetric. The active part of the wave function must then be *symmetric* in the orbital (ψ), spin (χ), and isospin (ϕ) DF's taken together. This construction is described elsewhere [30] in some detail, but the resultant structure for different states is summarized in the standard notation

$$|56\rangle^q = \psi^s \chi^s \phi^s, \quad |56\rangle^d = \psi^s(\chi_c \phi_c^* + \chi_c^* \phi_c)/\sqrt{2}, \tag{4.21}$$

$$|70\rangle^q = \chi^s(\psi_c \phi_c^* + \psi_c^* \phi_c)/\sqrt{2}, \tag{4.22}$$

$$|70\rangle^d=(\psi_c\chi_c\phi_c+\psi_c^*\chi_c^*\phi_c^*)/\sqrt{2} \quad (3.8)\;, \tag{4.23}$$

$$|20\rangle^q=\psi_a\chi^s\phi_a,\quad |20\rangle^d=\psi_a(\chi_c\phi_c^*-\chi_c^*\phi_c)/\sqrt{2}\;. \tag{4.24}$$

The superscripts **q** and **d** on the SU(6) states stand for quartet ($S=\frac{3}{2}$) and doublet ($S=\frac{1}{2}$) spin states; for each subspace, the scripts **s** (**a**) denote symmetric (antisymmetric) functions *(both real)*, while the script c denotes *complex* functions of [2,1] symmetry. The resultant wave functions (4.21)–(4.24) have definite S_3-symmetry properties, being either wholly real (S_3 symmetric) or wholly imaginary (S_3 antisymmetric). The construction of the orbital function (ψ^s,ψ_c,ψ_c^*) having correct S_3-symmetry properties is greatly facilitated by considering states of highest seniority ($N=L$) which are simulated through appropriate powers of Z_+ and Z_+^* [30], and noting that $\mathbf{Z}\cdot\mathbf{Z}^*$, $Z_+Z_+^*$, and Z_+^3 are all S_3 invariant, while (Z_+,Z_+^2) transform like (Z_+,Z_+^*). The angular momenta carried by these basic units are in conformity with the definition (4.4) of angular momenta in the complex basis. The *natural*-parity states (1.1) of maximal seniority may now be written as

$$|56^+;70^-;70^+;56^-\rangle$$
$$=(2Z_+Z_+^*)^l[1;Z_+;Z_+^2;Z_+^3]\psi_0 \tag{4.25}$$

as solutions of (3.15), where $l=0,1,2,3,\ldots$; $\psi_0=e^{-\mathbf{Z}\cdot\mathbf{Z}^*}$ is the ground-state wave function ($N=0$). The L^P value of these respective states are

$$L^P=2l^+,(2l+1)^-,(2l+2)^+,(2l+3)^-\;. \tag{4.26}$$

The two lowest *orbital* states of $N=L$ in the **56**$^-$ sequence are

$$|\mathbf{56},3^-\rangle=Z_+^3\psi_0,\quad |\mathbf{56},5^-\rangle=(2Z_+Z_+^*)Z_+^3\psi_0\;, \tag{4.27}$$

while the corresponding $N=3$ state of lower seniority ($L=N-2$) is

$$|\mathbf{56},1^-\rangle=2\mathbf{Z}^2Z_+\psi_0$$
$$(\text{or } 2\mathbf{Z}^{*2}Z_+^*\psi_0 \text{ for the c.c. state})\;. \tag{4.28}$$

In contrast, a radially excited (**70**,1^-) state of $N=3$ is

$$|\mathbf{70},1^-\rangle_R=(2\mathbf{Z}\cdot\mathbf{Z}^*-4)\cdot(Z_+)\psi_0\;. \tag{4.29}$$

This last is a particular case of the general result that a *single* radial excitation of each of the species listed in (4.25) gives rise to a multiplicative factor $(2\mathbf{Z}\cdot\mathbf{Z}^*-2l-3-n)$ to the corresponding wave function, with $n=0,1,2,3$, respectively. For completeness a (**70**,0^+) state can be read off from (4.28) by omitting the factor Z_+, while still another type of (**56**,0^+) state making its first debut at $N=4$ has the form $\mathbf{Z}^2\mathbf{Z}^{*2}\phi_0$. (These scalar excitations must be distinguished from conventional radial excitations.) Finally the basic building block for the unnatural-parity states (1.2) is $\boldsymbol{\zeta}=i\mathbf{Z}\times\mathbf{Z}^*$ which is a fully antisymmetric S_3 singlet. The corresponding structure of such states of highest seniority are

$$|\mathbf{20}^+;\mathbf{70}^-;\mathbf{70}^+;\mathbf{20}^-\rangle=\zeta_+[1;Z_+;Z_+^2;Z_+^3]\psi_0\;. \tag{4.30}$$

However, these states will not be considered in this paper.

The 3D normalizations for the natural-parity states (4.25) are compactly expressed by [30]

$$N_{ln}^{-2}=\pi^3\Gamma(l+1)\Gamma(l+n+1)/2^n\;, \tag{4.31}$$

where $n=0,1,2,3$ for the four listed cases, respectively. The derivation is outlined in Appendix A. This completes the essential ingredients for the evaluation of F_{OGE}, Eq. (4.20), in the complex basis which is described in Appendix B 2.

For purposes of (N,N_a) assignments, it is useful to note that the orbital structures (4.25) for $N=L$ states carry direct information about their $N_a=(N_c-N_c^*)$ status since each factor Z_+ carries one unit of N_c and a factor Z_+^* carries one unit of N_c^*. Further $N_a=1$ or 2 are associated with $|70\rangle$ states of either parity, while $N_a=0,3$ are associated with $|56\rangle$ states of parity (+) or (−), respectively. These considerations (modulo 2 units) apply automatically to the N_a assignments for lower seniority ($N>L$) states as well.

V. NUMERICAL RESULTS AND DISCUSSION

Equations (4.18)–(4.20) describe our net result for the mass (M) determination through an inversion of the nonlinear function $F(M,N)$ for given N [26]. Alternatively, this function which may be regarded as a sort of "figure of merit" for the model has the following significance: If the experimental masses are employed to calculate $F(M,N)$, its comparison with the expected value $N+3$ gives a direct measure of the agreement or otherwise of the model with the data. Therefore, before offering the predictions of the model in respect of **56**$^-$ states (Table II), it is necessary first to provide a reasonable calibration of its "tool," as outlined in Secs. II–IV, through a comparison with a fairly representative cross section of well-established resonances as given in Table I. In this respect a crucial role is played by the quantum number N_a which did not have any counterpart in our earlier formulation in terms of the real (ξ,η) basis [26] nor in other investigations of baryon spectroscopy so far [3,6,14]. Indeed this quantum number provides a good index of the extent to which the huge HO degeneracy gets lifted due to the various momentum and spin-dependent terms in the BS kernel: see, e.g., the structure of $\hat{Q}_B$, Eq. (3.12). More importantly, the general effect of N_a is strongly reflected in the expectation values of the F_{OGE} term, Eq. (4.20), for the different states involved. Further removal of degeneracy depends on the mixing of "allowed" states brought about by the full OGE package, Eq. (3.13), an exercise in which the N_a dependence, especially of the lower seniority states ($L<N$), again plays a very useful role for identification of the appropriate candidates for mixing. More specifically the N_a quantum number brings out very directly even the Coulomb term in (3.13) which has hitherto been regarded as playing a rather passive role in bringing about mixing among anything but radially excited baryonic states differing by $\Delta N=2$ (unlike for example, the more active roles of the Fermi-Breit and tensor

TABLE I. Comparison of BS model [21,22] with data [1] for main sequence resonances of highest seniority ($N=L$). For the meaning of $F(M,N)$, see text. $F_m(M,N)$ is the value of $F(M,N)$ after taking account of mixing among appropriate SU(6)×O(3) states, and indicates the expected improvement in M(th) through a comparison with its unmixed value $F(M,N)$ vis-à-vis $N+3$.

State	SU(6)	(N,N_a)	$F(M,N)$	$(N+3)$	M(th)	$F_m(M,N)$
$N(938)$	$(56,0^+)$	0,0	2.9936	3	941	
$\Delta(1232)$	$(56,0^+)$	0,0	3.0370	3	1222	
$D_{13}(1520)$	$(70,1^-)$	1,1	4.0368	4	1509	
$S_{31}(1620)$	$(70,1^-)$	1,1	4.5546	4	1759	4.2311
$D_{15}(1675)$	$(70,1^-)$	1,1	4.0012	4	1675	
$F_{15}(1680)$	$(56,2^+)$	2,0	4.6909	5	1764	4.7815
$D_{33}(1700)$	$(70,1^-)$	1,1	4.6489	4	1805	4.3095
$S_{31}(1900)$	$(70,1^-)_R$ [a]	3,1	5.5991	6	2015	5.9226
$D_{33}(1940)$	$(70,1^-)_R$	3,1	5.6093	6	2050	5.9487
$F_{37}(1950)$	$(56,2^+)$	2,0	5.1090	5	1922	
$F_{17}(1990)$	$(70,2^+)$	2,2	5.1954	5	1937	
$F_{15}(2000)$	$(70,2^+)$	2,2	5.6832	5	2096	5.6056
$G_{17}(2190)$	$(70,3^-)$	3,1	6.1401	6	2150	
$G_{19}(2250)$	$(70,3^-)$	3,1	5.8668	6	2286	
$H_{19}(2220)$	$(56,4^+)$	4,0	6.8674	7	2256	
$H_{39}(2300)$	$(56,4^+)$	4,0	6.9519	7	2314	
$H_{3,11}(2420)$	$(56,4^+)$	4,0	7.0773	7	2400	
$I_{1,11}(2600)$	$(70,5^-)$	5,1	8.0124	8	2597	
$K_{3,15}(2950)$	$(56,6^+)$	6,0	8.9084	9	2974	

[a]The subscript R stands for first radial excitation.

terms [32] which connect the states of the same N), now comes to the forefront by connecting several states of the same N, but with *different* N_a values.

Table I summarizes our results for a good cross section of the known N,Δ states. Both the $F(M,N)$ values for the experimental masses, as well as the predicted masses obtained by direct inversion of Eq. (4.18) are shown. The whole range of agreement is rather good, all the way up to $N=6$, when the allowed variations in the M values are taken into account. The effects of mixing via the Coulomb term are illustrated for two pairs:

$$D'_{33}(1700) \text{ vs } D''_{33}(1940) ,$$

$$S'_{31}(1620) \text{ vs } S''_{31}(1900)$$

through a simple prescription in which the two lower states retain their standard ($N=1$) assignments, while the two higher states are their respective radial excitations ($N=3$) [33]. (This is an example of the active role of the Coulomb term noted above.) On the other hand, the mass prediction of $F_{15}(56,2^+)$ is moderately improved by mixing with its $(70,2^+)$ counterpart via the (more conventional) Fermi-Breit term of Eq. (3.13). A more detailed list of comparison of theory with data may be found in [34] (whose methodology has been summarized in Appendix B), but the overall calibration provided by Table I should suffice to lend some credibility to this formalism before extrapolation to the less familiar 56^- states.

Table II represents our findings in the same format in

TABLE II. Predictions of (56, odd$^-$) assignments for Δ states, vis-à-vis (70^-) assignments where applicable. Two sample cases of nucleon states are included for comparison.

State	SU(6)	(N,N_a)	$F(M,N)$	$(N+3)$	M(th)
$D_{13}(2080)$	$(70,1^-)_R$ [a]	3,1	5.9735	6	2088
$D_{13}(2080)$	$(56,1^-)$	3,3	6.2208	6	2012
$S_{11}(2090)$	$(70,1^-)_R$	3,1	6.1249	6	2054
$S_{11}(2090)$	$(56,1^-)$	3,3	6.3694	6	1975
$D_{35}(1930)$	$(70,3^-)$	3,1	5.5125	6	2074
$D_{35}(1930)$ [b]	$(70,2^-)$	3,1	5.1694	6	2164
$D_{35}(1930)$	$(56,3^-)$	3,3	5.4296	6	2105
$D_{35}(1930)$	$(56,1^-)$	3,3	5.2954	6	2128
$S_{31}(2150)$	$(56,1^-)$	3,3	6.3718	6	2037
$S_{31}(2150)$	$(70,1^-)_R$	3,1	6.4607	6	2016
$G_{37}(2200)$	$(56,3^-)$	3,3	6.0563	6	2184
$G_{37}(2200)$	$(70,3^-)$	3,1	6.1753	6	2151
$G_{39}(2400)$	$(56,3^-)$	3,3	6.4485	6	2274
$I_{3,13}(2750)$	$(56,5^-)$	5,3	8.1406	8	2712

[a]The subscript R stands for first radial excitation.
[b]Unnatural-parity assignment for this case is considered only for illustrative purposes.

respect of the Δ^- states under study. Before discussing these results, a general feature about the spectra is in order: A (**56**, odd$^-$) state tends to yield a higher mass than the corresponding (**70**, odd$^-$) state, when the excitation quantum No. N is the same for both. (For even$^+$ states, a similar statement is true with the roles of **56** and **70** interchanged.) Next we remark that certain states such as G_{39} and $I_{3,13}$, for which the neighboring (**70**, odd$^-$) assignments are simply not available, seem to measure up fairly well to (**56**, odd$^-$) assignments with $N=3$ and 5, respectively, taking account of the uncertainties in the masses determined from partial-wave analysis [9]. The case of G_{37} suggests a better proximity to $(\mathbf{56},3^-)$ than to $(\mathbf{70},3^-)$, with mass ordering governed by the rule just stated. A precision fit is also possible with a mixture of these SU(6) multiplets brought out by the tensor term of the OGE [32], were it not for the low-star status of this state which makes such an exercise rather premature, pending a better understanding of its production mechanism.

Table II also shows the corresponding status of yet another Δ-resonance $S_{31}(2150)$ which seems to fit a $(\mathbf{56},1^-)$ assignment somewhat better than a $(\mathbf{70},1^-)_R$ assignment, Eq. (4.29). Since mixing cannot be ruled out here, we have for completeness also included two sample nucleonic candidates $D_{13}(2080)$ and $S_{11}(2090)$ in this analysis, and this brings out a comparable performance of $(\mathbf{56},1^-)$ and $(\mathbf{70},1^-)_R$ assignments. As regards the famous $D_{35}(1930)$ state [6,7] our analysis suggests that a $(\mathbf{56},1^-)$ pushes its mass up to 2128 MeV, compared to $(\mathbf{56},3^-)$ which gives 2105 MeV, and $(\mathbf{70},3^-)$ which gives the nearest value of 2074 MeV (still above the mark by ≈ 140 MeV).

So far we have relied only on the predicted mass positions of the 56^- states, together with their most likely SU(6)×O(3) status to determine the J^P assignments of the observed states. This is not to rule out the possibilities of mixing among competing **70** states, some of which are already illustrated in Table II. Further uncertainties in the masses are caused by "mass shifts" due to strong couplings to inelastic channels, as emphasized by Cutkosky and Hoehler in their respective discussions on the partial-wave analysis [9]. Therefore, additional support for SU(6)×O(3) assignments must come from other angles such as decay rates [13] to preferred channels, as well as production (or formation) amplitudes from accessible channels. As already noted in Sec. I, however, a complete numerical assessment on these lines involves considerable algebraic investment (without commensurate returns), the more so with more elaborate models. In particular, within the present BS formalism which provides mass positions with considerable accuracy (without free parameters), a complete evaluation of decay amplitudes requires 4D overlap integrals involving quark triangle loops [19], and cannot be compressed within the scope of the present paper (which is already long). A more realistic alternative in terms of 3D overlap integrals which is available in the existing literature [14] is unfortunately not relevant for the $\mathbf{56}^-$ states under study, since these papers [14] have been mostly concerned about calibration of their respective models with respect to data on the more prominent resonances (among which the $\mathbf{56}^-$ states have not figured).

"Direct" vs "Recoil" Terms in Single-Quark Transitions. Nevertheless, it is possible to make some general statements about the coupling status of $\mathbf{56}^-$ resonances to the principal production $(N\pi, N\gamma)$ and decay $(N\pi, N\eta, N\rho, \pi\Delta)$ channels, vis-à-vis those of the more prominent ones $(\mathbf{56}^+, \mathbf{70}^-)$, in a semianalytic fashion, through a comparison of the relative strengths with which their respective wave functions, Eq. (4.25), couple to these channels. Indeed, ignoring the internal structure of the pion, the operator for the Galilean invariant pseudoscalar interaction of quark No. 3 is [35,8]

$$\sigma^{(3)}\cdot(\mathbf{k}-\mathbf{p}_3\omega_k m_q^{-1}) , \tag{5.1}$$

where the second term represents the quark recoil effect [35]. A very similar structure also holds [8] for the em interaction in the manner of FKR [13]. If we now consider the (πN) production channel, the orbital matrix element for transition to a state ψ_L, as adapted to the complex basis, is of the form [8]

$$\int d^3Z d^3Z^* \psi_L^\dagger(Z,Z^*)\sigma\cdot[\mathbf{k}+i\omega_k(Z-Z^*)/m_q\sqrt{2}]\psi_0(Z-i\mathbf{k}/\sqrt{2},Z^*+i\mathbf{k}/\sqrt{2}) , \tag{5.2}$$

which is expressible in terms of a tensor $B_{(i)}^L$ of rank L in the general form [8]

$$[\sigma\cdot\mathbf{k}F_D(k^2)k_{i_1}+F_R(k^2)\sigma_{i_1}]k_{i_2}\cdots k_{i_L}B_{i_1 i_2\cdots i_L}^L \tag{5.3}$$

representing the direct (D) and recoil (R) form factors $F_{D,R}$, respectively. Here the final-state wave function ψ_L is any one of the types (4.25) corresponding to the respective L values given by Eq. (4.26). Therefore, an analytical comparison of the relative strengths of the transition amplitudes for the production of, say, the $(\mathbf{70},L^-)$ resonances with $L=2l+1$, versus the corresponding $\mathbf{56}^-$ states of *natural* parity under study $(L=2l+3)$, is available from a comparison of their respective wave functions which, from Eq. (4.25), differ essentially by a factor Z_+^2 (Z_+^{*2}) or its suitably stepped down counterpart. Such a factor, in turn, does *not* produce any extra suppression in the matrix elements for 56^- production over and above the normal centrifugal (k) effects already "budgeted" for the corresponding 70^- production in which the direct (D) and recoil (R) terms differ by *two* units of k factors. This is in sharp contrast to the corresponding mechanism of transition to *unnatural-parity* (20^+) states which involve an additional factor $\zeta = iZ\times Z^*$, Eq. (4.30), and therefore do *not* couple to the $(N\pi)$ channel via the pseudoscalar operator in this lowest order involving single quark transitions [10,35]. In other words, a strong *angular selection rule* is operative against transitions to unnatural-parity (20^+) states, a feature which was referred to as an S_3-symmetry selection rule $\Delta S_3=2$ in Sec. I [8].

We have thus demonstrated analytically that the production of 56^- states through the $N\pi$ channel is no more inhibited than that of the 70^- states which obey the "normal" selection rules [10,35,8]. A similar FKR-type [13] mechanism [8] is easily seen to hold for their relative production strengths through the $N\gamma$ channel. As regards the decay channel, an additional facility is available from the recoil term in Eq. (5.3) which predicts *enhanced* $L-1$ wave decays [35] to the $N\eta$ channel by virtue of the appearance of the factor ω_k in Eq. (5.1) due to Galilean invariance. This means that the decay channel $N\eta$ can provide an additional sensitivity for the possible detection of 56^- states on lines similar to the enhanced $N\eta$ mode for the famous $S_{11}(1535)$ state [35], over and above the more or less standard [14] channel preferences, such as stronger coupling to $\Delta\pi$ than to $N\pi$, which can again be attributed to a parallel role of the recoil term, viz., to overcome the centrifugal (k) barrier effects to the extent of *two* units as compared to the direct term [8].

To summarize, this limited exercise has been intended for a search for the locations of (**56**, odd$^-$) states as the only surviving members of the natural-parity series (1.1) still begging experimental support [1]. The search which has been confined to Δ states to avoid uncomfortable mixing effects, has been carried out within a complex HO basis which not only provides a very compact representation of S_3 symmetry, but is also realizable as solution of a dynamical equation (BSE) based on a vectorlike harmonic confinement [17–22]. The numerical predictions of the mass locations (Table II) are preceded by a prior contact with a representative cross section of known (N,Δ) resonances (Table I), thus serving as a precalibration of the dynamical model [21] employed for the purpose. The results suggest a rather natural place for several new Δ^- resonances in terms of appropriate (**56**, odd$^-$) quantum numbers which are fairly unique in some cases like G_{39} and $I_{3,13}$, and compete very favorably with parallel (**70**, odd$^-$) assignments where the latter cannot be ruled out. It is hoped that this kind of dynamical analysis which supplements the traditional partial-wave techniques [9] will stimulate more vigorous searches for these outstanding members of the natural-parity family (1.1). Further calculational details, including effects of smaller corrections such as $O(\delta^2/M^2)$, the nondiagonal terms of the $\hat{Q}_B$ operator, Eq. (3.13), etc., may be found in [34].

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APPENDIX A

Here we outline a method of integration for the normalization of the spatial wave functions (4.25) and (4.28)–(4.30) in the six-dimensional (Z, Z^*) space. The volume measure in the 6D (Z, Z^*) space may be expressed in the spherical basis as

$$\begin{aligned} d^6\tau &= d^3Z d^3Z^* \\ &= (dZ_+ dZ_-^*)(dZ_- dZ_+^*)(dZ_3 dZ_3^*) \end{aligned} \tag{A1}$$

where the six elements on the RHS have been rearranged into three sets of *real* 2D volumes since $(dZ_+ dZ_-^*)$, $(dZ_- dZ_+^*)$ and $(dZ_3 dZ_3^*)$ each form complex conjugate pairs. Now put

$$\begin{aligned} \sqrt{2}(Z_+; Z_-^*) &= \mathcal{R}_1 e^{\pm i\vartheta_1} , \\ \sqrt{2}(Z_-; Z_+^*) &= \mathcal{R}_2 e^{\pm i\vartheta_2} , \\ \sqrt{2}(Z_3; Z_3^*) &= \mathcal{R}_3 e^{\pm i\vartheta_3} . \end{aligned} \tag{A2}$$

Then the volume element (A1) becomes

$$d^2\tau = \mathcal{R}_1 d\mathcal{R}_1 d\vartheta_1 \cdot \mathcal{R}_2 d\mathcal{R}_2 d\vartheta_2 \cdot \mathcal{R}_3 d\mathcal{R}_3 d\vartheta_3 , \tag{A3}$$

where

$$0 \le \mathcal{R}_{1,2,3} \le \infty, \quad 0 \le \vartheta_{1,2,3} \le 2\pi \tag{A4}$$

and

$$\mathcal{R}_1^2 + \mathcal{R}_2^2 + \mathcal{R}_3^2 = 2Z \cdot Z^* \equiv \mathcal{R}^2 \quad (S_3 \text{ invariant}) . \tag{A5}$$

Since the phase angles $\vartheta_1, \vartheta_2, \vartheta_3$ (not quite Euler angles) will not appear in the squared moduli of the wave functions, these can be integrated out to give

$$d^6\tau = \pi^3 d\mathcal{R}_1^2 d\mathcal{R}_2^2 d\mathcal{R}_3^2 . \tag{A6}$$

Further integration may be sometimes facilitated by the spherical polar transformation [see Eq. (A5)]

$$\mathcal{R}_1 = \mathcal{R} \sin\vartheta \cos\phi, \quad \mathcal{R}_2 = \mathcal{R} \sin\vartheta \sin\phi, \quad \mathcal{R}_3 = \mathcal{R} \cos\vartheta \tag{A7}$$

and the obvious limits

$$0 \le \mathcal{R} \le \infty, \quad 0 \le \vartheta \le \pi/2, \quad \phi \le \pi/2 \tag{A8}$$

by virtue of the positivity of each $\mathcal{R}_i$.

We now express the natural-parity sequence (4.25) of maximal seniority in compact notation as

$$\psi = N_{\text{ln}} (2Z_+ Z_+^*)^l Z_+^n \exp(-Z \cdot Z^*) , \tag{A9}$$

where $n = 0, 1, 2, 3$ correspond to $|56, 2l^+\rangle$, $|70, 2l+1^-\rangle$, $|70, 2l+2^+\rangle$, and $|56, 2l+3^-\rangle$ states, respectively, and the normalizer N_{ln} is defined as

$$N_{\text{ln}}^{-2} = \int d^6\tau [\mathcal{R}_1^2 \mathcal{R}_2^2]^l (\mathcal{R}_1^2/2)^n e^{-\mathcal{R}^2} . \tag{A10}$$

Using the measure (A6), the integrals are elementary and the result is compactly expressed as

$$N_{\text{ln}}^{-2} = \pi^3 \Gamma(l+1)\Gamma(l+n+1)/2^n . \tag{A11}$$

These results agree with those given in Ref. [26] for the real (ξ, η) representations of the corresponding states.

For the unnatural-parity sequence (4.30), there is an extra multiplicative factor $(\zeta_+ \zeta_-^*)$ as a part of the integrand of (A10), which is expressible as

$$\langle \zeta_+ \zeta_-^* \rangle = [(\mathcal{R}^2/2)^2 - (\mathcal{R}_3^2/2)^2 - \mathcal{R}_1^2 \mathcal{R}_2^2] . \tag{A12}$$

Denoting the corresponding normalizers by $\tilde{N}_{ln}$, we have (again with $n = 0, 1, 2, 3$)

$$\begin{aligned} \tilde{N}_{ln}^{-2} = \frac{\pi^3}{2^n} & \frac{\Gamma(l+1)\Gamma(l+n+1)}{12} \\ & \times [(l+n+1)(n+2) + (l+1)(l+4)] . \end{aligned} \tag{A13}$$

Finally we list a few other cases of immediate interest. One radial (R) excitation of the natural-parity sequence gives an extra multiplicative factor $(\mathcal{R}^2 - 2l - n - 3)$ in the integrand of (A10). So the corresponding normalizer N_{lnR} can be expressed as

$$N_{lnR}^{-2} = \pi^3 (2l+n+4)\Gamma(l+1)\Gamma(l+n+1)/2^n . \tag{A14}$$

A similar procedure can be used to calculate the normalizers of some lower seniority states as well. Some cases of immediate interest are

$$N[|56, 1^-\rangle = 2Z_+ Z^2 \psi_0] = 5\pi^3 , \tag{A15}$$

$$N[|70, 0^+\rangle = Z^2 \psi_0] = 3\pi^3/2 . \tag{A16}$$

The reciprocity between momentum and coordinate spaces that is implied in a HO description allows the entire derivation given above to be reinterpreted in coordinate space terms via the correspondence

$$Z_i \Longleftrightarrow \zeta_i \equiv i\partial_{Z_i^*}, \quad Z_i^* \Longleftrightarrow \zeta_i^* \equiv i\partial_{Z_i} , \tag{A17}$$

where [see Eq. (4.3)]

$$\sqrt{2}\zeta_i = u_i + iv_i, \quad \sqrt{2}\zeta_i^* = u_i - iv_i , \tag{A18}$$

and

$$u_i = i\beta\partial_{\xi_i}, \quad v_i = i\beta\partial_{\eta_i} . \tag{A19}$$

We shall make use of the correspondence (A17) in Appendix B in connection with the evaluation of the expectation values of the OGE term for different states, which are more easily facilitated in coordinate space.

APPENDIX B

The broad procedure for the evaluation of mass (M) spectra through an inversion of Eq. (4.18), on the lines of a similar investigation [26] under the null-plane ansatz has already been described in Sec. IV of the text for the present treatment under the covariant instantaneity ansatz. In this appendix we merely summarize some quantitative steps designed to fill in some gaps in the said procedure so as to enable the interested reader to "see through" the calculations somewhat more closely without much effort. To keep the description within reasonable bounds, no attempt is made to provide any detailed justification [34] for the steps involved, but the nature and sources of the approximations employed are adequately indicated. As already explained in the text, the spectral predictions of NPA and CIA are identical in respect of the two-body ($q\bar{q}$) states but are expected to differ slightly for three-body (qqq) states. Since this comparison is parameter dependent we specify at the outset that the basic constants (ω_0, C_0, m_q) now being employed for the qqq spectra under CIA are the same as those used for the $q\bar{q}$ spectra [27] under NPA (equivalent to a CIA treatment for two-body states), so as to emphasize the unified nature of the investigation, viz.,

$$\omega_0 = 158 \text{ MeV}, \quad C_0 = 0.270, \quad m_q = 265 \text{ MeV} . \tag{B1}$$

The quantity β^2 is therefore fully specified via Eq. (4.1) apart from its (implicit) dependence on the mass M of the states involved.

1. Calculation of F_{con} by "reference spectrum"-like method

The central task of this investigation is the inversion of Eq. (4.18), viz., $F(M,N) = N+3$, for an explicit determination of the mass (M) corresponding to a given state (N). $F(M,N)$ in turn consists of two pieces F_{con}, Eqs. (3.15) and (4.19), and F_{OGE}, Eqs. (3.13) and (4.20), of which only the latter requires an explicit knowledge of the wave function, while the former already contains the eigenvalues explicitly, except for quantities like $\langle \omega_i \rangle$ which first need to be properly defined. For the evaluation of F_{con} we therefore use a method somewhat analogous (but not quite identical) to the "reference spectrum method" of Bethe and co-workers [36] in the context of the nuclear many-body problem. In the present context the position of the "reference point" employed for $\langle \omega_i \rangle$ may be illustrated with respect to the No. (12) subsystem with No. 3 as a spectator. Indeed from the various definitions given in Secs. II–IV of the text, we have, in the (12;3) basis,

$$\langle \omega_{1,2}^2 \rangle = \langle m_q^2 + (\xi^2+\eta^2)/2 + (\xi^2-\eta^2)/4 \pm \sqrt{3}2\xi\cdot\eta/4 \rangle , \tag{B2}$$

where the S_3-invariant quantity $\langle \xi^2+\eta^2 \rangle$ in turn may be replaced by $\beta^2(N+3)$ for the state under study, while the other two terms have zero expectation from symmetry. Thus,

$$\langle \omega_{1,2}^2 \rangle = m_q^2 + \beta^2(N+3)/2 \equiv \langle \omega \rangle^2 . \tag{B3}$$

(Note that β^2 is itself M dependent as well as $\langle \omega \rangle$ dependent, so that a considerable degree of self-consistency is involved in the final determination of M.)

We must also keep in mind the smallness ansatz for δ/M (see text), where $\delta = M - \omega_1 - \omega_2 - \omega_3$ which has been used as the basis of the structure of the master equation (3.15). This ansatz has been numerically checked to give 1–2 % contributions from effects of $O(\delta^2/M^2)$ in typical cases [34], and gives rise to the useful equation

$$\langle \omega_{1,2} \rangle = (M - \langle \omega \rangle)/2 , \tag{B4}$$

which has been anticipated for effecting some simplification leading to Eqs. (4.1), (4.19), and (4.20) of the text. We also note the result

$$\langle \nu_3 \rangle = \langle \omega_3/M \rangle = \langle \omega \rangle / M . \tag{B5}$$

This "reference spectrum" basis works rather well for high excitations ($N \geq 4$), in as much as correction terms like $\omega_i - \langle \omega \rangle$ produce quite rapid convergence. However, for $N < 4$, one encounters some problems of numerical accuracy (i.e., larger corrections from $\omega_i - \langle \omega \rangle$ terms) necessitating a slightly different form of strategy, viz., a shift in the position of the reference point $\langle \omega \rangle$, based on a somewhat *slower motion* of the spectator. For the lowest state $N=0$, one may consider the extreme limit of instantaneous rest ($\hat{p}_3 = 0$) for the spectator (3) when the (12) pair is in mutual interaction. In this limit the equivalent of (B2) becomes

$$\begin{aligned} \langle \omega_{1,2}^2 \rangle &= m_q^2 + 3\langle \xi^2 \rangle /4 \\ &\Longrightarrow m_q^2 + 3\beta^2(N+3)/8 \equiv \langle \omega \rangle^2 . \end{aligned} \tag{B6}$$

The difference between (B3) and (B6) is quite small for $N \geq 4$ states but less trivial for the lower N states. The implication of $\hat{p}_3 = 0$ on (B5) is now

$$\langle \nu_3 \rangle = m_q / M . \tag{B7}$$

The difference between (B5) and (B7), on the one hand, and between (B3) and (B6) on the other, gives a rough indication of the shift in the reference point (on a sliding scale) as one goes from $N=0$ to $N \geq 4$. Using these considerations, the "reference" value for the common factor $(1-\langle \nu_3 \rangle)^2$ which is involved in all the basic quantities, viz., β^2, $\bar{\alpha}_s$, Eq. (4.1), and F_{con}, Eq. (4.19), takes the forms

$$\begin{aligned} &N=0,1: \quad (1-\langle \nu_3 \rangle)^2 = (1-m_q/M)^2 , \\ &N=2,3: \quad (1-\langle \nu_3 \rangle)^2 = (1-m_q/M)(1-\langle \omega \rangle/M) , \\ &N \geq 4: \quad (1-\langle \nu_3 \rangle)^2 = (1-\langle \omega \rangle/M)^2 . \end{aligned} \tag{B8}$$

The value of $\langle \omega \rangle$ may be taken as in (B6) for the sake of uniformity, since for $N \geq 4$ states the difference between (B3) and (B6) is anyway small. These expressions ensure minimal corrections due to deviations from the respective reference values.

For completeness, we also record the full expression for $\langle \alpha_s \rangle$ [27]:

$$\langle \alpha_s \rangle = \alpha_s(M - \langle \omega \rangle) = \frac{12\pi}{33-2f}\left[\ln \frac{(M-\langle \omega \rangle)^2}{\Lambda^2}\right]^{-1} \quad (\Lambda = 200\ \text{MeV}) , \tag{B9}$$

where the value of $\langle \omega \rangle$ is taken again as in (B6) except only for the nucleon, where a marginal change $[\frac{3}{8} \rightarrow \frac{9}{32}$ in the RHS of Eq. (B6)] is necessary to ensure stability of the inversion process for such a low mass state. For the actual results given in Tables I and II we have of course added the corrections due to deviations from the above reference points upto $O(\omega_i - \langle \omega \rangle)^2$ and $O(\delta^2)$ and found quite rapid convergence. This completes the specification of the F_{con} part of Eq. (4.18).

2. Calculation of F_{OGE} in complex HO basis

We now outline the calculation, in the complex HO basis, of the second component F_{OGE} of $F(M,N)$, Eq. (4.18), before the full expression can be inverted to obtain an explicit solution for M. To that end we first recall Eq. (3.13) for the full OGE term W_{OGE} as it appears as an additive contribution to the RHS of the basic dynamical Eq. (3.19), thus specifying its relative normalization with respect to W_{con}. Note, however, the presence of a Δ term in Eq. (3.15) for W_{con}: When transferred to the LHS, there arises a resolvent

$$[1 - 2C_0 \alpha_s M(1-\langle \nu \rangle)^2/(M - \langle \omega \rangle)]^{-1}$$

whose effect has already been included in the definition of $\bar{\alpha}_s$ in terms of α_s, Eq. (4.1), where the quantity α_s is seen as a multiplicative factor in the definition of W_{con}, Eq. (3.11). The same resolvent will also modify the multiplicative factor α_s of W_{OGE}, thus giving rise to the defining Eq. (4.20) for F_{OGE} where

$$\langle \overline{W}_{\text{OGE}} \rangle = \langle W_{\text{OGE}} \rangle \bar{\alpha}_s / \alpha_s . \tag{B10}$$

To calculate $\langle W_{\text{OGE}} \rangle$ it is enough to consider only the (12) pair, using the same reference spectrum basis as described for F_{con} and multiplying the result by 3. The resulting expression for $\overline{W}_{\text{OGE}}$ in operator form (except for the factor $\langle \omega \rangle$ where it appears), but neglecting the noncentral terms, works out straightforwardly as

$$\overline{W}_{\text{OGE}} = \frac{4M\bar{\alpha}_s \beta\sqrt{3}}{2}\left[\frac{1}{u} + \frac{3\beta^2}{4\langle \omega \rangle^2}\xi_i \frac{1}{u}\xi_i + \frac{\pi\beta^2}{4\langle \omega \rangle^2}(3 - 2\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\delta^3(\mathbf{u})\right] . \tag{B11}$$

Here we have used a mixed representation wherein $\mathbf{u}$ $(= u_i)$ is a normalized relative *coordinate* for the (12) pair, viz.,

$$u = \sqrt{u_i u_i}, \quad \beta \mathbf{r}_{12} = 2\mathbf{u}/\sqrt{3}, \quad \xi_i = -i\beta\partial_{u_i} , \tag{B12}$$

where ξ_i is the corresponding momentum and the various expectation values are taken with respect to the states involved, after taking due care of the *order* in which the ξ_i and u_i variables appear in (B11). Further the quantity $\langle \omega \rangle$ continues to be given by (B6). At this stage it is more convenient to specialize to the coordinate representation for the various wave functions, and for this the momentum-space $(\boldsymbol{\xi},\boldsymbol{\eta})$ structure of the wave functions given in Appendix A is already available. However, as earlier noted at the end of Appendix A, the momentum wave functions in the complex basis can be adapted almost verbatim to the corresponding coordinate space expressions via the correspondence (A16). Indeed in order not to have to repeat the various definitions in Appendix A, it is more convenient to regard the variables Z_i and Z_i^* given therein as the corresponding coordinates $i\partial_{Z_i^*}$ and $i\partial_{Z_i}$ themselves, so that the variable u^2 can now be expressed in the complex basis as

$$\begin{aligned} 2u^2 &= Z^2 + Z^{*2} + 2Z\cdot Z^* \\ &= \mathcal{R}^2 + 2\mathcal{R}_1\mathcal{R}_2\cos(\vartheta_1+\vartheta_2) + \mathcal{R}_3^2\cos(2\vartheta_3) . \end{aligned} \tag{B13}$$

The last two terms are not only small compared to $\mathcal{R}^2$ but contribute only in second order on taking the expectation values over the angles. Thus, to sufficient accuracy, we have

$$\frac{1}{\sqrt{2}u} \Longrightarrow \frac{1}{\mathcal{R}} + \frac{3}{16\mathcal{R}^5}(4\mathcal{R}_1^2\mathcal{R}_2^2 + \mathcal{R}_3^2) . \tag{B14}$$

The calculation of the RHS is easily facilitated by the method of Appendix A employed for the various normalizations, and yields the result

$$\left\langle \frac{1}{u} \right\rangle = \sqrt{2}\frac{\Gamma(2l+n+5/2)}{\Gamma(2l+n+3)} \times \left[1 + \frac{3}{8}\frac{1+2(l+1)(l+n+1)}{(2l+n+4)(2l+n+3)}\right] . \tag{B15}$$

For the second term of (B11), we have the complex representation

$$\xi_i \frac{1}{u}\xi_i = -\frac{1}{\sqrt{2}}\left[\partial_{Z_i}\frac{1}{u}\partial_{Z_i^*} + \partial_{Z_i^*}\frac{1}{u}\partial_{Z_i}\right] , \tag{B16}$$

where u is now given by (B13).

Similarly for the third term, we have

$$\delta^3(\mathbf{u}) = \delta^3[(Z + Z^*)/\sqrt{2}] . \tag{B17}$$

The rest of the calculation is lengthy but straightforward and will lead to explicit formulas of the type (B15), following the method of Appendix A. The results are

$$\left\langle \xi_i \frac{1}{u}\xi_i \right\rangle = \frac{1}{\sqrt{2}}\frac{\Gamma(2l+n+3/2)}{\Gamma(2l+n+3)}[2(l+n) + (2l+n+\tfrac{3}{2})(l(l+n-1) - n + 5/2)] \tag{B18}$$

and

$$\langle \delta^3(\mathbf{u}) \rangle = \frac{2}{\pi^2} \frac{\Gamma(n+2l+1)}{\Gamma(l+1)\Gamma(n+l+1)} \frac{\Gamma(2l+n+3/2)}{(4l+2n+1)!!} . \tag{B19}$$

All the expressions [(B15), (B18), and (B19)] are valid only for natural-parity states. While (B19) is exact, (B15) and (B18) are based on the approximation (B14), thus accounting for a mismatch of a π factor between the two groups. We have also checked these results against an alternative formulation in terms of creation and annihilation operators a, a^*, $a^\dagger$, $a^{*\dagger}$ defined in Eq. (4.7), and found the necessary consistency [34], but these details are omitted for brevity.

3. Mixing of states via OGE terms

We now illustrate the mixing aspects as between different states, brought about by both spin-dependent *and* spin-independent OGE terms in the full package (B11). While the role of the former has been well recognized in the literature [32], that of the latter has been relatively obscure, but stems rather naturally from the present classification scheme which assigns a new quantum number N_a to the various states involved. For the mixing of two states, one has to evaluate the full (2×2) matrix of $F_{\rm OGE}$, defined through Eq. (4.20) and (B11), corresponding to both on- and off-diagonal elements, and then add the same to the (already diagonal) part $F_{\rm con}$, Eq. (4.19), in order to get the full $F(M,N)$ matrix to be subsequently diagonalized.

We now give an example of mixing between two states bearing the same quantum number N_a, viz., $|70\rangle^d$ and its radial excitation $|70\rangle^d_R$ (e.g., D'_{33} versus D''_{33}; S'_{31} versus S''_{31}). Both the states have the same qqq wave functions except for their respective spatial parts. The complete wave functions for the two states may be written compactly as

$$\begin{bmatrix} |70\rangle^d \\ |70\rangle^d_R \end{bmatrix} = \frac{1}{\sqrt{2}}\chi_c\phi_c \begin{bmatrix} \psi_c \\ \psi_{cR} \end{bmatrix} + \frac{1}{\sqrt{2}}\chi_c^*\phi_c^* \begin{bmatrix} \psi_c^* \\ \psi_{cR}^* \end{bmatrix} , \tag{B20}$$

where the spin-isospin functions are defined as in Eqs. (4.21)–(4.24). The evaluation of the spatial matrix elements is a straightforward extension of the steps indicated in (B11)–(B17), while the spin-isospin matrix elements may be simplified through the use of the orthonormality of $|\chi_c\phi_c\rangle$ and $|\chi_c^*\phi_c^*\rangle$ as well as the result

$$\sigma_1\cdot\sigma_2 \begin{bmatrix} \chi_c \\ \chi_c^* \end{bmatrix} = \begin{bmatrix} -\chi_c - 2\chi_c^* \\ -\chi_c^* - 2\chi_c \end{bmatrix} . \tag{B21}$$

As to the off-diagonal elements of M, $\bar{\alpha}_s$, β^2, $\langle\omega\rangle^2$ the most natural assignments are the geometrical means of the corresponding values in the two different states. For completeness the full wave functions for the D'_{33} and D''_{33} states are

$$|D'_{33};D''_{33}\rangle = \sqrt{2}Z_+ e^{-Z\cdot Z^*}(1;Z\cdot Z^*-2)\chi^c_{1/2}\phi_c , \tag{B22}$$

$$|S'_{31};S''_{31}\rangle = (Z_+\chi^c_{-1/2} + iZ_3^*\chi^c_{1/2})e^{-Z\cdot Z^*}(1;Z\cdot Z^*-2)\phi_c , \tag{B23}$$

where the subscripts in χ stand for the respective m_s values.

As another example, we illustrate the mixing between two states having *different* N_a quantum numbers, viz., $|56\rangle^d$ and $|70\rangle^d$, whose prototypes are $F_{15}(1680)$ and $F'_{15}(2000)$, respectively, and whose wave functions are given by (4.21) and (4.23), respectively. In this case, only the last term (Fermi-Breit) of $F_{\rm OGE}$, Eq. (B11), will contribute, and the rest of the procedure is the same as indicated above, using the maximally stretched spatial wave functions:

$$|\psi^s(F_{15});\psi_c(F'_{15})\rangle = (2Z_+Z_+^*;2Z_+^2)e^{-Z\cdot Z^*} . \tag{B24}$$

We note in the passing that some spin-dependent corrections which arise also from $F_{\rm con}$ [26] have been routinely included in the calculations and found to provide $\lesssim 5\%$ contribution to $F(M,N)$. Mixing among other admissible states can be treated in a similar fashion.

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