Matrix representation of the nonlocal kinetic energy operator, the spinless Salpeter equation and the Cornell potential

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A new procedure for solving the spinless Salpeter equation is developed. This procedure is implemented with the Cornell potential, where all of the required matrix elements can be calculated from analytic expressions in a convenient basis. Beginning with analytic results for the square of the momentum operator, the matrix elements of the nonlocal kinetic energy operator are obtained from an algorithm that computes the square root of the square of the relativistic kinetic energy operator. Results calculated with the spinless Salpeter equation are compared with those obtained from Schrödinger's equation for heavy-quark systems, heavy-light systems, and light-quark systems. In each case the Salpeter energies agree with experiment substantially better than the Schrödinger energies.

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I. INTRODUCTION

Since the pioneering work of Stanley and Robson [1] and of Godfrey and Isgur [2], it has been apparent that calculations of mesons containing a light quark should simultaneously address a number of relativistic corrections. These include the effects of relativistic kinematics [3], which require a nonlocal kinetic energy operator, as well as momentum-dependent corrections to the potential energy operator, which introduce nonlocal modifications of the relative coordinate and adjustments to the strengths of various parts of the potential. Since the comprehensive calculations of Godfrey and Isgur [2] agreed with a broad spectrum of experimental data, one might infer that a solid foundation for the treatment of relativistic corrections had been established. However, the subsequent calculation of Gara *et al.* [4], which was based on a reduced Bethe-Salpeter analysis of relativistic potential and kinetic energy effects, did not find this to be the case. Indeed, they discovered that the trend of their $2S$-$1S$ separations for decreasing constituent masses was contrary to the behavior of the observed separations. They were able to trace this disturbing trend to the behavior of relativistic corrections to the potential for scalar exchange. The general conclusions of Gara *et al.* about the difficulty of finding a consistent relativistic extension of the nonrelativistic quark model were supported in a more recent calculation by Lucha, Rupprecht, and Schöberl [5].

In view of this confusing picture and the difficulties inherent in finding a coherent viewpoint to address questions of relativistic consistency, it is important to see if one can find support for an experimental signature of the relativistic kinetic energy operator [6,7]. To this end, we have followed the scenario used by Jacobs, Olsson, and Suchyta (JOS) [6], that is, to conduct a competition between Schrödinger's equation and the spinless Salpeter equation, using the Cornell potential. In each case, the parameters are determined independently. We extended the original scope of the JOS calculation from heavy-quark systems to include heavy-light systems in Ref. [7]. In the present work we present additional results for the heavy-quark systems and the heavy-light systems and extend the scope of the comparison between the two wave equations to light-quark systems.

The energy eigenvalues and the eigenfunctions of Refs. [1] and [2] were obtained by diagonalizing the Hamiltonian operator after calculating the matrix elements of both the kinetic and potential energy operators in a harmonic oscillator basis. Following JOS, in the calculation reported in Ref. [7] we used a basis in which each radial function consisted of a centrifugal barrier factor, a common exponential function and a Laguerre polynomial. In all of the Salpeter calculations just discussed, the matrix elements of the kinetic energy operator were evaluated by numerical integrations in momentum space. Below we develop a procedure which *will dispense with the necessity of these numerical integrations for obtaining the kinetic energy matrix elements.*

The new procedure begins with analytic results for the square of the momentum operator in the Laguerre polynomial basis [7]. To this matrix we add a diagonal matrix to include the effects of the square of the rest mass. Thus, we have analytic expressions for the matrix elements of the square of the kinetic energy operator. In order to find the matrix elements of the kinetic energy operator, we implement an algorithm to take the square root of the squared operator. Our algorithm, which is based on diagonalizing a real symmetric matrix, is in essence the same algorithm used by Nickisch, Durand, Durand, and Gara [8,9] and was discussed in an earlier report [10]. Below, we show that this procedure leads to stable values for the matrix elements representing the kinetic energy operator. These matrix elements for the kinetic energy operator are uniquely determined by the Laguerre polynomial basis, and thus they may be used as part of the procedure to determine the elements of the Hamiltonian operator for arbitrary (well-behaved) potentials. Since the analytic expressions for the square of the momentum operator are straightforward to evaluate and

the size of the matrix required to take the square root (30×30 or 40×40) is not too large, the implementation of the new procedure should result in a substantial savings in computer time in most cases.

In the case of the Cornell potential, we have shown [7] that one can also obtain analytic expressions for both the linear and Coulomb parts of the Cornell potential, regardless of the value of the angular momentum. When the analytic results for the potential are combined with results of the square root algorithm, it is apparent that *all of the matrix elements necessary to solve the spinless Salpeter equation with a Cornell potential can be generated analytically.*

Further, I should like to note that there is considerable interest [11–14] in approaching the spinless Salpeter equation from momentum space. There, in order to solve the Cornell potential problem, one has to develop a procedure for handling the singular kernels [11] that arise from both the Coulomb and the linear potentials. Although we have not had any direct experience with the singular integral equations, it is likely that our present matrix mechanics method is easier to implement.

Our method for solving the spinless Salpeter equation is presented in Sec. II. Explicit expressions for all of the requisite matrix elements are listed there as well as some of the procedures for verifying the correctness of our results. Our results for the spin-averaged energies of the Υ system, charmonium, the B-flavor mesons, the charmed mesons, and the light mesons are presented in Sec. III. In each sector we show that the Salpeter eigenvalues agree with experiments better than the Schrödinger eigenvalues. The advantage becomes more pronounced as the systems become lighter, as one might expect. In Sec. IV we discuss some of the problems presented by allowing the coupling constant to run and some possible future applications of our method.

II. SOLUTION OF THE SPINLESS SALPETER EQUATION

The spinless Salpeter equation [6,15] for a quark and antiquark in a mutual orbit in their center-of-momentum system may be expressed

$$H\Psi_n=E_n\Psi_n \ , \tag{1}$$

$$H=\sqrt{m_1^2+p^2}+\sqrt{m_2^2+p^2}+V(r)+V_{\rm SD}(\mathbf{r}) \ , \tag{2}$$

where E_n denotes the total energy of the system, $V(r)$ is the central potential, and $V_{\rm SD}(\mathbf{r})$ includes the effects of all spin-dependent potentials. Since our point of comparison with experiment is the spin-averaged energies, we will neglect the effects of $V_{\rm SD}$. The Cornell potential [16] is given by

$$V(r)=Ar-\kappa/r+C \ , \tag{3}$$

where A is the string constant and κ is the Coulomb parameter. The additive constant C is zero in the heavy-quark sector. It is mildly flavor dependent in the heavy-light sector and flavor dependent in the light-quark sector. This flavor dependence is not a surprise, since it was necessary to add such a constant for each quark-antiquark system in the original calculations [16]. Since we are interested in identifying a set of circumstances under which we can compare Schrödinger's and Salpeter's equations, most of our results will be based on strict flavor independence of A and κ. However, we also present some results in the heavy-quark sector where the parameter κ is allowed to run, in order to see if the required behavior of κ is compatible with asymptotic freedom [5].

To solve Eqs. (1) and (2) we choose the method of Rayleigh, Ritz, and Galerkin [17], where one expands the wave function in terms of a complete set of basis functions. We take this basis set [18] to be

$$R_{nl}(r)=N_{nl}\beta^{3/2}(2\beta r)^l e^{-\beta r}L_n^{2l+2}(2\beta r) \ , \tag{4}$$

where

$$N_{nl}^2=\frac{8(n!)}{\Gamma(n+2l+3)} \ , \tag{5}$$

l denotes the angular momentum, L_n^{2l+2} denotes an associated Laguerre polynomial [19,20], and β is a parameter that sets the scale of length. Thus, the determination of the energy eigenvalues requires the diagonalization of a Hamiltonian matrix whose elements are calculated with the functions of Eqs. (4) and (5). We have shown [7] that it is possible to obtain accurate results for the low-lying S and P states with a range of possible scale lengths β. However, for the Υ system the optimum value of β, which requires the smallest number of basis states, is near 2.0 GeV. For charmonium and the light-quark systems the optimum values of β are somewhat smaller.

In Ref. [7] we have shown that the matrix elements of the nonrelativistic kinetic energy operator may be evaluated analytically. If the result for the nonrelativistic kinetic energy operator there is multiplied by twice the reduced mass, then the matrix elements for the square of the momentum operator take the form

$$(p^2)_{nn'}=-\frac{\beta^2N_{nl}N_{n'l}}{2}\times\left[\frac{1}{4}\delta_{nn'}\frac{\Gamma(n+2l+3)}{n!}-\sum_{i=0}^{n}(l+1+i)\frac{\Gamma(i+2l+2)}{i!}\right] , \tag{6}$$

where n must be restricted to values smaller than n'. For S, P, and D states the results are especially simple: that is,

$$(p^2)_{nn'}=\beta^2\left[\frac{(n+1)(n+2)}{(n'+1)(n'+2)}\right]^{1/2}\left[2+\frac{4n}{3}-\delta_{nn'}\right] \quad (l=0) \ , \tag{7}$$

$$(p^2)_{nn'}=\beta^2\left[\frac{(n+1)(n+2)(n+3)(n+4)}{(n'+1)(n'+2)(n'+3)(n'+4)}\right]^{1/2}\left[2+\frac{4n}{5}-\delta_{nn'}\right] \quad (l=1)\ , \tag{8}$$

$$(p^2)_{nn'}=\beta^2\left[\frac{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{(n'+1)(n'+2)(n'+3)(n'+4)(n'+5)(n'+6)}\right]^{1/2}\left[2+\frac{4n}{7}-\delta_{nn'}\right] , \quad (l=2)\ , \tag{9}$$

where $n \leq n'$.

Equations (6)–(9) are the starting point for the algorithm to determine the matrix elements of the kinetic energy operator of Eq. (2). The theoretical underpinnings of the procedure are based on well-known properties of matrices and are described in some detail in Refs. [8,9]. One begins the procedure by adding m^2 times the unit matrix to the matrix elements of p^2 to form the matrix elements of the square of the kinetic energy operator,

$$(E_k^2)_l=p_l^2+m^2 \quad \text{(step 1)}\ . \tag{10}$$

Use a library subroutine [EVCSF in the International Mathematical and Scientific Library (IMSL)] to diagonalize the matrix representative of the square of the kinetic energy operator (step 2). The operator $(E_k^2)_l$ is related to the diagonal matrix Λ formed from the eigenvalues by the similarity transformation

$$(E_k^2)_l=U\Lambda U^{-1}\ , \tag{11}$$

where U is the matrix of normalized eigenvectors. Form the square root of the diagonal matrix Λ by taking the positive square roots of each of its eigenvalues (step 3). Use the similarity transformation of Eq. (11) to restore the square root matrix to the original basis: that is,

$$(E_k)_l=U\Lambda^{1/2}U^{-1} \quad \text{(step 4)}\ . \tag{12}$$

The operator computed with Eq. (12) is the desired representative of the relativistic kinetic energy operator for a particle of mass m. The four-step algorithm must be carried out for each of the constituent quarks of Eq. (2). The kinetic energy calculation is readily implemented as a FORTRAN program on a Vax 8650.

It is important to ascertain the role of matrix size in the results for the matrix elements of the relativistic energy operator of step 4. In order to investigate concerns about stability of these results, we have undertaken a comparison of results obtained with 20×20, 30×30, 40×40, and 60×60 matrices for the Υ system where $m_1=m_2$. Results from the first column of the matrix representation of the kinetic energy operator are presented in Table I for the first three cases and compared with the results of a numerical integration in momentum space, which was used in the earlier calculation [7]. Since the results of the 40×40 calculation are almost identical to those of the numerical integration, it is clear that a matrix of this size should suffice to give accurate results for the relativistic kinetic energy operator, although substantially smaller matrices may suffice for some applications [21]. The accuracy of the 40×40 calculation is also supported by comparison with the results of the 60×60 calculation.

We have also shown [7] that all of the potential energy matrix elements of Eqs. (2) and (3) can be obtained analytically. For arbitrary values of angular momentum, the matrix elements of the linear potential may be expressed as a tridiagonal matrix: namely,

$$\begin{aligned}\langle Ar\rangle_{nn'}=\frac{A}{2\beta}[&(2l+3+2n)\delta_{nn'}\\&-\sqrt{n'(n'+2l+2)}\,\delta_{n',n+1}\\&-\sqrt{n(n+2l+2)}\,\delta_{n',n-1}]\ .\end{aligned} \tag{13}$$

The matrix elements for the Coulomb potential may also be expressed in an analytic form. For S, P, and D states, the expressions simplify to

$$\langle \kappa/r\rangle_{nn'}=\kappa\beta\left[\frac{(n+1)(n+2)}{(n'+1)(n'+2)}\right]^{1/2} \quad (l=0)\ , \tag{14}$$

$$\langle \kappa/r\rangle_{nn'}=\frac{\kappa\beta}{2}\left[\frac{(n+1)(n+2)(n+3)(n+4)}{(n'+1)(n'+2)(n'+3)(n'+4)}\right]^{1/2} \quad (l=1)\ , \tag{15}$$

$$\langle \kappa/r\rangle_{nn'}=\frac{\kappa\beta}{3}\left[\frac{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{(n'+1)(n'+2)(n'+3)(n'+4)(n'+5)(n'+6)}\right]^{1/2} \quad (l=2)\ , \tag{16}$$

where $n \leq n'$. When the matrix elements of Eqs. (13)–(16) are combined with those of the four-step algorithm above, all of the matrix elements required to diagonalize the Hamiltonian operator of the spinless Salpeter equation of Eq. (1) have been determined. This second diagonalization is also readily carried out with EVCSF.

The role of the size of the Hamiltonian matrix in determining the low-lying S eigenvalues is explored in Fig. 1. There, it is clear that these eigenvalues are accurate for 8×8 or 10×10 matrices, if $\beta=2.0$ GeV, which is near the optimum value [22] for the Υ system. Use of a larger value of β, for example, $\beta=5.0$ GeV, requires larger

TABLE I. Matrix elements of the kinetic energy operator for $\beta=2.0$ GeV and $m=5.0$ GeV.

Element	Matrix size 20×20	30×30	40×40	Numerical Result
$(E_k)_{00}$	5.3613	5.3613	5.3613	5.3613
$(E_k)_{10}$	0.3861	0.3861	0.3861	0.3861
$(E_k)_{20}$	0.2325	0.2325	0.2324	0.2324
$(E_k)_{30}$	0.1540	0.1539	0.1539	0.1539
$(E_k)_{40}$	0.1086	0.1085	0.1085	0.1085
$(E_k)_{50}$	0.0802	0.0801	0.0801	0.0801
$(E_k)_{60}$	0.0614	0.0613	0.0613	0.0613
$(E_k)_{70}$	0.0485	0.0483	0.0483	0.0483
$(E_k)_{80}$	0.0392	0.0390	0.0390	0.0390

Hamiltonian matrices to achieve the same accuracy. In order to provide a comfortable margin for error, we have used a 20×20 Hamiltonian matrix to obtain the results presented below.

III. RESULTS

Cornell potential eigenvalues for heavy-quark systems are presented in Table II and compared with spin-averaged measurements of the low-lying S, P, and D energies [23]. Since the energies of the singlet S states of the Υ system have not been measured, determining the spin

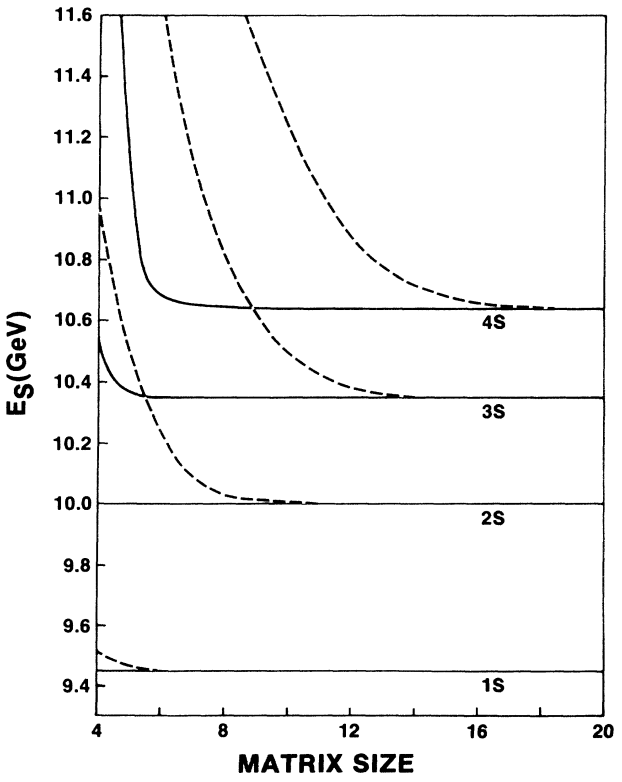

FIG. 1. Low-lying S eigenvalues as a function of matrix size from Salpeter's equation. The full curves were obtained with $\beta=2.0$ GeV and the dashed curves with $\beta=5.0$ GeV. The mass is 4.731 GeV and the potential parameters are the same as those used in Table II below.

TABLE II. Spin-averaged energies for heavy-quarkonium systems. The results for Υ were obtained with $\beta=2.0$ GeV and those for charmonium were obtained with $\beta=1.5$ GeV. The ranges of error include errors from the measurements and uncertainties in the procedures used to determine the spin averages. The quantities χ^2 are computed with the states below the flavor thresholds.

States	Schrödinger (MeV)	Salpeter (MeV)	Experiment[a] (MeV)
$b\bar{b}$ states			
$1S$	9448	9448	9448±5
$2S$	10007	9999	10017±5
$3S$	10356	10351	10351±5
$4S$	10642	10639	10580±?
$5S$	10894	10892	10865±?
$6S$	11123	11122	11019±?
$1P$	9901	9900	9900±1
$2P$	10261	10262	10260±1
$1D$	10148	10150	
$c\bar{c}$ states			
$1S$	3067	3067	3067±2
$2S$	3693	3668	3663±5
$3S$	4164	4112	4040±?
$4S$	4568	4486	4415±?
$1P$	3497	3504	3525±1
$2P$	3991	3970	
$1D$	3806	3811	3770±?
$2D$	4242	4216	4159±?
χ^2	827	459	
Parameters			
m_b (GeV)	4.749	4.7305	
m_c (GeV)	1.320	1.3195	
A (GeV^2)	0.191	0.203	
κ	0.472	0.437	

[a]Particle Data Group (Ref. [23]).

averages requires theoretical input. We take the hyperfine splittings of these states from two recent, comprehensive calculations [24] that give consistent results. Then we allow for about a 50% error in this result. The error bar for the $2S$ state of charmonium reflects the experimental uncertainties surrounding $\eta_c(2S)$. The quantity χ^2 listed in Table II is defined as

$$\chi^2=\sum_i \left[\frac{M_i^{\rm th}-M_i^{\rm expt}}{\Delta M_i}\right]^2, \tag{17}$$

where ΔM_i denote the error bars listed in Table II. The χ^2 comparison listed in Table II is based on those Υ and charmonium states below the flavor thresholds, since these states should be less contaminated with the complications of continuum effects. This quantity is almost a factor of 2 larger for the Schrödinger results than for the Salpeter results, and the difference arises almost entirely from charmonium. It is remarkable how closely the two calculated results for the $4S$, $5S$, and $6S$ states of the Υ system agree, which of course supports the usual claim that this system is a good example of a nonrelativistic system. The Salpeter results for the charmonium states above threshold are substantially better than their

Schrödinger counterparts. The Schrödinger results in Table II were also obtained by diagonalizing the nonrelativistic Hamiltonian matrix [7]. They were checked by comparing with the results of a numerical integration [25].

In Table III we present our results for the heavy-light systems, where the parameters A and κ are the same as those used in Table II. It was necessary to introduce the flavor-dependent constants C_D and C_B to obtain good fits to the D and B meson masses with reasonable values for the light-quark constituent masses. The agreement of our earlier Salpeter prediction [7] for the B_S mass with the recent measurements [26,27] is very gratifying. Including this B_S measurement in our χ^2 calculation for the heavy-light sector leads to an advantage of the Salpeter results over the Schrödinger that is a factor of about 5.

Using the values for the light-quark masses obtained in Table III, we have calculated energies of the low-lying states of light-quark systems in Table IV. In each case it was necessary to add a constant to bring the scale of energies into good agreement with experiment. The spin-averaged data for the $u\bar{u}$ or $d\bar{d}$ systems in Table IV are an average of the spin averages of the isoscalar and the isovector mesons in the light, unflavored sector. The error bars measure the differences between these two spin averages. The χ^2 value listed in Table IV for the Salpeter results is almost an order of magnitude better than the Schrödinger χ^2.

In Table V we present results from charmonium where the Coulomb parameter κ is allowed to run from its values used to fit the Υ system. We fix the value of κ in Table V by adjusting the $1P$ energy to agree with experi-

TABLE III. Spin-averaged energies for the heavy-light systems. Both the Salpeter and the Schrödinger results were obtained with $\beta=1.0$ GeV. The parameters C_D and C_B are the additive constants for the charmed sector and the bottom sector respectively. The units of all parameters listed are MeV.

States	Schrödinger (MeV)	Salpeter (MeV)	Experiment (MeV)
Charmed sector			
$D(1S)$	1973	1973	1973+3[a]
$D(2S)$	2757	2615	
$D(1P)$	2474	2457	2437±10
$D(2P)$	3125	2960	
$D_S(1S)$	2075	2075	2075±2
$D_S(2S)$	2771	2713	
$D_S(1P)$	2534	2544	2550±10
$D_S(2P)$	3099	3045	
Bottom sector			
$B(1S)$	5313	5313	5313±3
$B(2S)$	6066	5892	
$B(1P)$	5799	5780	
$B(2P)$	6420	6011	
$B_S(1S)$	5383	5404	5410±10[b]
$B_S(2S)$	6038	5982	
$B_S(1P)$	5824	5858	
$B_S(2P)$	6348	6300	
χ^2	23.6	4.7	
m_u	325	150	
m_S	602	364	
C_D	−441	−244	
C_B	−470	−213	

[a]Particle Data Group (Ref. [23]).
[b]Recent measurement from the CERN e^+e^- collider LEP (Ref. [26]).

TABLE IV. Spin-averaged energies for light quark-antiquark systems. Both the Schrödinger and Salpeter results were obtained with $\beta=1.0$ GeV. The light-quark masses are the same as those listed in Table III.

States	Schrödinger (MeV)	Salpeter (MeV)	Experiment[a] (MeV)
$u\bar{u}$ or $d\bar{d}$ states			
$1S$	681	703	667±57[b]
$2S$	1577	1416	1397±28
$1P$	1240	1240	1240±15
$1D$	1692	1642	1666±22
Strange sector			
$K(1S)$	794	794	794±1
$K(2S)$	1626	1495	1412±50
$K(1P)$	1320	1315	1383±50
$K(1D)$	1738	1703	1713±50
$s\bar{s}$ states			
$1S$	1004	1004	1004±1
$2S$	1759	1695	1665±65[c]
$1P$	1490	1508	1507±15
$1D$	1869	1885	1850±50
χ^2	66.5	7.4	
Parameters			
$C_{u\bar{u}}$	−934	−599	
C_K	−987	−624	
$C_{s\bar{s}}$	−913	−527	

[a]Particle Data Group (Ref. [23]).
[b]These errors reflect the differences between the spin averages of the light isoscalars and the isovectors.
[c]This number is based on the assumption that the $2S$ hyperfine splitting for ϕ is no larger than the $1S$.

TABLE V. Spin-averaged energies of charmonium with a running coupling constant.

States	Schrödinger (MeV)	Salpeter (MeV)	Experiment (Mev)
$c\bar{c}$ states			
$1S$	3067	3067	3067±2
$2S$	3715	3681	3663±5
$3S$	4190	4129	4040±?
$4S$	4595	4504	4415±?
$1P$	3525	3525	3525±1
$2P$	4023	3993	
$1D$	3841	3837	3770±?
$2D$	4277	4242	4159±?
χ^2_{heavy}	115	29.9	
χ^2_{char}[a]	145	29.1	
m_c (GeV)	1.3555	1.344	
A (GeV^2)	0.191	0.203	
κ_{char}	0.561	0.491	

[a]In this calculation of χ^2 the experimental errors of the states above threshold are taken to be the widths of the respective states.

ment, thereby eliminating a large contribution to the χ^2 calculations of Table II. It is interesting to note that the variation of κ in Table V is what one would expect from asymptotic freedom [5]. One set of the χ^2 values listed in Table V is obtained from both the Υ and the charmonium states below the flavor thresholds. The other χ^2 values are obtained from all the charmonium states. In each case the Salpeter results for χ^2 are substantially better.

IV. DISCUSSION AND OUTLOOK

In each sector that we have examined, the heavy sector, the heavy-light sector, and the light sector, we have found that the eigenvalues obtained with Salpeter's equation agree with the spin-averaged data better than those obtained with Schrödinger's equation. We have found this to be the case when the string tension and Coulomb parameter are taken to be universal and in a preliminary study of heavy-quark systems where the Coulomb parameter is allowed to run in a manner consistent with asymptotic freedom. In view of the difficulties that Lucha, Ruprecht, and Schöberl [5] encountered when attempting to reconcile their results with asymptotic freedom, it is important to assess our prospects of encountering similar difficulties. If we follow the strategy used in Table V, that is, we assume that the string constant is universal, but we allow the Coulomb parameter κ to run in order to adjust the calculated Salpeter $1P$-$1S$ differences to agree with the measured values, then we can decide what kind of adjustment is necessary for each of the lighter systems. Since the $1P$-$1S$ differences listed in Table IV for kaons and for the $u\bar{u}$ system are somewhat too small, these two systems would require a larger value of κ (than the Υ system), which would be consistent with asymptotic freedom. The $1P$-$1S$ differences listed in Tables III and IV for the D_S mesons and the $s\bar{s}$ system agree very well with the measured differences, which is not consistent with expectations based on asymptotic freedom. Further, the Salpeter result listed in Table III for the D mesons is about 2 MeV larger than the measured $1P$-$1S$ difference. Thus, the promising observation about the Coulomb parameter and asymptotic freedom in Table V is not supported by a more comprehensive analysis of our results for systems with light quarks. Of course, all of the discrepancies between the Salpeter results of Tables III and IV ($+20$ to -68 MeV) and experiments are relatively small and thus may be swamped by relativistic potential effects. In any event, it seems premature to pursue this question of consistency of asymptotic freedom beyond the heavy-quark systems without careful attention to relativistic potential corrections.

The advantages of the results of Salpeter's equation over Schrödinger's presented above may be viewed as an indication of a signature for the relativistic kinetic energy operator, although we have only considered the Cornell potential. It would be interesting to know if other static potentials in widespread use [24,25,28–31] would give a similar indication of the importance of using relativistic kinematics. Perhaps such considerations could be used to classify some of the alternative forms of the quark-antiquark potentials into a group that produces better results with the Salpeter equation and a group that does not. Of course, any such classification may prove to be premature, because relativistic potential effects may very well lead to a different grouping of potentials when a higher degree of relativistic consistency is considered.

After completing this study of the effects of relativistic kinematics, the most compelling issue would seem to be a careful consideration of relativistic potential corrections in order to address the dichotomy of viewpoints represented by a natural interpretation of the work of Godfrey and Isgur [2] on the one hand and the results of Gara *et al.* [4] and Lucha, Rupprecht, and Schöberl [5] on the other. If one uses the reduced Bethe-Salpeter equation as a context for the study of relativistic potential corrections, then one will encounter many nonlocal corrections to the potential energy, including a term of the form

$$\frac{1}{4}\left[1+\frac{m_1}{\sqrt{m_1^2+p^2}}\right]V(r)\left[1+\frac{m_2}{\sqrt{m_2^2+p^2}}\right]. \quad (18)$$

In order to calculate the matrix elements of this operator, one must insert two complete sets of states. Thus, this operator involves the multiplication of three matrices. If one works in the basis of Eq. (4), then it should be possible to obtain representatives for the momentum-dependent operators from the analytic expressions of Eqs. (6)–(9). It will be important to study the role of matrix size in obtaining stable results for the momentum-dependent operators of Eq. (18) and for accurately carrying out the matrix multiplications there.

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