One-loop vertex function in Yennie gauge QED

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In this paper we give a convenient expression for the one-loop vertex correction in Yennie gauge QED. We use dimensional regularization to regulate ultraviolet and potential infrared divergences. We study the infrared behavior of the vertex correction in a class of simple covariant gauges, and verify that $\beta = 2/(1-2\epsilon)$ is the appropriate choice for the gauge parameter of the Yennie gauge (where $n = 4 - 2\epsilon$ is the dimensionality of spacetime).

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I. INTRODUCTION

Gauge invariance, and the consequent gauge independence of physical quantities, is a central feature of QED. The effort required for the calculation of physical effects in QED is strongly dependent on the choice of gauge, especially when bound states are involved. The Yennie gauge [1–3] has been found useful for the calculation of radiative corrections to bound-state properties because of its attractive infrared properties [4–7].

The Yennie gauge is one of the "simple covariant" gauges, in which the photon propagator has the general form [8]

$$D_\beta^{\mu\nu}(k) = -(1/k^2)(g^{\mu\nu} + \beta k^\mu k^\nu / k^2) \quad . \tag{1}$$

The Yennie gauge is conventionally specified by a value for the gauge parameter of $\beta = 2$. However, when using dimensional regularization it is necessary to take

$$\beta = 2/(1-2\epsilon) \tag{2}$$

(where $n = 4 - 2\epsilon$ is the dimension of spacetime) in order to preserve the favorable infrared properties of the Yennie gauge [9]. This was demonstrated by a careful examination of the infrared behavior of the electron self-energy function. An independent approach is to look at the gauge and infrared dependence of the electron propagator using functional methods. The value of the gauge parameter which eliminates all gauge-dependent infrared divergences is given by Eq. (2) [10].

The Yennie gauge has been, and will undoubtedly continue to be, of great use for the study of bound state properties in QED. In particular, the form of the one-loop vertex correction will be required for many calculations. In this paper, we obtain a convenient form for the one-loop vertex function in dimensionally regularized Yennie gauge QED. The calculations that we do are mostly elementary, but there are subtleties in the handling of the infrared sensitive terms. We demonstrate that the infrared sensitive terms in fact vanish in the Yennie gauge specified by Eq. (2). Along the way, we calculate the one-loop renormalization constant, and show that it is independent of gauge in this regularization scheme. Our results should be of use in calculations of radiative corrections to bound-state properties.

II. DERIVATION OF THE UNSUBTRACTED VERTEX FUNCTION

The one-loop contribution to the vertex function is pictured in Fig. 1. Explicitly, it is

$$\Lambda^\lambda_{(1)}(p',p) = \int (d\ell)'_n [-ie(n)\gamma_\mu] \frac{i}{\gamma(p'+\ell)-m} \gamma^\lambda \frac{i}{\gamma(p+\ell)-m} [-ie(n)\gamma_\nu] i D_\beta^{\mu\nu}(\ell) \quad , \tag{3}$$

where $(d\ell)'_n = d^n\ell/(2\pi)^n$ and $e(n) = e\mu^\epsilon$, with μ an arbitrary mass scale. We introduce the fine structure constant $\alpha = e^2/(4\pi)$, and rewrite the vertex correction as

$$\begin{aligned} \Lambda^\lambda_{(1)}(p',p) &= \left(\frac{\alpha}{4\pi}\right)(4\pi\mu^2)^\epsilon \int \frac{d^n\ell}{i\pi^{n/2}} \frac{\gamma_\mu[\gamma(p'+\ell)+m]\gamma^\lambda[\gamma(p+\ell)+m]\gamma_\nu}{[(\ell+p')^2-m^2][(\ell+p)^2-m^2]\ell^2} \left[g^{\mu\nu} + \beta\frac{\ell^\mu\ell^\nu}{\ell^2}\right] \\ &= \Lambda^\lambda_{(1)F}(p',p) + \beta\Lambda^\lambda_{(1)G}(p',p) \quad . \end{aligned} \tag{4}$$

We will calculate the "Feynman" and "gauge" parts of Λ separately.

We use Feynman parameters x and u to combine denominators, associating xu with $(\ell + p')^2 - m^2$, $x(1-u)$ with $(\ell + p)^2 - m^2$, and $1 - x$ with ℓ^2. Integration limits for the parametric integrals are always from 0 to 1. The momentum integrations will be done using the standard dimensionally regularized integration formulas derived from

$$\int \frac{d^n\ell}{i\pi^{n/2}} \frac{1}{(-\ell^2 + 2\ell Q + M^2)^\xi} = \frac{1}{\Gamma(\xi)} \frac{\Gamma(\xi - n/2)}{\Delta^{\xi - n/2}} \quad , \quad (5)$$

where

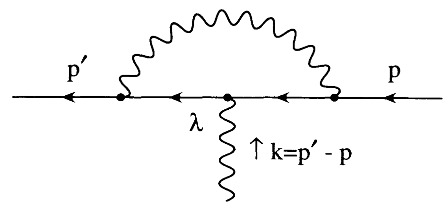

FIG. 1. The one-loop vertex correction. (The external electron and photon legs are not considered to be part of the vertex function.)

$$\Delta = Q^2 + M^2 \quad . \quad (6)$$

The Feynman part of the vertex correction has the form

$$\Lambda^\lambda_{(1)F}(p', p) = \left(\frac{\alpha}{4\pi}\right)(4\pi\mu^2)^\epsilon \int dx du x \left\{ \bar{A}^\lambda \frac{\Gamma(1+\epsilon)}{\Delta^{1+\epsilon}} + 2(1-\epsilon)^2 \gamma^\lambda \frac{\Gamma(\epsilon)}{\Delta^\epsilon} \right\} \quad , \quad (7)$$

with

$$\bar{A}^\lambda = -\gamma^\alpha[\gamma(p' + Q) + m]\gamma^\lambda[\gamma(p + Q) + m]\gamma_\alpha \quad , \quad (8a)$$

$$Q = -x[up' + (1-u)p] = -xq \quad , \quad (8b)$$

$$\Delta = x^2q^2 + xu(m^2 - p'^2) + x(1-u)(m^2 - p^2) = xH \quad , \quad (8c)$$

$$H = xm^2 - xu(1-u)k^2 + (1-x)\left[u(m^2 - p'^2) + (1-u)(m^2 - p^2)\right] \quad . \quad (8d)$$

The gauge part of the vertex correction is

$$\Lambda^\lambda_{(1)G}(p', p) = \left(\frac{\alpha}{4\pi}\right)(4\pi\mu^2)^\epsilon \int \frac{d^n\ell}{i\pi^{n/2}} \frac{\gamma\ell[\gamma(p' + \ell) + m]\gamma^\lambda[\gamma(p + \ell) + m]\gamma\ell}{[(\ell + p')^2 - m^2][(\ell + p)^2 - m^2]\ell^4}$$
$$= \left(\frac{\alpha}{4\pi}\right)(4\pi\mu^2)^\epsilon \int dx du x(1-x) \left\{ \bar{B}^\lambda_0 \frac{\Gamma(2+\epsilon)}{\Delta^{2+\epsilon}} - \frac{1}{2}\bar{B}^\lambda_1 \frac{\Gamma(1+\epsilon)}{\Delta^{1+\epsilon}} + \frac{1}{4}\bar{B}^\lambda_2 \frac{\Gamma(\epsilon)}{\Delta^\epsilon} \right\} \quad . \quad (9)$$

The lower index on the $\bar{B}^\lambda$ quantities indicates the number of contractions. The zero-contraction term is

$$\bar{B}^\lambda_0 = \gamma Q[\gamma(p' + Q) + m]\gamma^\lambda[\gamma(p + Q) + m]\gamma Q \quad . \quad (10)$$

The one-contraction term is

$$\bar{B}^\lambda_1 = -\bar{A}^\lambda - 2(1-\epsilon)\gamma Q\gamma^\lambda\gamma Q$$
$$+ (4 - 2\epsilon)\gamma^\lambda[\gamma(p + Q) + m]\gamma Q + (4 - 2\epsilon)\gamma Q[\gamma(p' + Q) + m]\gamma^\lambda$$
$$+ \gamma^\alpha[\gamma(p' + Q) + m]\gamma^\lambda\gamma_\alpha\gamma Q + \gamma Q\gamma^\alpha\gamma^\lambda[\gamma(p + Q) + m]\gamma_\alpha \quad , \quad (11)$$

and the two-contraction term is

$$\bar{B}^\lambda_2 = 4(2-\epsilon)(3-\epsilon)\gamma^\lambda \quad . \quad (12)$$

A parametric form for the unsubtracted one-loop vertex correction is obtained as the sum of Eqs. (7) and (9). It is

$$\Lambda^\lambda_{(1)}(p', p) = \left(\frac{\alpha}{4\pi}\right)(4\pi\mu^2)^\epsilon \int dx du x \left\{ \beta(1-x)\bar{B}^\lambda_0 \frac{\Gamma(2+\epsilon)}{\Delta^{2+\epsilon}} + \left(\bar{A}^\lambda - \frac{\beta}{2}(1-x)\bar{B}^\lambda_1\right) \frac{\Gamma(1+\epsilon)}{\Delta^{1+\epsilon}} \right.$$
$$\left. + \left[2(1-\epsilon)^2 + \beta(2-\epsilon)(3-\epsilon)(1-x)\right]\gamma^\lambda \frac{\Gamma(\epsilon)}{\Delta^\epsilon} \right\} \quad . \quad (13)$$

III. THE ONE-LOOP RENORMALIZATION CONSTANT

The full vertex function $\Gamma^\lambda(p', p)$ is related to the vertex correction $\Lambda^\lambda(p', p)$ by

$$\Gamma^\lambda(p', p) = \gamma^\lambda + \Lambda^\lambda(p', p) \quad , \quad (14)$$

where $\Lambda^\lambda(p', p)$ includes vertex corrections of all orders. The full vertex function contains a divergent part described by the renormalization constant Z_1, which is defined through the zero momentum transfer limit of Γ^λ for electrons on the mass shell [11]:

$$\bar{u}(p)\Gamma^\lambda(p,p)u(p) = (Z_1)^{-1}\bar{u}(p)\gamma^\lambda u(p) \quad . \tag{15}$$

The free Dirac spinors satisfy

$$\bar{u}(p)(\gamma p - m) = 0 = (\gamma p - m)u(p) \quad , \tag{16}$$

so one has

$$\Gamma^\lambda(p',p) \to (Z_1)^{-1}\gamma^\lambda \quad , \tag{17}$$

where the arrow indicates the limit that $k = p' - p \to 0$, $p^2 \to m^2$, and $\gamma p \to m$ on the right or left. If, in this limit,

$$\Lambda^\lambda(p',p) \to L\gamma^\lambda \quad , \tag{18}$$

then one has

$$Z_1 = (1+L)^{-1} \quad . \tag{19}$$

We will calculate the one-loop contribution $L_{(1)}$ to L.

It is convenient to start with an evaluation of the zero momentum transfer, mass-shell limits of the factors $\bar{A}^\lambda$, $\bar{B}_0^\lambda$, and $\bar{B}_1^\lambda$. They are easily seen to be

$$\bar{A}^\lambda \to \tilde{A}^\lambda = \left[-4(1-x) + 2(1-\epsilon)x^2\right]m^2\gamma^\lambda \quad , \tag{20a}$$

$$\bar{B}_0^\lambda \to \tilde{B}_0^\lambda = x^2(2-x)^2 m^4\gamma^\lambda \quad , \tag{20b}$$

$$\bar{B}_1^\lambda \to \tilde{B}_1^\lambda = 4\left[1 - (3-\epsilon)x(2-x)\right]m^2\gamma^\lambda \quad . \tag{20c}$$

Then one has

$$\begin{aligned}\Lambda^\lambda_{(1)}(p',p) = \left(\frac{\alpha}{4\pi}\right)(4\pi\mu^2)^\epsilon \int dx du x\Bigg(&\beta(1-x)\left(\bar{B}_0^\lambda - \tilde{B}_0^\lambda\right)\frac{\Gamma(2+\epsilon)}{\Delta^{2+\epsilon}} \\ &+ \left[\left(\bar{A}^\lambda - \tilde{A}^\lambda\right) - \frac{\beta}{2}(1-x)\left(\bar{B}_1^\lambda - \tilde{B}_1^\lambda\right)\right]\frac{\Gamma(1+\epsilon)}{\Delta^{1+\epsilon}} + \beta(1-x)x^2(2-x)^2m^4\gamma^\lambda\frac{\Gamma(2+\epsilon)}{\Delta^{2+\epsilon}} \\ &+ \left\{\left[-4(1-x)+2(1-\epsilon)x^2\right] - 2\beta(1-x)\left[1-(3-\epsilon)x(2-x)\right]\right\}m^2\gamma^\lambda\frac{\Gamma(1+\epsilon)}{\Delta^{1+\epsilon}} \\ &+ \left[2(1-\epsilon)^2 + \beta(2-\epsilon)(3-\epsilon)(1-x)\right]\gamma^\lambda\frac{\Gamma(\epsilon)}{\Delta^\epsilon}\Bigg) \quad . \end{aligned} \tag{21}$$

In this form, it is clear that only the last three terms contribute to L.

The renormalization constant $L_{(1)}$ is obtained from $\Lambda^\lambda_{(1)}(p',p)$ by taking the limit shown in Eq. (18). In this limit $\Delta = xH \to x^2m^2$, so

$$\begin{aligned}L_{(1)} = \left(\frac{\alpha}{4\pi}\right)\left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \int dx x\Bigg(&\beta(1-x)x^2(2-x)^2\frac{\Gamma(2+\epsilon)}{(x^2)^{2+\epsilon}} \\ &+ \left\{\left[-4(1-x)+2(1-\epsilon)x^2\right] - 2\beta(1-x)\left[1-(3-\epsilon)x(2-x)\right]\right\}\frac{\Gamma(1+\epsilon)}{(x^2)^{1+\epsilon}} \\ &+ \left[2(1-\epsilon)^2 + \beta(2-\epsilon)(3-\epsilon)(1-x)\right]\frac{\Gamma(\epsilon)}{(x^2)^\epsilon}\Bigg) \\ = \left(\frac{\alpha}{4\pi}\right)\left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \left(\frac{3-2\epsilon}{1-2\epsilon}\right)\Gamma(\epsilon) \quad . \end{aligned} \tag{22}$$

The quantity $L_{(1)}$ is a dimensionless number which diverges like $1/\epsilon$ as $\epsilon \to 0$. It is independent of the gauge parameter β. The renormalization constant Z_1 calculated from Eq. (19) agrees to one loop with the wave-function renormalization constant Z_2 [12], as required by the Ward identity [11].

It is interesting to note that both ultraviolet and infrared divergences are regulated by the dimensional method. Consequently, the $1/\epsilon$ divergence in Eq. (22) has different meanings in different gauges. In the ultraviolet finite Landau gauge ($\beta = -1$), the whole $1/\epsilon$ divergence is due to the infrared. In the infrared safe Yennie gauge, the $1/\epsilon$ represents an ultraviolet divergence. In the Feynman gauge ($\beta = 0$), both ultraviolet and infrared divergences contribute to the $1/\epsilon$. A demonstration that Z_1 is gauge independent to all orders when using dimensional regularization is given in Ref. [13].

IV. THE YENNIE GAUGE PARAMETER

The requirement that $\beta = 2/(1-2\epsilon)$ in the Yennie gauge was shown in [9] by a study of the electron self-energy function, and in [10] on more general grounds. We will arrive at this conclusion again by careful examination of the infrared behavior of the vertex correction. The subtracted one-loop vertex correction function $\Lambda^\lambda_{(1)S}(p',p)$, defined through

$$\Lambda^\lambda_{(1)}(p',p) = L_{(1)}\gamma^\lambda + \Lambda^\lambda_{(1)S}(p',p) \quad , \tag{23}$$

is finite in the limit that the number of dimensions $n = 4 - 2\epsilon$ approaches 4. By its definition we know that

$$\Lambda^\lambda_{(1)S}(p',p) \to 0 \quad , \tag{24}$$

where the arrow represents the "mass-shell" limit as used in Eq. (17). The $\epsilon \to 0$ limit of $\Lambda^{\lambda}_{(1)S}(p',p)$ is the renormalized one-loop vertex correction function, and it also must go to zero in the mass-shell limit:

$$\left\{\lim_{\epsilon\to 0}\Lambda^{\lambda}_{(1)S}(p',p)\right\} \to 0 \quad . \tag{25}$$

This seemingly self-evident requirement leads to the restriction on β.

The subtracted one-loop vertex correction function, from Eqs. (21) and (22), has the form

$$\begin{aligned}
\Lambda^{\lambda}_{(1)S}(p',p) = {}& \left(\frac{\alpha}{4\pi}\right)\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}\int dxdux\Big\{\beta(1-x)\frac{1}{m^4}(\bar{B}^{\lambda}_0-\tilde{B}^{\lambda}_0)\left(\frac{m^2}{xH}\right)^{2+\epsilon}\Gamma(2+\epsilon)\\
&+\left[\frac{1}{m^2}(\bar{A}^{\lambda}-\tilde{A}^{\lambda})-\frac{\beta}{2}(1-x)\frac{1}{m^2}(\bar{B}^{\lambda}_1-\tilde{B}^{\lambda}_1)\right]\left(\frac{m^2}{xH}\right)^{1+\epsilon}\Gamma(1+\epsilon)\\
&+\beta(1-x)x^2(2-x)^2\gamma^{\lambda}\left[\left(\frac{m^2}{xH}\right)^{2+\epsilon}-\left(\frac{1}{x^2}\right)^{2+\epsilon}\right]\Gamma(2+\epsilon)\\
&+\{[-4(1-x)+2(1-\epsilon)x^2]-2\beta(1-x)[1-(3-\epsilon)x(2-x)]\}\\
&\qquad\times\gamma^{\lambda}\left[\left(\frac{m^2}{xH}\right)^{1+\epsilon}-\left(\frac{1}{x^2}\right)^{1+\epsilon}\right]\Gamma(1+\epsilon)\\
&+\left[2(1-\epsilon)^2+\beta(2-\epsilon)(3-\epsilon)(1-x)\right]\gamma^{\lambda}\left[\left(\frac{m^2}{xH}\right)^{\epsilon}-\left(\frac{1}{x^2}\right)^{\epsilon}\right]\Gamma(\epsilon)\Big\} \quad .
\end{aligned} \tag{26}$$

The third term of $\Lambda^{\lambda}_{(1)S}(p',p)$ can be rewritten using the integral identity

$$\begin{aligned}
&\int dxdux\left\{\beta(1-x)x^2(2-x)^2\left[\left(\frac{m^2}{xH}\right)^{2+\epsilon}-\left(\frac{1}{x^2}\right)^{2+\epsilon}\right]\Gamma(2+\epsilon)\right\}\\
&=\int dxdux\Big\{\beta(1-x)x^2(2-x)^2\left(1-\frac{H_1}{m^2}\right)\left(\frac{m^2}{xH}\right)^{2+\epsilon}\Gamma(2+\epsilon)\\
&\quad+\beta[4(1-\epsilon)-8(2-\epsilon)x+5(3-\epsilon)x^2-(4-\epsilon)x^3]\left[\left(\frac{m^2}{xH}\right)^{1+\epsilon}-\left(\frac{1}{x^2}\right)^{1+\epsilon}\right]\Gamma(1+\epsilon)\Big\} \quad ,
\end{aligned} \tag{27}$$

where

$$H = H_0 + xH_1 \quad , \tag{28a}$$
$$H_0 = u(m^2-p'^2)+(1-u)(m^2-p^2) \quad , \tag{28b}$$
$$H_1 = m^2-u(1-u)k^2-u(m^2-p'^2)-(1-u)(m^2-p^2) \quad . \tag{28c}$$

Using Eq. (27) in Eq. (26), one has

$$\begin{aligned}
\Lambda^{\lambda}_{(1)S}(p',p) = {}& \left(\frac{\alpha}{4\pi}\right)\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}\int dxdux\Big\{\beta(1-x)\\
&\times\left[\frac{1}{m^4}(\bar{B}^{\lambda}_0-\tilde{B}^{\lambda}_0)+x^2(2-x)^2\left(1-\frac{H_1}{m^2}\right)\gamma^{\lambda}\right]\left(\frac{m^2}{xH}\right)^{2+\epsilon}\Gamma(2+\epsilon)\\
&+\left[\frac{1}{m^2}(\bar{A}^{\lambda}-\tilde{A}^{\lambda})-\frac{\beta}{2}(1-x)\frac{1}{m^2}(\bar{B}^{\lambda}_1-\tilde{B}^{\lambda}_1)\right]\left(\frac{m^2}{xH}\right)^{1+\epsilon}\Gamma(1+\epsilon)\\
&+\{[-4+2\beta(1-2\epsilon)](1-x)+[2(1-\epsilon)-\beta(3-\epsilon)]x^2+\beta(2-\epsilon)x^3\}\\
&\times\gamma^{\lambda}\left[\left(\frac{m^2}{xH}\right)^{1+\epsilon}-\left(\frac{1}{x^2}\right)^{1+\epsilon}\right]\Gamma(1+\epsilon)\\
&+\left[2(1-\epsilon)^2+\beta(2-\epsilon)(3-\epsilon)(1-x)\right]\gamma^{\lambda}\left[\left(\frac{m^2}{xH}\right)^{\epsilon}-\left(\frac{1}{x^2}\right)^{\epsilon}\right]\Gamma(\epsilon)\Big\} \quad .
\end{aligned} \tag{29}$$

The only part of Eq. (29) that does not satisfy condition (25) is the term proportional to

$$I_{\mathrm{IR}} = \left[4-2\beta(1-2\epsilon)\right]K_{\mathrm{IR}}, \tag{30}$$

where

$$K_{\mathrm{IR}} = \int dxdux\left[\left(\frac{m^2}{xH}\right)^{1+\epsilon}-\left(\frac{1}{x^2}\right)^{1+\epsilon}\right] \quad . \tag{31}$$

In the zero momentum transfer mass-shell limit clearly $K_{IR} \to 0$. However, in the small ϵ limit K_{IR} has a $1/\epsilon$ divergence which does not vanish on shell. The choice $\beta = 2$ would render I_{IR} finite but nonzero in the $\epsilon \to 0$ limit. The choice $\beta = 2 + 4\epsilon + O(\epsilon^2)$ would cause I_{IR} to vanish in the $\epsilon \to 0$ limit. However, the infrared divergence in K_{IR} would still be present, just concealed. Only when $\beta = 2/(1-2\epsilon)$ does the infrared divergence completely disappear.

V. EQUIVALENT EXPRESSIONS FOR THE SUBTRACTED VERTEX FUNCTION

The subtracted vertex function in the Yennie gauge [with $\beta = 2/(1-2\epsilon)$] can be expressed in various equivalent ways. We will quantify some aspects of this freedom by introducing three parameters, which multiply three identities. Two of these are given by

$$0 = \int dxdux\Big\{-x^n\Big[\Big(\frac{m^2}{xH}\Big)^{1+\epsilon} - \Big(\frac{1}{x^2}\Big)^{1+\epsilon}\Big]\Gamma(1+\epsilon) + x^{n-1}\Big(x - \frac{H}{m^2}\Big)\Big(\frac{m^2}{xH}\Big)^{1+\epsilon}\Gamma(1+\epsilon) + \epsilon x^{n-2}\Big[\Big(\frac{m^2}{xH}\Big)^{\epsilon} - \Big(\frac{1}{x^2}\Big)^{\epsilon}\Big]\Gamma(\epsilon)\Big\} \tag{32}$$

for $n = 2$ and 3, which hold point by point in x, u space. Another is the integral identity [akin to Eq. (27)]

$$0 = \int dxdux\Big\{-(1-x)x^2\Big[\Big(\frac{m^2}{xH}\Big)^{1+\epsilon} - \Big(\frac{1}{x^2}\Big)^{1+\epsilon}\Big]\Gamma(1+\epsilon) + (1-x)x^2\Big(1 - \frac{H_1}{m^2}\Big)\Big(\frac{m^2}{xH}\Big)^{1+\epsilon}\Gamma(1+\epsilon) + [(2-\epsilon) - (3-\epsilon)x]\Big[\Big(\frac{m^2}{xH}\Big)^{\epsilon} - \Big(\frac{1}{x^2}\Big)^{\epsilon}\Big]\Gamma(\epsilon)\Big\} \quad . \tag{33}$$

On multiplying Eq. (32) with $n = 2$ by a, Eq. (32) with $n = 3$ by b, and Eq. (33) by c, and adding to Eq. (29), one obtains the form

$$\Lambda^{\lambda}_{(1)S}(p',p) = \Big(\frac{\alpha}{4\pi}\Big)\Big(\frac{4\pi\mu^2}{m^2}\Big)^{\epsilon}\int dxdux^{-\epsilon}\Big\{\beta x(1-x)\frac{1}{m^4}B^{\lambda}\Big(\frac{m^2}{H}\Big)^{2+\epsilon}\Gamma(2+\epsilon) + \frac{1}{m^2}A^{\lambda}\Big(\frac{m^2}{H}\Big)^{1+\epsilon}\Gamma(1+\epsilon) + \xi_B\gamma^{\lambda}\Big[\Big(\frac{m^2}{H}\Big)^{1+\epsilon} - \Big(\frac{1}{x}\Big)^{1+\epsilon}\Big]\Gamma(1+\epsilon) + \xi_A\gamma^{\lambda}\Big[\Big(\frac{m^2}{H}\Big)^{\epsilon} - \Big(\frac{1}{x}\Big)^{\epsilon}\Big]\Gamma(\epsilon)\Big\}, \tag{34}$$

where

$$B^{\lambda} = \frac{1}{x^2}(\bar{B}_0^{\lambda} - \tilde{B}_0^{\lambda}) + (2-x)^2m^2(m^2 - H_1)\gamma^{\lambda} \quad , \tag{35a}$$

$$A^{\lambda} = (\bar{A}^{\lambda} - \tilde{A}^{\lambda}) - \frac{\beta}{2}(1-x)(\bar{B}_1^{\lambda} - \tilde{B}_1^{\lambda}) + (a+bx)x(xm^2 - H)\gamma^{\lambda} + c(1-x)x^2(m^2 - H_1)\gamma^{\lambda} \quad , \tag{35b}$$

$$\xi_B = [2(1-\epsilon) - \beta(3-\epsilon) - a - c]x^2 + [\beta(2-\epsilon) - b + c]x^3 \quad , \tag{36a}$$

$$\xi_A = [2(1-\epsilon)^2 + \beta(2-\epsilon)(3-\epsilon) + \epsilon a + (2-\epsilon)c]x + [-\beta(2-\epsilon)(3-\epsilon) + \epsilon b - (3-\epsilon)c]x^2 \quad . \tag{36b}$$

The difference terms $(m^2/H)^{n+\epsilon} - (1/x)^{n+\epsilon}$ are awkward to use in actual calculations. They can be replaced by more tractable forms. One approach is to introduce an extra parameter through

$$\Big[\Big(\frac{m^2}{H}\Big)^{1+\epsilon} - \Big(\frac{1}{x}\Big)^{1+\epsilon}\Big]\Gamma(1+\epsilon) = \Big(x - \frac{H}{m^2}\Big)\int_0^1 dz\Big(\frac{m^2}{\bar{H}}\Big)^{2+\epsilon}\Gamma(2+\epsilon) \tag{37a}$$

$$\Big[\Big(\frac{m^2}{H}\Big)^{\epsilon} - \Big(\frac{1}{x}\Big)^{\epsilon}\Big]\Gamma(\epsilon) = \Big(x - \frac{H}{m^2}\Big)\int_0^1 dz\Big(\frac{m^2}{\bar{H}}\Big)^{1+\epsilon}\Gamma(1+\epsilon) \quad , \tag{37b}$$

where

$$\begin{aligned}\bar{H} &= xm^2 + z(H - xm^2) \\ &= xm^2 - xzu(1-u)k^2 + (1-x)z[u(m^2 - p'^2) + (1-u)(m^2 - p^2)] \quad .\end{aligned} \tag{38}$$

Alternatively, one could integrate by parts over x using

$$\int dxx^{n-\epsilon}\Big[\Big(\frac{m^2}{H}\Big)^{1+\epsilon} - \Big(\frac{1}{x}\Big)^{1+\epsilon}\Big]\Gamma(1+\epsilon) = \int dx\Big\{u(1-u)\frac{k^2}{m^2}\Big(\frac{m^2}{D}\Big)^{2+\epsilon} - x^{n-\epsilon}\frac{H_0}{m^2}\Big(\frac{m^2}{H}\Big)^{2+\epsilon}\Big\}\frac{\Gamma(2+\epsilon)}{n-2\epsilon} \tag{39a}$$

$$\int dxx^{n-\epsilon}\Big[\Big(\frac{m^2}{H}\Big)^{\epsilon} - \Big(\frac{1}{x}\Big)^{\epsilon}\Big]\Gamma(\epsilon) = \int dx\Big\{u(1-u)\frac{k^2}{m^2}\Big(\frac{m^2}{D}\Big)^{1+\epsilon} - x^{n-\epsilon}\frac{H_0}{m^2}\Big(\frac{m^2}{H}\Big)^{1+\epsilon}\Big\}\frac{\Gamma(1+\epsilon)}{n+1-2\epsilon} \quad , \tag{39b}$$

where

$$D = m^2 - xu(1-u)k^2 \quad , \tag{40}$$

which hold for $n = 2, 3$ and $n = 1, 2$, respectively.

Finally, it is often convenient to expand A^λ and B^λ in terms of elementary γ matrix factors like $\gamma^\lambda(\gamma p - m)$ and $(\gamma p' - m)\gamma^\lambda$ [14]. One has

$$B^\lambda = L_B(\gamma p' - m)\gamma^\lambda(\gamma p - m) - 2mq^\lambda(\gamma p' - m)(\gamma p - m) + (\gamma p' - m)M_B^\lambda + M_B'^\lambda(\gamma p - m) + N_B\gamma^\lambda \quad , \qquad (41)$$

and

$$A^\lambda = L_A(\gamma p' - m)\gamma^\lambda(\gamma p - m) + (\gamma p' - m)M_A^\lambda + M_A'^\lambda(\gamma p - m) + N_A\gamma^\lambda - 2x[1 + (1-\epsilon)x](1-2u)mk^\lambda + 2x[1 - (1-\epsilon)x]mi\sigma^{\lambda\kappa}k_\kappa \quad , \qquad (42)$$

where $v = 1 - u$, $q = up' + vp$, $q^2 = H_1$, and

$$L_B = (1-x)q^2 \quad , \qquad (43a)$$
$$M_B^\lambda = (2pq - xq^2)m\gamma^\lambda - 2v(m^2 - p^2)q^\lambda - 2v(2pq - xq^2)p^\lambda \quad , \qquad (43b)$$
$$M_B'^\lambda = (2p'q - xq^2)m\gamma^\lambda - 2u(m^2 - p'^2)q^\lambda - 2u(2p'q - xq^2)p'^\lambda \quad , \qquad (43c)$$
$$N_B = (2p'q - xq^2)(2pq - xq^2) - (2-x)^2m^2q^2 + u(m^2 - p'^2)(2pq - xq^2) + v(m^2 - p^2)(2p'q - xq^2) \quad , \qquad (43d)$$

$$L_A = -2(1+\epsilon)(1-x) + \beta(1-x)[-(1+\epsilon) + (3-\epsilon)x] \quad , \qquad (44a)$$
$$M_A^\lambda = \Big\{-2 + 2\epsilon x + 2(1-\epsilon)x^2 + \beta(1-x)[-1 + (3-\epsilon)x]\Big\}m\gamma^\lambda + \Big\{-4(1-\epsilon)xu(1-xu)\Big\}p'^\lambda + \Big\{4(1-x) + 4(1-\epsilon)x^2uv + \beta(1-x)[2 - 2(3-\epsilon)xv]\Big\}p^\lambda \quad , \qquad (44b)$$
$$M_A'^\lambda = \Big\{-2 + 2\epsilon x + 2(1-\epsilon)x^2 + \beta(1-x)[-1 + (3-\epsilon)x]\Big\}m\gamma^\lambda + \Big\{4(1-x) + 4(1-\epsilon)x^2uv + \beta(1-x)[2 - 2(3-\epsilon)xu]\Big\}p'^\lambda + \Big\{-4(1-\epsilon)xv(1-xv)\Big\}p^\lambda \quad , \qquad (44c)$$
$$N_A = (1-x)\Big\{2 - [2(1-\epsilon) + a]xu - (b-c)x^2u + \beta[1 - (3-\epsilon)x(1+u) + 2(3-\epsilon)x^2u]\Big\}(m^2 - p'^2) + (1-x)\Big\{2 - [2(1-\epsilon) + a]xv - (b-c)x^2v + \beta[1 - (3-\epsilon)x(1+v) + 2(3-\epsilon)x^2v]\Big\}(m^2 - p^2) + \Big\{2(1-x) + [2(1-\epsilon) + a + c]x^2uv + (b-c)x^3uv + \beta(1-x)[1 - (3-\epsilon)x + 2(3-\epsilon)x^2uv]\Big\}k^2 \quad . \qquad (44d)$$

Additional freedom in the exact expression for the vertex correction is implied by the identity

$$\int dx(1-x)x^{n-\epsilon}\frac{q^2}{m^2}\left(\frac{m^2}{H}\right)^{2+\epsilon}\Gamma(2+\epsilon) = \int dx[(n-\epsilon) - (n+1-\epsilon)x]x^{n-1-\epsilon}\left(\frac{m^2}{H}\right)^{1+\epsilon}\Gamma(1+\epsilon) \quad , \qquad (45)$$

which allows contributions to be shifted back and forth between B^λ and A^λ. Equation (45) can be used to simplify the expressions for L, M^λ, M'^λ, and N. The alternative expressions are [15]

$$L_B = 2q^2 \quad , \qquad (46a)$$
$$M_B^\lambda = 2pq\,m\gamma^\lambda - 2v(m^2 - p^2)q^\lambda - 4vpq\,p^\lambda \quad , \qquad (46b)$$
$$M_B'^\lambda = 2p'q\,m\gamma^\lambda - 2u(m^2 - p'^2)q^\lambda - 4up'q\,p'^\lambda \quad , \qquad (46c)$$
$$N_B = 4(p'q\,pq - m^2q^2) + 2upq\,(m^2 - p'^2) + 2vp'q\,(m^2 - p^2) \quad , \qquad (46d)$$

$$L_A = -2(1+\epsilon)(1-x) - 2\beta(1-2x) \quad , \qquad (47a)$$
$$M_A^\lambda = \Big\{-2 + 2\epsilon x + 2(1-\epsilon)x^2 - \beta(1-2x)\Big\}m\gamma^\lambda + \Big\{-4(1-\epsilon)xu(1-xu)\Big\}p'^\lambda + \Big\{4(1-x) + 4(1-\epsilon)x^2uv + 2\beta[1 - x(1+v)]\Big\}p^\lambda \quad , \qquad (47b)$$
$$M_A'^\lambda = \Big\{-2 + 2\epsilon x + 2(1-\epsilon)x^2 - \beta(1-2x)\Big\}m\gamma^\lambda + \Big\{4(1-x) + 4(1-\epsilon)x^2uv + 2\beta[1 - x(1+u)]\Big\}p'^\lambda + \Big\{-4(1-\epsilon)xv(1-xv)\Big\}p^\lambda \quad , \qquad (47c)$$
$$N_A = \Big\{(1-x)[2 - [2(1-\epsilon) + a]xu - (b-c)x^2u] + \beta[1 - x(2+u) + (3-\epsilon)x^2u - (2-\epsilon)x^3u]\Big\}(m^2 - p'^2) + \Big\{(1-x)[2 - [2(1-\epsilon) + a]xv - (b-c)x^2v] + \beta[1 - x(2+v) + (3-\epsilon)x^2v - (2-\epsilon)x^3v]\Big\}(m^2 - p^2) + \Big\{2(1-x) + [2(1-\epsilon) + a + c]x^2uv + (b-c)x^3uv + \beta[1 - 2x + (3-\epsilon)x^2uv - (2-\epsilon)x^3uv]\Big\}k^2 \quad . \qquad (47d)$$

Perhaps the simplest form of all is achieved when $a = 2(1-\epsilon) - \beta$, $b = 0$, and $c = -\beta(2-\epsilon)$. In that case one has $\xi_B = 0$, $\xi_A = 2(1-\epsilon)(1+\beta)x$, and

$$N_A = \Big\{2(1-x)[1-2(1-\epsilon)xu] + \beta(1-2x)\Big\}(m^2 - p'^2) + \Big\{2(1-x)[1-2(1-\epsilon)xv] + \beta(1-2x)\Big\}(m^2 - p^2) + \Big\{2[1-x+2(1-\epsilon)x^2uv] + \beta(1-2x)\Big\}k^2 \quad . \tag{48}$$

VI. CONCLUSION

Our main goal was to present a form of the one-loop vertex correction in the Yennie gauge that is useful for calculations. That form is given in Eq. (34) along with the accompanying definitions. We give explicit expressions for the terms B^λ, A^λ, ξ_B, and ξ_A that occur in the vertex correction, and discuss the lack of uniqueness in their definitions. Along the way, we have rederived the gauge invariant one-loop contribution to the renormalization constant Z_1, and have provided a new demonstration that the gauge parameter must take the value $\beta = 2/(1-2\epsilon)$ in order to eliminate all infrared divergences.

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