Analytic structure of the full fermion propagator in quenched and unquenched QED

P. Maris*
Institute for Theoretical Physics, University of Groningen, Nijenborgh 4, NL-9747 AG Groningen, The Netherlands
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We study the analytic structure of the electron propagator in the entire complex p^2 plane, using the Dyson-Schwinger equation. It is shown that in the usual ladder approximation there are two complex conjugate branch points, both in quenched and in unquenched strong coupling QED. There is, however, an essential difference between the quenched and the unquenched approximation: using the unquenched approximation, the branch points seem to approach the real axis in the continuum limit, in contrast with what happens in the quenched approximation.

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I. INTRODUCTION

The behavior of the full fermion propagator plays an essential role in quantum field theories. Some aspects of the fermion propagator can be studied in standard perturbation theory, but other methods have to be used in order to study nonperturbative phenomena, such as dynamical symmetry breaking and confinement. One way of doing this is to analyze the Dyson-Schwinger equation. This integral equation is very useful for studying dynamical chiral symmetry breaking and it has been used for this purpose for a long time [1–3]. For a recent review about dynamical chiral symmetry breaking, see [4].

In order to find nontrivial solutions of this integral equation, several approximations have to be made. The standard approximation is the ladder approximation, in which the full gauge boson propagator and the full vertex are replaced by the bare ones. Then the equation is transformed to Euclidean space, and analyzed numerically. This approach has been very successful in the Euclidean ultraviolet region. Starting with a chiral Lagrangian, it leads to dynamical chiral symmetry breaking if the coupling is above a certain critical value. Of course different choices in approximating the Dyson-Schwinger equation sometimes lead to slightly different results, but the general behavior of the solutions is similar. There is however at least one problem with this approach: it leads to complex singularities in the full fermion propagator, both in QED and in QCD [5–9].

A. The analytic structure of the propagator

The analytic structure of the bare fermion propagator in momentum space is well known: it has a single pole at the bare mass of the fermion. The integration path one encounters in all kinds of calculations, goes around this singularity, and therefore we can perform the usual Wick rotation from Minkowski space to Euclidean space. In perturbation theory, the full fermion propagator has a similar structure, at least on the first Riemann sheet: a single pole at the physical mass of the particle, and a more complicated structure for momenta beyond some threshold energy for multiparticle production. If we are dealing with massless particles, as in QED, where we have massless photons, this single pole becomes a logarithmic branch point. We therefore expect a full electron propagator with a singularity at the physical mass of the electron, which is located on the real axis in the timelike region at $p^2_{\rm Mink} = m^2_{\rm phys}$, and a logarithmic branch cut along the real axis, beyond this singularity.

However, it turns out that in the ladder approximation there are complex singularities: more than 15 years ago there was a first analysis of the analytic structure of the electron propagator in truncated four-dimensional QED [5], which obtained complex singularities. The origin of these singularities is not known, but they were generally believed to be artifacts of the approximation, and they are not taken very seriously. Therefore they have been neglected for about 10 years. A few years ago this unsolved mystery has attracted some more attention, especially in QCD [6–8]. The suggestion has been made that they are not artifacts of the approximation, but a property of the full theory connected with confinement [6,10]: if the quarks have no mass singularity in the timelike region, they can never be on mass shell, and thus never be observed. In that case however, it should be possible to find different analytic structures in confining and nonconfining theories. We have analyzed the analytic structure of the fermion propagator in QED in more detail [9], in order to get insight into the origin and physical meaning, if any, of these complex singularities. In the next sections we will discuss the main results in four-dimensional QED, first in the quenched approximation (Sec. II), and then with the one-loop vacuum polarization (Sec. III).

B. The Dyson-Schwinger equation

The Dyson-Schwinger equation for fermions with a bare mass zero is

$$S^{-1}(p) = \not p - ie^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu S(k) \Gamma_\nu(p,k) D^{\mu\nu}(q) \qquad (1)$$

with $q_\mu = (p-k)_\mu$. The full fermion propagator in this equation, $S(p)$, can be written as

$$S^{-1}(p) = \beta(p^2)\left[\not p - m(p^2)\right], \qquad (2)$$

*Present address: Department of Physics, Nagoya University, Nagoya 464-01, Japan.

where $m(p^2)$ is called the dynamical mass function.

We use the so-called ladder approximation for the vertex, in which the full vertex $\Gamma^\mu(p,k)$ is replaced by the bare vertex γ^μ. This approximation is generally believed to be reliable in the Landau gauge only, and therefore we will use that gauge. Recently, it has been shown that the ladder approximation in the Landau gauge gives almost the same result as a more elaborate vertex ansatz in a general gauge [3]. The full photon propagator in the Landau gauge is

$$D^{\mu\nu}(q) = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{1}{q^2+\Pi(q^2)}\,. \tag{3}$$

The vacuum polarization $\Pi(q)$ can be expressed in terms of the full electron propagator and the full vertex function; however we will start with the quenched approximation, in which the vacuum polarization is neglected.

Furthermore we will keep $\beta(p^2)$ equal to one for all momenta, not only in the quenched approximation, but also if we take into account the effects of the vacuum polarization. That means that we have to deal with only one integral equation for the dynamical mass function of the fermions:

$$m(p^2) = -3\,ie^2\int\frac{d^4k}{(2\pi)^4}\frac{m(k^2)}{k^2-m^2(k^2)}\frac{1}{q^2+\Pi(q^2)}\,. \tag{4}$$

This integral equation can be analyzed by making a Wick rotation from Minkowski space to Euclidean space, which is of course only allowed if there are no complex singularities in the complex plane; for the moment we will assume that that is the case.

In massless QED in four dimensions there is no intrinsic mass scale present in the Lagrangian, which means that the mass scale has to be defined by something else. Here we will use the (Euclidean) ultraviolet cutoff Λ, which is needed in the integral, to set our scale. To study chiral symmetry beaking it is more convenient to use a finite ultraviolet cutoff [2] rather than to work in the limit $\Lambda\to\infty$. In the end however we are interested in this continuum limit.

II. QUENCHED QED

In the quenched approximation we replace the full photon propagator by the bare one, and neglect all the effects of the vacuum polarization. With $\Pi(q^2)=0$ for all momenta, we can perform the angular integration analytically, which leads to

$$m(x) = \lambda\int_0^{\Lambda^2} dy\,\frac{1}{\max(x,y)}\frac{y\,m(y)}{y+m^2(y)} \tag{5}$$

with $x=-p^2$ in the Euclidean region. We have absorbed some numerical factors and the coupling constant in the new constant λ

$$\lambda = \frac{3\alpha}{4\pi} = \frac{3e^2}{(4\pi)^2}\,. \tag{6}$$

This integral equation can now be converted to a nonlinear, second-order differential equation with boundary conditions by differentiating twice with respect to x:

$$x m''(x) + 2m'(x) + \lambda\frac{m(x)}{x+m^2(x)} = 0\,. \tag{7}$$

A. Quasilinear equation

It is obvious that this differential equation, with its boundary conditions, possesses the trivial solution $m(x)=0$ for all momenta, and it turns out that this is the only solution for small values of the coupling. If the coupling becomes larger than the critical value $\lambda_c=\frac{1}{4}$, a nontrivial solution bifurcates from the trivial one, which can be studied using bifurcation theory [3, 11, 12]. This leads to a quasilinear equation which is similar to Eq. (5), the only difference being the replacement $y+m^2(y)\to y+m^2$ in the denominator of this equation. The nontrivial solution of this linearized equation, together with its boundary conditions, is the hypergeometric function

$$m(x) = m\,{}_2F_1(a_+,a_-,2;-x/m^2) \tag{8}$$

with $a_\pm = \frac{1}{2}\pm\sqrt{\frac{1}{4}-\lambda}$.

This solution can be continued into the entire complex plane, and its analytic structure is well known [13]. Around the origin the hypergeometric function is an analytic function, which can be written as a hypergeometric series which converges for all values $|x|<m^2$. Furthermore, this function has two singular points: one at $x=-m^2$ and one at $x=-\infty$; this last point however is not interesting because we use a finite cutoff Λ. At $x=-m^2$ the mass function has a branch point, and there the *linearized* propagator

$$S(p) = \frac{\not p + m(-p^2)}{p^2-m^2} \tag{9}$$

goes to infinity, which means that this point corresponds to the branch point at the physical mass of the electron. This branch point is caused by the zero of the denominator $x+m^2$ of the linearized kernel of the integral equation; for real values of m this only happens on the negative real axis. So this quasilinear equation leads *by construction* to the analytic structure of the mass function and the propagator that we expect on physical grounds, and to an electron with a physical mass m.

B. Nonlinear equation

After the reduction to a differential equation, Eq. (7), we can easily solve the nonlinear equation numerically [9]. It turns out that there is dynamical chiral symmetry breaking only if the coupling is larger than the critical value $\lambda_c=\frac{1}{4}$, as expected from the bifurcation analysis of the differential equation. Both the solution of the quasilinear equation and of the nonlinear equation behave very similarly near the critical value of the coupling constant, at least on the positive real axis.

Once we have solved the equation on the positive real axis, we can make an analytic continuation into the complex plane numerically. It turns out [9] that there is a pair of complex conjugate branch points $x_b,\bar{x}_b$ at which points the denominator of the propagator goes to zero

$$x_b + m^2(x_b) = 0\,. \tag{10}$$

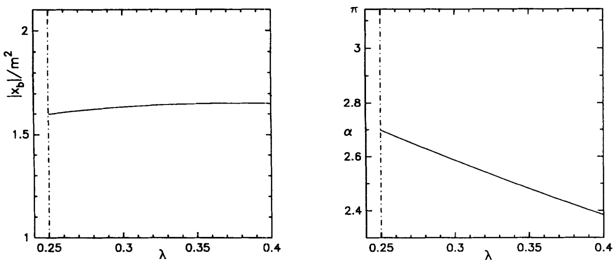

FIG. 1. The location of the branch points in massless quenched four-dimensional QED: the absolute value divided by $m^2(0)$ (left) and the phase α_b (right), both as a function of the coupling; the critical coupling is $\lambda_c = 0.25$.

This means that the propagator goes to infinity at these singularities, and it would correspond to a physical mass $m^2 = -x_b$ if these branch points were on the real axis. The branch points are essentially caused by the nonlinear structure of the Dyson-Schwinger equation: they are located at the zero of the denominator and they will not appear in a linearized approximation.

As the coupling tends to the critical value, the branch points approach the origin (with a fixed ultraviolet cutoff), because their absolute value is almost proportional to the mass $m(0)$, see Fig. 1 (left). However they do not approach the origin along the real axis [14]: the phase α of the branch point x_b stays smaller than π for all values of $\lambda > \lambda_c$, as can be seen from Fig. 1 (right). In the limit $\lambda \downarrow \lambda_c$ the phase goes toward 2.6985 rad.

Similar singularities can also be found using the analytical solution of the quasilinear equation. If we insert that solution in the full propagator

$$S(p) = \frac{\not p + m(-p^2)}{p^2 - m^2(-p^2)}\,, \tag{11}$$

we find two complex conjugate points where the propagator goes to infinity, located at the zeros of

$$x + \left[m\,{}_2F_1(a_+, a_-, 2; -x/m^2)\right]^2 = 0\,, \tag{12}$$

which can be calculated numerically. These complex singularities are located close to the ones found with the nonlinear differential equation, and at the critical coupling the phase of these points goes toward 2.7581 rad. That means that not only on the real axis, but also in the complex plane, the solutions of the nonlinear and the linearized equation have a similar behavior.

III. UNQUENCHED QED

One way to improve the usual ladder approximation is to take a dressed photon propagator instead of a bare one. The full vacuum polarization is

$$\Pi(p^2) = -\frac{ie^2 N_f}{3}\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}[\gamma_\mu S(q)\Gamma^\mu(p,k)S(k)], \tag{13}$$

where N_f is the number of fermion flavors. In order to calculate the leading contribution from this vacuum polarization, we make the same approximation for the vertex function as before: replace $\Gamma^\mu(p,k)$ by γ^μ. When we take bare, massless fermion propagators instead of the full ones we can calculate this vacuum polarization exactly; its leading logarithmic contribution is

$$\Pi(x) = -\tfrac{4}{9}\lambda N_f x \ln(\Lambda^2/x)\,. \tag{14}$$

The inclusion of this logarithmic part of the vacuum polarization leads to a correction of the bare propagator and gives the following integral equation for the mass function:

$$m(x) = \lambda \int_0^{\Lambda^2} dy\, \frac{y m(y)}{y + m^2(y)}\,\frac{2}{\pi}\int_0^\pi \frac{\sin^2\theta\, d\theta}{z - \Pi(z)} \tag{15}$$

with $z = x + y - 2\sqrt{x}\sqrt{y}\cos\theta$. The limit $N_f \downarrow 0$ corresponds to the quenched approximation discussed earlier. This integral equation cannot be reduced to a differential equation as before, and therefore we have to solve it as an integral equation. Note that there is not only a radial integration, but also an angular integration to perform numerically. We first solve the integral equation on the real axis. This leads again to a critical value for the coupling, but with one fermion flavor its value is roughly twice as large as in the quenched approximation:

$$\lambda_c = 0.4948\,. \tag{16}$$

This critical coupling increases with the number of fermion flavors, with two flavors it is

$$\lambda_c = 0.6733\,. \tag{17}$$

These values for the critical coupling are in good agreement with similar numerical and analytical analyses [15].

The analytic continuation of the integral equation is not so simple as in the case of quenched QED, where we have a differential equation. Now we have to solve an integral equation, which means that we have to rotate the whole equation into the complex plane over an angle α. Because we have a finite cutoff Λ^2 in our integral, we also have to take into account the contribution from the arc from $\Lambda^2 e^{i\alpha}$ to Λ^2. Then we can solve the integral equation along our new contour. In order to be sure that we deal with the analytic continuation of the solution on the real axis, we compare the value of $m(x)$ at $x = 0$, which is a common point for all our contours, with the original $m(0)$.

We have determined the phase of the branch points, caused by zeros of $x + m^2(x)$, as a function of the coupling constant for both one and two fermion flavors, see Fig. 2. Away from the critical coupling the branch points behave like the ones we have found in quenched QED: the phase increases with decreasing coupling and there seems to be only a numerical difference: they are located much closer to the negative real axis. However, in the limit $\lambda \downarrow \lambda_c$ there is an essential difference. In quenched QED the phase goes toward 2.6985 rad (see Fig. 1), whereas with the vacuum polarization the branch points seem to approach the real axis. Although it is very difficult to get accurate results close to the critical coupling, the results strongly suggest that in the continuum limit the phase

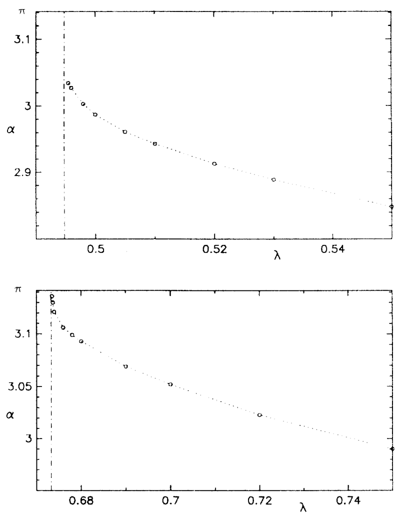

FIG. 2. The phase of the branch points as a function of the coupling in unquenched QED: with $N_f = 1$ (top) and with $N_f = 2$ (bottom).

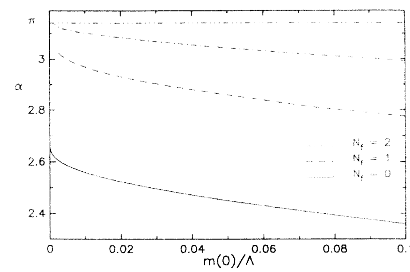

FIG. 3. The phase of the branch points as function of $m(0)$ for $N_f = 0,\ 1,\ 2$.

goes to π, the value which we would expect on physical grounds.

IV. DISCUSSION

The analytic structure is qualitatively the same in both quenched and unquenched QED, and at first hand it seems that there is only a quantitative difference [14]. In quenched QED the phase of the branch point is of order 2.4 to 2.6, and goes toward 2.7 rad if the coupling approaches the critical coupling, see Fig. 1, whereas with the one-loop vacuum polarization the phase is of order 3, see Fig. 2.

In order to compare both quenched and unquenched QED, in Fig. 3 we have plotted the phase of the branch point versus the value of the mass function at the origin. From this figure we can see that the inclusion of the vacuum polarization moves the branch points toward the negative real axis, where we would expect them to be on physical grounds; this indicates that the complex branch points are indeed artifacts of the approximations. This is also suggested by the behavior of the complex branch points in the continuum limit: a closer look at Fig. 3 reveals that there is an essential difference in the limit $\lambda \downarrow \lambda_c$ between quenched and unquenched QED.

The physically relevant region in strong coupling QED is the continuum limit $\Lambda \to \infty$ and $\lambda \downarrow \lambda_c$. In taking this limit, we keep the infrared mass $m(0)$ fixed, and in this way we can remove the (artificial) ultraviolet cutoff from the integral equation; the mass scale will then be set by the (fixed) infrared mass. It has been argued that in the broken phase the critical coupling acts as an ultraviolet fixed point of the theory, at least in the quenched ladder approximation [2]. It is therefore very interesting what happens in this continuum limit: as the coupling goes to the critical value and $m(0)/\Lambda$ vanishes, the phase of the branch points seems to approach π if one includes the vacuum polarization, in contrast to what happens in quenched QED, see Fig. 3. Thus in unquenched QED in the continuum limit the complex branch points become ordinary mass singularities in the timelike region.

This leads to the conclusion that the complex branch points are indeed an artifact of the approximations, at least in QED in four dimensions, and that the quenched approximation is not very reliable. Of course we have to study this in more detail, and there are several improvements still to be made: we should make better approximations for the full vertex function and we should also use the full fermion propagator instead of the bare one in the vacuum polarization. The gauge dependence of these complex branch points is also a very interesting point, which might be studied using a suitable ansatz for the full vertex function.

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[1] R. Jackiw and K. Johnson, Phys. Rev. D **8**, 2386 (1973); T. Maskawa and H. Nakajima, Prog. Theor. Phys. **52**, 1326 (1974); K. Higashijima, Prog. Theor. Phys. Suppl. **104**, 1 (1991).

[2] P.I. Fomin, V.P. Gusynin, V.A. Miransky, and Yu.A. Sitenko, Nuovo Cimento **6**, 1 (1983); V.A. Miransky and P.I. Fomin, Fiz. Elem. Chastits At. Yadra **16**, 469 (1985) [Sov. J. Part. Nucl. **16**, 203 (1985)]; V.A. Miransky, Phys. Lett. **165B**, 401 (1985); V.A. Miransky, Nuovo Cimento **90A**, 149 (1985).

[3] D. Atkinson, V.P. Gusynin, and P. Maris, Phys. Lett. B **303**, 157 (1993); D. Atkinson, J.C.R. Bloch, V.P. Gusynin, M.R. Pennington, and M. Reenders, *ibid.* **329**, 117 (1994).

[4] C.D. Roberts and A.G. Williams, Prog. Part. Nucl. Phys. **33**, 477 (1994).
[5] R. Fukuda and T. Kugo, Nucl. Phys. **B117**, 250 (1976); D. Atkinson and D.W.E. Blatt, *ibid.* **B151**, 342 (1979).
[6] S.J. Stainsby and R.T. Cahill, Phys. Lett. A **146**, 467 (1990).
[7] P. Maris and H.A. Holties, Int. J. Mod. Phys. A **7**, 5369 (1992).
[8] S.J. Stainsby and R.T. Cahill, Int. J. Mod. Phys. A **7**, 7541 (1992).
[9] P. Maris, Ph.D. thesis, University of Groningen, 1993.
[10] C.J. Burden, C.D. Roberts, and A.G. Williams, Phys. Lett. B **285**, 347 (1992); C.D. Roberts, A.G. Williams, and G. Krein, Int. J. Mod. Phys. A **7**, 5607 (1992).
[11] G.H. Pimbley, *Eigenfunction Branches of Nonlinear Operators, and Their Bifurcations*, Lecture Notes in Physics Vol. 104 (Springer-Verlag, Berlin, 1969).
[12] D. Atkinson, J. Math. Phys. **28**, 2494 (1987).
[13] A. Erdélyi, editor, *Higher Transcendental Functions*, Vol. I of Bateman Manuscript Project (McGraw-Hill, New York, 1953).
[14] P. Maris, in *Beyond the Standard Model III*, edited by S. Godfrey and P. Kalyniak (World Scientific, Singapore, 1993), p. 441.
[15] K. Kondo, Y. Kikukawa, and H. Mino, Phys. Lett. B **220**, 270 (1989); J. Oliensis and P.W. Johnson, Phys. Rev. D **42**, 656 (1990); V.P. Gusynin, Mod. Phys. Lett. A **5**, 133 (1990); P.E.L. Rakow, Nucl. Phys. **B356**, 27 (1991); K. Kondo and H. Nakatani, *ibid.* **B351**, 236 (1991); K. Kondo and H. Nakatani, Prog. Theor. Phys. **88**, 737 (1992); D. Atkinson, H.J. de Groot, and P.W. Johnson, Int. J. Mod. Phys. A **7**, 7629 (1992); K. Kondo, H. Mino, and H. Nakatani, Mod. Phys. Lett. A **7**, 1509 (1992).