Axially symmetric solutions in electroweak theory

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(Received 30 March 1994)

We present the general ansatz, the energy density, and the Chern-Simons charge for static axially symmetric configurations in the bosonic sector of the electroweak theory. Containing the sphaleron, the multisphalerons, and the sphaleron-antisphaleron pair at finite mixing angle, the ansatz further allows the construction of the sphaleron and multisphaleron barriers and of the bisphalerons at finite mixing angle. We conjecture that further solutions exist.

PACS number(s): 11.15.Kc, 12.15.−y

I. INTRODUCTION

In the electroweak theory several types of classical solutions are known. A decade ago the sphaleron solution of the electroweak theory was discovered [1] and constructed in the limit of vanishing mixing angle [2]. In this limit the sphaleron is spherically symmetric and parity reflection symmetric. Much later the sphaleron was constructed for the full electroweak theory with gauge group SU(2)⊗U(1) [3,4]. At finite mixing angle the sphaleron is only axially symmetric, but it retains its parity reflection symmetry. At the physical mixing angle the spherical approximation for the sphaleron is excellent [3,4].

Recently further solutions of the electroweak theory have been constructed, which are axially symmetric and symmetric under parity reflections. These are, on the one hand, the multisphaleron solutions [5] and, on the other hand, the sphaleron-antisphaleron pair [6,7]. The multisphaleron solutions carry Chern-Simons charge $N_{\rm CS} = n/2$, where n is an integer counting the winding of the fields in the azimuthal angle ϕ. The sphaleron has winding number $n = 1$. Like the sphaleron the multisphalerons are thus associated with fermion number violation [5]. In contrast the sphaleron-antisphaleron pair carries Chern-Simons charge $N_{\rm CS} = 0$ [7]. The ansatz for the sphaleron-antisphaleron pair can be generalized by realizing that it involves a winding of the fields in the angle θ. Denoting the corresponding winding number m, the sphaleron-antisphaleron pair has winding number $m = 2$, while the sphaleron has $m = 1$.

When constructing noncontractible loops in configuration space, the intermediate configurations between the vacua and the sphaleron, representing the sphaleron barrier, have fewer symmetries than the sphaleron, even for vanishing mixing angle [1,8]. Indeed in the limit of vanishing mixing angle the construction of the sphaleron barrier involves configurations which do not retain the discrete symmetry of the sphaleron, parity reflection symmetry. For finite mixing angle the sphaleron barrier has not yet been constructed.

Furthermore, for high values of the Higgs boson mass new classical solutions appear in the electroweak theory, the bisphalerons [9,10]. These solutions, constructed so far only for vanishing mixing angle, where they are spherically symmetric, are not invariant under parity, but occur as parity doublets. Like the sphaleron, at finite mixing angle they will retain only axial symmetry. The bisphalerons are lower in energy than the sphaleron [9,10]. This was demonstrated also in a perturbative analysis for finite mixing angle [11]. Therefore at large Higgs boson masses the lowest bisphalerons represent the top of the energy barrier between neighboring topologically distinct vacua. The construction of the bisphalerons at finite mixing angle is an outstanding problem.

In this paper we develop the formalism for the construction of general classical static configurations of the electroweak theory with axial symmetry. In Sec. II we present the general ansatz for the fields and the energy density obtained with this ansatz. Further we discuss the four residual gauge symmetries of the energy density and several choices of gauge. In Sec. III we discuss the classical solutions obtainable with this ansatz, the sphaleron and multisphalerons, the sphaleron-antisphaleron pair and their generalizations, as well as the bisphalerons and the barriers. In Sec. IV we present the Chern-Simons charge for the general ansatz. Further we evaluate the Chern-Simons charge for the multisphalerons and for the solutions which may be obtained with the generalized sphaleron-antisphaleron pair ansatz. We present our conclusions in Sec. V.

II. ANSATZ AND ENERGY DENSITY

Let us consider the bosonic sector of the Weinberg-Salam theory:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + (D_\mu\Phi)^\dagger(D^\mu\Phi) - \lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2 , \qquad (1)$$

with the SU(2) field strength tensor

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu \ , \tag{2}$$

with the U(1) field strength tensor

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{3}$$

and the covariant derivative for the Higgs field

$$D_\mu \Phi = \left(\partial_\mu - \frac{i}{2} g \tau^a W^a_\mu - \frac{i}{2} g' A_\mu \right)\Phi \ . \tag{4}$$

The gauge symmetry is spontaneously broken due to the nonvanishing vacuum expectation value v of the Higgs field,

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{5}$$

leading to the boson masses

$$M_W = \frac{1}{2} g v \ , \quad M_Z = \frac{1}{2}\sqrt{(g^2 + g'^2)} v \ , \quad M_H = v\sqrt{2\lambda} \ . \tag{6}$$

The mixing angle θ_W is determined by the relation $\tan\theta_W = g'/g$ and the electric charge is $e = g \sin\theta_W$.

A. Axially symmetric ansatz

Let us introduce the set of orthonormal vectors [12,13,5]

$$\mathbf{u}_1^{(n)}(\phi) = (\cos n\phi, \sin n\phi, 0) \ ,$$

$$\mathbf{u}_2^{(n)}(\phi) = (0, 0, 1) \ ,$$

$$\mathbf{u}_3^{(n)}(\phi) = (\sin n\phi, -\cos n\phi, 0) \tag{7}$$

and the matrices

$$G_i^{(n)}(\phi) = u_i^{a(n)}(\phi)\tau^a \ , \tag{8}$$

where τ^a are the Pauli matrices and ϕ is the azimuthal angle defined via

$$\begin{aligned}(x, y, z) &= (\rho\cos\phi, \rho\sin\phi, z) \\ &= (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) \ .\end{aligned} \tag{9}$$

The static axially symmetric ansatz for the SU(2) gauge fields, the U(1) gauge field, and the Higgs field is then given by

$$\begin{aligned} W_i(\mathbf{r}) &= W_i^a(\mathbf{r})\tau^a = u_j^{i(1)}(\phi) G_k^{(n)}(\phi) w_j^k(\rho, z) \ , \\ W_0(\mathbf{r}) &= W_0^a(\mathbf{r})\tau^a = 0 \ , \end{aligned} \tag{10}$$

$$\begin{aligned} A_i(\mathbf{r}) &= u_j^{i(1)}(\phi) a_j(\rho, z), \\ A_0(\mathbf{r}) &= 0 \ , \end{aligned} \tag{11}$$

$$\Phi(\mathbf{r}) = \frac{v}{\sqrt{2}} \left[h_0(\rho, z) + i h_j(\rho, z) G_j^{(n)}\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{12}$$

where the indices i, j, k, and a run from 1 to 3.

This ansatz contains 16 arbitrary real functions of the variables ρ and z. The ansatz is axially symmetric; i.e., a rotation around the z axis can be compensated by a gauge transformation. [For the Higgs field the compensating gauge transformation is an element of the diagonal group U(1)⊕U(1), the first U(1) being the subgroup of SU(2) generated by the matrix $G_2^{(n)}$.]

B. Energy functional

The resulting axially symmetric energy functional E,

$$E = \frac{1}{2} \int (E_w + E_a + v^2 E_h)\, d\phi\, \rho d\rho\, dz \ , \tag{13}$$

then has the contributions

$$\begin{aligned} E_w = & \left(\partial_\rho w_3^1 + \frac{1}{\rho}(n w_1^3 + w_3^1) - g(w_3^2 w_1^3 - w_3^3 w_1^2)\right)^2 + \left(\partial_\rho w_3^2 + \frac{1}{\rho} w_3^2 - g(w_1^1 w_3^3 - w_3^1 w_1^3)\right)^2 \\ & + \left(\partial_\rho w_3^3 + \frac{1}{\rho}(w_3^3 - n w_1^1) - g(w_3^1 w_1^2 - w_1^1 w_3^2)\right)^2 + \left(\partial_z w_3^1 + n\frac{w_2^3}{\rho} - g(w_3^2 w_2^3 - w_3^3 w_2^2)\right)^2 \\ & + [\partial_z w_3^2 - g(w_3^3 w_2^1 - w_2^3 w_3^1)]^2 + \left(\partial_z w_3^3 - n\frac{w_2^1}{\rho} - g(w_2^2 w_3^1 - w_3^2 w_2^1)\right)^2 + [\partial_\rho w_2^3 - \partial_z w_1^3 - g(w_1^2 w_2^1 - w_1^1 w_2^2)]^2 \\ & + [\partial_z w_1^2 - \partial_\rho w_2^2 - g(w_1^3 w_2^1 - w_1^1 w_2^3)]^2 + [\partial_z w_1^1 - \partial_\rho w_2^1 - g(w_2^3 w_1^2 - w_2^2 w_1^3)]^2 \ , \end{aligned} \tag{14}$$

$$E_a = \left(\partial_\rho a_3 + \frac{1}{\rho} a_3\right)^2 + (\partial_z a_3)^2 + (\partial_\rho a_2 - \partial_z a_1)^2 \ , \tag{15}$$

$$E_h = \left(\partial_\rho h_0 + \frac{g}{2}(h_1 w_1^1 + h_2 w_1^2 + h_3 w_1^3) - \frac{g'}{2}(h_2 a_1)\right)^2 + \left(\partial_\rho h_1 + \frac{g}{2}(h_3 w_1^2 - h_2 w_1^3 - h_0 w_1^1) + \frac{g'}{2}(h_3 a_1)\right)^2$$
$$+ \left(\partial_\rho h_2 + \frac{g}{2}(h_1 w_1^3 - h_3 w_1^1 - h_0 w_1^2) + \frac{g'}{2}(h_0 a_1)\right)^2 + \left(\partial_\rho h_3 + \frac{g}{2}(h_2 w_1^1 - h_1 w_1^2 - h_0 w_1^3) - \frac{g'}{2}(h_1 a_1)\right)^2$$
$$+ \left(\partial_z h_0 + \frac{g}{2}(h_1 w_2^1 + h_2 w_2^2 + h_3 w_2^3) - \frac{g'}{2}(h_2 a_2)\right)^2 + \left(\partial_z h_1 + \frac{g}{2}(h_3 w_2^2 - h_2 w_2^3 - h_0 w_2^1) + \frac{g'}{2}(h_3 a_2)\right)^2$$
$$+ \left(\partial_z h_2 + \frac{g}{2}(h_1 w_2^3 - h_3 w_2^1 - h_0 w_2^2) + \frac{g'}{2}(h_0 a_2)\right)^2 + \left(\partial_z h_3 + \frac{g}{2}(h_2 w_2^1 - h_1 w_2^2 - h_0 w_2^3) - \frac{g'}{2}(h_1 a_2)\right)^2$$
$$+ \left(\frac{g}{2}(h_1 w_3^3 - h_0 w_3^2 - h_3 w_3^1) + \frac{g'}{2} h_0 a_3\right)^2 + \left(\frac{g}{2}(h_0 w_3^3 - h_2 w_3^1 + h_1 w_3^2) + \frac{g'}{2} h_1 a_3 - \frac{n h_1}{\rho}\right)^2$$
$$+ \left(\frac{g}{2}(-h_1 w_3^1 - h_2 w_3^2 - h_3 w_3^3) + \frac{g'}{2} h_2 a_3\right)^2 + \left(\frac{g}{2}(h_3 w_3^2 - h_0 w_3^1 - h_2 w_3^3) + \frac{g'}{2} h_3 a_3 - \frac{n h_3}{\rho}\right)^2$$
$$+ \frac{\lambda v^2}{2}(h_0^2 + h_1^2 + h_2^2 + h_3^2 - 1)^2 . \quad (16)$$

C. Residual gauge symmetries

The energy functional is invariant under a large class of gauge transformations, which keep the ansatz form invariant. These gauge transformations are given by

$$\begin{aligned} U_0(\mathbf{r}) &= \exp[i\Gamma_0(\rho,z)] ,\\ U_1(\mathbf{r}) &= \exp[i\Gamma_1(\rho,z)G_1^{(n)}(\phi)] ,\\ U_2(\mathbf{r}) &= \exp[i\Gamma_2(\rho,z)G_2^{(n)}(\phi)] ,\\ U_3(\mathbf{r}) &= \exp[i\Gamma_3(\rho,z)G_3^{(n)}(\phi)] . \end{aligned} \quad (17)$$

1. *Transformation properties of the fields*

Considering first the transformation U_0, the components of the Abelian gauge field a_i transform as

$$a_1' = a_1 + \frac{2}{g'}\frac{\partial\Gamma_0}{\partial\rho} , \quad a_2' = a_2 + \frac{2}{g'}\frac{\partial\Gamma_0}{\partial z} , \quad a_3' = a_3 \quad (18)$$

and the Higgs field components

$$\begin{pmatrix} h_3 \\ h_1 \end{pmatrix} , \quad \begin{pmatrix} h_0 \\ h_2 \end{pmatrix} \quad (19)$$

transform as doublets with angle Γ_0, and the SU(2) fields are invariant under U_0.

Considering next the three Abelian transformations U_i generated by $G_i^{(n)}$, the Abelian gauge field is invariant under these transformations, while the components of the non-Abelian gauge fields w_a^b and of the Higgs field h_0 and h_i form various multiplets.

Under the transformation $U_3 = \exp(i\Gamma_3 G_3^{(n)})$ the SU(2) gauge field components

$$\begin{pmatrix} w_1^1 \\ w_1^2 \end{pmatrix} , \quad \begin{pmatrix} w_2^1 \\ w_2^2 \end{pmatrix} , \quad \begin{pmatrix} w_3^1 \\ w_3^2 - \frac{n}{g\rho} \end{pmatrix} \quad (20)$$

transform as doublets with angle $2\Gamma_3$, w_3^3 is invariant, and the two remaining components (w_1^3, w_2^3) transform as a two-dimensional gauge field

$$\left(w_1^3\right)' = w_1^3 + \frac{2}{g}\frac{\partial\Gamma_3}{\partial\rho} , \quad \left(w_2^3\right)' = w_2^3 + \frac{2}{g}\frac{\partial\Gamma_3}{\partial z} . \quad (21)$$

The Higgs field components

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} , \quad \begin{pmatrix} h_3 \\ h_0 \end{pmatrix} \quad (22)$$

transform as doublets with angle Γ_3.

Analogously, under the transformations $U_1 = \exp(i\Gamma_1 G_1^{(n)})$ and $U_2 = \exp(i\Gamma_2 G_2^{(n)})$ similar schemes occur with

$$\begin{pmatrix} w_1^2 \\ w_1^3 \end{pmatrix} , \quad \begin{pmatrix} w_2^2 \\ w_2^3 \end{pmatrix} , \quad \begin{pmatrix} w_3^2 - \frac{n}{g\rho} \\ w_3^3 \end{pmatrix} , \quad w_3^1 , \quad \begin{pmatrix} w_1^1 \\ w_2^1 \end{pmatrix} , \quad (23)$$

and

$$\begin{pmatrix} w_1^3 \\ w_1^1 \end{pmatrix} , \quad \begin{pmatrix} w_2^3 \\ w_2^1 \end{pmatrix} , \quad \begin{pmatrix} w_3^3 \\ w_3^1 \end{pmatrix} , \quad w_3^2 , \quad \begin{pmatrix} w_1^2 \\ w_2^2 \end{pmatrix} , \quad (24)$$

respectively, for the SU(2) gauge field components, and

$$\begin{pmatrix} h_1 \\ h_0 \end{pmatrix} , \quad \begin{pmatrix} h_2 \\ h_3 \end{pmatrix} \quad (25)$$

and

$$\begin{pmatrix} h_2 \\ h_0 \end{pmatrix} , \quad \begin{pmatrix} h_3 \\ h_1 \end{pmatrix} , \quad (26)$$

respectively, for the Higgs field components.

2. *Choices of gauge*

In order to construct classical solutions, the four residual gauge degrees of freedom need to be fixed. There appear to be many different ways to fix these four gauge degrees of freedom. However, from our experience in constructing the sphaleron at finite mixing angle, we know that care must be taken to choose a gauge where the classical solutions are regular [3,4,14].

(*a*) *Coulomb gauges.* For the single residual gauge degree of freedom present for the sphaleron at finite mixing angle U_3 [15], we chose the Coulomb gauge for the two-dimensional gauge field

$$\frac{\partial w_1^3}{\partial \rho} + \frac{\partial w_2^3}{\partial z} = 0 \tag{27}$$

since it lead to regular classical solutions [3,4,14]. We therefore suggest as a probably good choice of gauge the Coulomb gauge for all four two-dimensional gauge fields, i.e., in addition to Eq. (27),

$$\frac{\partial w_1^1}{\partial \rho} + \frac{\partial w_2^1}{\partial z} = 0 , \quad \frac{\partial w_1^2}{\partial \rho} + \frac{\partial w_2^2}{\partial z} = 0 ,$$
$$\frac{\partial a_1}{\partial \rho} + \frac{\partial a_2}{\partial z} = 0 . \tag{28}$$

In the general case such a choice of gauge leaves 16 unknown functions to be determined numerically.

(*b*) *Other gauges.* Another way of fixing the gauge consists of eliminating one or more functions, leaving a smaller number of unknown functions to be determined numerically. Appearing attractive at first sight, such gauge choices may prove to be singular [14].

Let us nevertheless consider such choices briefly. For instance, setting the angular part of the Higgs field in a canonical position, we could obtain the *physical gauge* or the *hedgehog gauge.* In the *physical gauge* the Higgs field is specified only by h_0, while $h_1 = h_2 = h_3 = 0$. In the *hedgehog gauge* the Higgs field is specified only by the function h, defined via $h_1 = h\sin\theta$, $h_2 = h\cos\theta$, while $h_0 = h_3 = 0$. Fixing three of the four degrees of freedom, both these gauges are known to be singular for the sphaleron at finite mixing angle [14].

Another possibly better choice could be to only assume $h_3 = 0$ and supplement this gauge choice with the Coulomb gauge for the remaining three degrees of freedom. Note that h_3 vanishes in all known classical solutions.

III. CLASSICAL SOLUTIONS

All known static (three-dimensional) classical solutions can be obtained from the general static ansatz. This ansatz further allows us to construct the sphaleron barrier at finite mixing angle, to generalize the bisphalerons, known at vanishing mixing angle, to finite mixing angle, and to possibly construct new solutions.

A. Barriers and bisphalerons

Until now, vacuum to vacuum paths passing the sphaleron have been constructed only at vanishing mixing angle. Since they involve parity-violating configurations, the general axially symmetric ansatz must be taken to obtain such paths at finite mixing angle. The general ansatz is also necessary to obtain the barriers associated with multisphalerons, as well as for the construction of bisphalerons at finite mixing angle.

1. Parametrization of the general ansatz

In order to compare with the known spherical barrier and to take out the trivial angular dependence (on the angle θ) we parametrize the axial functions in spherical coordinates as follows:

$$\begin{aligned}
w_1^3 &= \frac{2}{gr}F_1(r,\theta)\cos\theta , \qquad & w_2^3 &= -\frac{2}{gr}F_2(r,\theta)\sin\theta , \\
w_3^1 &= -\frac{2n}{gr}F_3(r,\theta)\cos\theta , \qquad & w_3^2 &= \frac{2n}{gr}F_4(r,\theta)\sin\theta , \\
w_1^2 &= \frac{2}{gr}H_1(r,\theta)\sin\theta\cos\theta , & & \\
w_2^1 &= \frac{2}{gr}H_2(r,\theta)\sin\theta\cos\theta , & & \\
w_1^1 &= \frac{2}{gr}[H_3(r,\theta)\sin^2\theta + H_4(r,\theta)] , & & \\
w_2^2 &= \frac{2}{gr}[H_3(r,\theta)\cos^2\theta + H_4(r,\theta)] , & & \\
w_3^3 &= \frac{2n}{gr}H_5(r,\theta) , & & \\
h_1 &= F_5(r,\theta)\sin\theta , & h_2 &= F_6(r,\theta)\cos\theta , \\
h_3 &= H_6(r,\theta)\sin\theta , & h_0 &= H_7(r,\theta) , \\
a_1 &= \frac{2}{g'r}H_8(r,\theta)\sin\theta\cos\theta , & a_2 &= \frac{2}{g'r}H_9(r,\theta) , \\
a_3 &= \frac{2}{g'r}F_7(r,\theta)\sin\theta . & &
\end{aligned} \tag{29}$$

This parametrization is a generalization of the parametrization used for the sphaleron at finite mixing angle, containing in addition to the seven functions $F_i(r,\theta)$ the nine functions $H_i(r,\theta)$. The factors of $\sin\theta$ and $\cos\theta$ in the above parametrization are chosen in accordance with the known spherical configurations, the sphaleron, the sphaleron barrier, and the bisphalerons, where the functions $F_i(r,\theta)$ and $H_i(r,\theta)$ reduce to functions of the radial coordinate r alone, as shown below.

2. Recovering spherical symmetry

In the limit $\theta_W = 0$ the sphaleron, the configurations along the sphaleron barrier, and the bisphalerons are spherically symmetric. The Abelian gauge potential can consistently be set to zero, i.e., in terms of the above parametrization (29),

$$F_7(r,\theta) = 0 , \quad H_8(r,\theta) = 0 , \quad H_9(r,\theta) = 0 . \tag{30}$$

The general spherically symmetric ansatz, necessary to obtain the sphaleron barrier and the bisphalerons, is given by

$$W_i^a = \frac{1-f_A(r)}{gr}\epsilon_{aij}\hat{\mathbf{r}}_j + \frac{f_B(r)}{gr}(\delta_{ia} - \hat{\mathbf{r}}_i\hat{\mathbf{r}}_a) + \frac{f_C(r)}{gr}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_a \ ,$$
$$W_0^a = 0 \ , \tag{31}$$
$$\Phi = \frac{v}{\sqrt{2}}[H(r) + i\tau \cdot \hat{\mathbf{r}}K(r)]\begin{pmatrix}0\\1\end{pmatrix} \ . \tag{32}$$

Comparing with the general axially symmetric ansatz we find $n = 1$ and

$$\begin{aligned} &F_1(r,\theta) = F_2(r,\theta) = F_3(r,\theta) = F_4(r,\theta) = \frac{1-f_A(r)}{2} \ , \\ &F_5(r,\theta) = F_6(r,\theta) = K(r) \ , \\ &H_1(r,\theta) = H_2(r,\theta) = H_3(r,\theta) = \frac{f_C(r) - f_B(r)}{2} \ , \\ &H_4(r,\theta) = H_5(r,\theta) = \frac{f_B(r)}{2} \ , \\ &H_6(r,\theta) = 0 \ , \quad H_7(r,\theta) = H(r) \ . \end{aligned} \tag{33}$$

The functions $f_B(r)$, $f_C(r)$, and (in the usual parametrization) $H(r)$ represent the parity-violating terms, present in the configurations along the sphaleron barrier and the bisphalerons, which generalize to seven functions $H_i(r,\theta)$ in the axially symmetric ansatz. The spherically symmetric ansatz has a residual gauge symmetry, which can be fixed, for instance, by requiring $f_C(r) = 0$. The spherically symmetric parity conserving sphaleron solution has $f_B(r) = f_C(r) = H(r) = 0$, corresponding to the vanishing of all functions $H_i(r,\theta)$.

B. Solutions with mirror symmetry

Besides being axially symmetric, the sphaleron at finite mixing angle [3,4], the multisphalerons [5], and the sphaleron-antisphaleron pair [6,7] have discrete symmetries. Supplementing the axial invariance of the fields by the discrete mirror symmetry

$$M_{xz} \otimes C \otimes (-\mathbb{1})_{\text{custodial}} \ , \tag{34}$$

where the first factor represents reflection through the xz plane and the second factor denotes charge conjugation

$$W_\mu^c = -W_\mu^T \ , \quad \Phi^c = \Phi^* \ , \quad A_\mu^c = -A_\mu \ , \tag{35}$$

leads to the simplifying conditions [1,13,3,4]

$$w_1^1 = w_1^2 = w_2^1 = w_2^2 = w_3^3 = 0 \ , \tag{36}$$

$$h_3 = h_0 = 0 \ , \tag{37}$$

$$a_1 = a_2 = 0 \ , \tag{38}$$

corresponding to $H_i(r,\theta) = 0$, $i = 1,\dots,9$. The known axially symmetric solutions are additionally invariant under parity.

1. Sphaleron and multisphalerons

The sphaleron at finite mixing angle and the multisphalerons are described by the seven axial functions $F_i(r,\theta)$ of Eqs. (29) [3–5]. The sphaleron and multisphaleron functions satisfy

$$F_a(r,\theta) = F_a(r,\pi-\theta) \ . \tag{39}$$

The solutions are invariant under $P \otimes -\mathbb{1}_{\text{custodial}}$, where the second factor is necessary because the classical Higgs field is parity odd in the gauge used.

The Higgs fields of the sphaleron and of the multisphalerons (S) assume the asymptotic forms

$$\begin{aligned} \Phi_S &= \frac{v}{\sqrt{2}}U_S(\infty)\begin{pmatrix}0\\1\end{pmatrix} \\ &= i\frac{v}{\sqrt{2}}[\sin\theta G_1^{(n)}(\phi) + \cos\theta G_2^{(n)}(\phi)]\begin{pmatrix}0\\1\end{pmatrix} \ . \end{aligned} \tag{40}$$

The gauge fields become pure gauge configurations at infinity:

$$W_i(\infty) = -\frac{2i}{g}\partial_i U_S(\infty)U_S^\dagger(\infty) \ . \tag{41}$$

Thus the boundary conditions for the functions $F_i(r,\theta)$ are [3–5]

$$\begin{aligned} r = 0: &\quad F_i(r,\theta)|_{r=0} = 0, \quad i = 1,\dots,7 \\ r \to \infty: &\quad F_i(r,\theta)|_{r=\infty} = 1, \quad i = 1,\dots,6, \\ &\quad F_7(r,\theta)|_{r=\infty} = 0 \ , \\ \theta = 0: &\quad \partial_\theta F_i(r,\theta)|_{\theta=0} = 0, \quad i = 1,\dots,7 \ , \\ \theta = \pi/2: &\quad \partial_\theta F_i(r,\theta)|_{\theta=\pi/2} = 0, \quad i = 1,\dots,7 \ . \end{aligned} \tag{42}$$

2. Sphaleron-antisphaleron pair

The sphaleron-antisphaleron pair [6,7] is also axially symmetric and parity invariant. But in contrast with the sphaleron the Higgs field is even under parity.

Klinkhamer denoted the field components as

$$\begin{aligned} &w_1^3 = -\frac{\alpha_1}{\rho} \ , \quad w_2^3 = -\frac{\alpha_0}{z} \ , \quad w_3^1 = \frac{\alpha_2}{\rho} \ , \quad w_3^2 = \frac{\alpha_3}{\rho} \ , \\ &h_1 = \beta_1 \ , \qquad h_2 = -\beta_2 \ , \\ &a_3 = \frac{g'^2}{g^2}\frac{\alpha_4}{\rho} \ . \end{aligned} \tag{43}$$

He parametrized the gauge field components in terms of the angle θ analogous to Eqs. (29), leading to the relations for the gauge field functions

$$\begin{aligned} &F_1 = 2\frac{r^2}{a}f_1^{\text{Kl}} \ , \quad F_2 = 2\frac{r^2}{a}f_0^{\text{Kl}} \ , \\ &F_3 = 2\frac{r^2}{a}f_2^{\text{Kl}} \ , \quad F_4 = 2\frac{r^2}{a}f_3^{\text{Kl}} \end{aligned} \tag{44}$$

with $a = r^2 + r_a^2$ and r_a an arbitrary scale parameter, while he parametrized the Higgs field components differently:

$$h_1 = \frac{r^2}{a} h_1^{\rm Kl} \sin 2\theta \ , \quad h_2 = h_2^{\rm Kl} \ . \tag{45}$$

This parametrization lead to θ-dependent boundary conditions for the functions $f_2^{\rm Kl}, f_3^{\rm Kl}$, and $h_2^{\rm Kl}$.

The Higgs field of the sphaleron-antisphaleron pair (S^*) assumes the asymptotic form

$$\Phi_{S^*} = \frac{v}{\sqrt{2}} U_{S^*}(\infty) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = i\frac{v}{\sqrt{2}}[\sin 2\theta G_1^{(1)}(\phi) + \cos 2\theta G_2^{(1)}(\phi)] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{46}$$

while the gauge fields become pure gauge configurations:

$$W_i(\infty) = -\frac{2i}{g} \partial_i U_{S^*}(\infty) U_{S^*}^\dagger(\infty) \ . \tag{47}$$

Therefore another parametrization appears to be natural:

$$\begin{aligned} w_1^3 &= \frac{4}{gr}\tilde{F}_1(r,\theta)\cos\theta, \quad w_2^3 = -\frac{4}{gr}\tilde{F}_2(r,\theta)\sin\theta \ , \\ w_3^1 &= -\frac{4}{gr}\tilde{F}_3(r,\theta)\cos\theta\cos 2\theta, \\ w_3^2 &= \frac{4}{gr}\tilde{F}_4(r,\theta)\cos\theta\sin 2\theta \ , \\ h_1 &= \tilde{F}_5(r,\theta)\sin 2\theta, \quad h_2 = \tilde{F}_6(r,\theta)\cos 2\theta \ . \end{aligned} \tag{48}$$

In terms of this parametrization the functions $\tilde{F}_i(r,\theta)$, $i = 1, \ldots, 6$, approach one at infinity [16].

3. Generalization of the sphaleron-antisphaleron pair ansatz

Generalizing the ansatz for the sphaleron-antisphaleron pair to arbitrary integers m, we require for the Higgs field the asymptotic form

$$\Phi_{S_m^*} = \frac{v}{\sqrt{2}} U_{S_m^*}(\infty) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = i\frac{v}{\sqrt{2}}[\sin m\theta G_1^{(1)}(\phi) + \cos m\theta G_2^{(1)}(\phi)] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{49}$$

and for the gauge fields the pure gauge configurations

$$W_i(\infty) = -\frac{2i}{g} \partial_i U_{S_m^*}(\infty) U_{S_m^*}^\dagger(\infty) \ , \tag{50}$$

leading to the general parametrization

$$\begin{aligned} w_1^3 &= \frac{2m}{gr}\tilde{F}_1(r,\theta)\cos\theta, \quad w_2^3 = -\frac{2m}{gr}\tilde{F}_2(r,\theta)\sin\theta \ , \\ w_3^1 &= -\frac{2m}{gr}\tilde{F}_3(r,\theta)\frac{\sin m\theta}{m\sin\theta}\cos m\theta, \\ w_3^2 &= \frac{2m}{gr}\tilde{F}_4(r,\theta)\frac{\sin m\theta}{m\sin\theta}\sin m\theta \ , \\ h_1 &= \tilde{F}_5(r,\theta)\sin m\theta, \quad h_2 = \tilde{F}_6(r,\theta)\cos m\theta \ . \end{aligned} \tag{51}$$

In terms of this parametrization the functions $\tilde{F}_i(r,\theta)$, $i = 1, \ldots, 6$, approach one at infinity [17]. One further step is to include both integers n and m in the ansatz; i.e., use the gauge transformation $U_{S_{n,m}}$ for the fields at infinity:

$$\Phi_{S_{n,m}} = \frac{v}{\sqrt{2}} U_{S_{n,m}}(\infty) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = i\frac{v}{\sqrt{2}}[\sin m\theta G_1^{(n)}(\phi) + \cos m\theta G_2^{(n)}(\phi)] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{52}$$

and

$$W_i(\infty) = -\frac{2i}{g} \partial_i U_{S_{n,m}}(\infty) U_{S_{n,m}}^\dagger(\infty) \ . \tag{53}$$

IV. CHERN-SIMONS CHARGE

The Chern-Simons current K_μ is not conserved; its divergence $\partial^\mu K_\mu$ represents the U(1) anomaly of the baryon current. Classical configurations are characterized by their Chern-Simons charge. The SU(2) part of the Chern-Simons charge is given by

$$\begin{aligned} N_{\rm CS} &= \int d^3 r K^0 \\ &= -\frac{g^2}{64\pi^2} \int d^3 r \epsilon_{ijk} {\rm Tr}\Big(F_{ij} W_k + i\frac{g}{3} W_i W_j W_k\Big) \\ &= \frac{1}{2\pi^2} \int d^3 r Q(\rho, z) \ . \end{aligned} \tag{54}$$

The proper gauge for evaluating the Chern-Simons charge is the gauge where the gauge field is given by

$$W_i(\infty) = -\frac{2i}{g} \partial_i U(\infty) U^\dagger(\infty) \ , \tag{55}$$

with $U(\infty) = 1$. Then this Chern-Simons charge of the configurations corresponds to their baryonic charge, when the U(1) field does not contribute to the baryon number [5].

A. General axially symmetric ansatz

The general axially symmetric ansatz leads to a Chern-Simons charge characterized by

$$\begin{aligned} -\frac{4}{g^3}Q(\rho,z) &= w_1^1 w_2^2 w_3^3 + w_1^2 w_2^3 w_3^1 + w_1^3 w_2^1\left(w_3^2 - \frac{n}{g\rho}\right) - w_1^1 w_2^3\left(w_3^2 - \frac{n}{g\rho}\right) - w_1^3 w_2^2 w_3^1 - w_1^2 w_2^1 w_3^3 \\ &\quad -\frac{1}{g}\left[w_3^1(\partial_z w_1^1 - \partial_\rho w_2^1) + \left(w_3^2 - \frac{n}{g\rho}\right)(\partial_z w_1^2 - \partial_\rho w_2^2) + w_3^3(\partial_z w_1^3 - \partial_\rho w_2^3)\right] \ . \end{aligned} \tag{56}$$

This expression must be supplemented by the appropriate gauge transformation to obtain the Chern-Simons charge of the configurations forming the sphaleron barrier at finite mixing angle and the multisphaleron barriers and to obtain the Chern-Simons charge of the bisphalerons.

B. Sphaleron and multisphalerons

For the sphaleron and multisphalerons the Chern-Simons density is proportional to

$$Q(\rho,z) = n\frac{\sin^2\Omega}{r^2}\frac{\partial\Omega}{\partial r} + n\frac{\partial}{\partial z}\Big(\frac{z}{4r^3}F_1\sin(2\Omega)\Big) + \frac{n}{\rho}\frac{\partial}{\partial\rho}\Big(\frac{\rho^2}{4r^3}F_2\sin(2\Omega)\Big) + \frac{\cos^2\theta}{4r^2}\frac{\partial}{\partial r}[F_3\sin(2\Omega)]$$
$$+\frac{\sin^2\theta}{4r^2}\frac{\partial}{\partial r}\Big(F_4\sin(2\Omega)\Big) + \frac{1}{2\rho}\frac{\partial}{\partial z}\Big(\frac{z\rho^2}{r^3}(F_3-F_4)\frac{\partial\Omega}{\partial\rho}\Big) - \frac{1}{2\rho}\frac{\partial}{\partial\rho}\Big(\frac{z\rho^2}{r^3}(F_3-F_4)\frac{\partial\Omega}{\partial z}\Big)\ , \quad (57)$$

where, analogous to Ref. [2], we incorporated the effect of a gauge transformation of the form

$$U_S(\mathbf{r}) = \exp[i\Omega(r,\theta)(\sin\theta G_1^{(n)} + \cos\theta G_2^{(n)})] \quad (58)$$

and kept all derivative terms. The proper boundary conditions are $\Omega(0) = 0$ and $\Omega(\infty) = \pi/2$ [2]. Only the first term of $Q(\rho,z)$ determines the Chern-Simons charge since the derivative terms do not contribute due to the boundary conditions for the functions $F_i(r,\theta)$ [see Eqs. (42) [3–5]] and for $\Omega(r,\theta)$.

We find, for the multisphalerons the Chern-Simons charge,

$$N_{\rm CS} = n/2\ , \quad (59)$$

independently of the Higgs boson mass and of the mixing angle, reproducing the well-known Chern-Simons charge of the sphaleron, $N_{\rm CS} = 1/2$. This Chern-Simons charge of the sphaleron and of the multisphalerons corresponds to their baryonic charge $Q_B = n/2$ since the U(1) field does not contribute to their baryon number [5].

C. Generalization of the sphaleron-antisphaleron pair ansatz

For the sphaleron-antisphaleron pair another gauge transformation must be chosen to evaluate the Chern-Simons charge:

$$U_{S^*}(\mathbf{r}) = \exp[i\Omega(r,\theta)(\sin 2\theta G_1^{(1)} + \cos 2\theta G_2^{(1)})]\ , \quad (60)$$

with boundary conditions $\Omega(0) = 0$ and $\Omega(\infty) = \pi/2$, since this solution approaches infinity differently. In the following we present the Chern-Simons density directly for the generalized pair ansatz Eqs. (49)–(51), using the gauge transformation

$$U_{S^*_m}(\mathbf{r}) = \exp[i\Omega(r,\theta)(\sin m\theta G_1^{(1)} + \cos m\theta G_2^{(1)})]\ . \quad (61)$$

The Chern-Simons density is proportional to

$$Q(r,\theta) = \frac{m\sin m\theta}{\sin\theta}\frac{\sin^2\Omega}{r^2}\frac{\partial\Omega}{\partial r} + \frac{\partial}{\partial z}\Big(\frac{m\sin m\theta}{\sin\theta}\frac{\cos\theta}{4r^2}\tilde F_1\sin(2\Omega)\Big) + \frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\frac{m\sin m\theta}{\sin\theta}\frac{\sin^2\theta}{4r}\tilde F_2\sin(2\Omega)\Big)$$
$$+\frac{m\sin m\theta}{\sin\theta}\left(\frac{\cos^2 m\theta}{4r^2}\frac{\partial}{\partial r}[\tilde F_3\sin(2\Omega)] + \frac{\sin^2 m\theta}{4r^2}\frac{\partial}{\partial r}[\tilde F_4\sin(2\Omega)]\right)$$
$$+\frac{1}{2\rho}\frac{\partial}{\partial z}\Big(\cos m\theta\sin^2 m\theta(\tilde F_3-\tilde F_4)\frac{\partial\Omega}{\partial\rho}\Big) - \frac{1}{2\rho}\frac{\partial}{\partial\rho}\Big(\cos m\theta\sin^2 m\theta(\tilde F_3-\tilde F_4)\frac{\partial\Omega}{\partial z}\Big)\ , \quad (62)$$

where we kept all derivative terms. With the proper boundary conditions for the functions $\tilde F_i$, and for the gauge function $\Omega(0) = 0$ and $\Omega(\infty) = \pi/2$, again only the first term of $Q(\rho,z)$ determines the Chern-Simons charge since the derivative terms do not contribute. We find the Chern-Simons charge

$$N_{\rm CS} = \frac{1-\cos m\pi}{4} = \begin{cases} \frac{1}{2} & \text{if } m \text{ odd} \\ 0 & \text{if } m \text{ even}\ . \end{cases} \quad (63)$$

V. CONCLUSIONS

We have presented the general ansatz, the energy density, and the Chern-Simons charge for static axially symmetric configurations in the bosonic sector of the electroweak theory. The ansatz contains the known axially symmetric solutions with parity reflection symmetry, the sphaleron, the multisphalerons, and the sphaleron-antisphaleron pair at finite mixing angle. It further allows for the construction of configurations without parity reflection symmetry, such as the sphaleron and multisphaleron barriers at finite mixing angle and the bisphalerons at finite mixing angle. The leading correction to the spherical bisphalerons was obtained in a perturbative calculation in θ_W [11]. The change of the sphaleron barrier due to the finite mixing angle as well as the barriers associated with the multisphalerons have not yet been obtained. The construction of the multisphaleron barriers will allow the investigation of the fermion level crossing phenomenon for vacuum to vacuum transitions via multisphalerons.

The numerical construction of these barriers or of the bisphalerons at finite mixing angle now appears to be straightforward, at least in the Coulomb gauges, but numerically involved, because a large system of up to 16 partial nonlinear differential equations must be solved simultaneously.

The multisphalerons are characterized by an integer winding number n, describing the winding the fields with respect to the angle ϕ. Their Chern-Simons charge is given by $N_{\rm CS} = n/2$. The sphaleron has winding number $n = 1$. Since the bisphalerons bifurcate from the sphaleron at large Higgs boson masses, we expect that corresponding n bisphalerons exist, bifurcating from the multisphalerons with winding number n. Using the formalism derived in this paper, these solutions can numerically be searched for. A stability analysis of the multisphaleron solutions may be helpful in determining the critical values of the Higgs boson mass.

Besides the winding in the angle ϕ, a winding in the angle θ with winding number m can be considered. The sphaleron-antisphaleron pair represents a solution with winding number $m = 2$. We have generalized the ansatz for the sphaleron-antisphaleron pair to allow for arbitrary integer winding number m. The Chern-Simons charge of solutions with odd m is $N_{\rm CS} = 1/2$, while the Chern-Simons charge of solutions with even m vanishes, $N_{\rm CS} = 0$. The sphaleron has winding number $m = 1$. We conjecture that solutions with winding number $m > 2$ exist. Further there may be solutions with both winding numbers excited, $n > 1$ and $m > 1$. The numerical construction of such solutions may turn out to be complicated, though only seven functions are involved.

Finally, all these solutions may bifurcate and general bisphalerons with $m > 1$ and with $n > 1$ and $m > 1$ may exist for large Higgs boson masses. The construction of such solutions provides a great numerical challenge.

[1] N. S. Manton, Phys. Rev. D **28**, 2019 (1983).

[2] F. R. Klinkhamer and N. S. Manton, Phys. Rev. D **30**, 2212 (1984).

[3] B. Kleihaus, J. Kunz, and Y. Brihaye, Phys. Lett. B **273**, 100 (1991).

[4] J. Kunz, B. Kleihaus, and Y. Brihaye, Phys. Rev. D **46**, 3587 (1992).

[5] B. Kleihaus and J. Kunz, Phys. Lett. B **329**, 61 (1994).

[6] F. R. Klinkhamer, Phys. Lett. B **246**, 131 (1990).

[7] F. R. Klinkhamer, Nucl. Phys. **B410**, 343 (1993).

[8] T. Akiba, H. Kikuchi, and T. Yanagida, Phys. Rev. D **38**, 1937 (1988).

[9] J. Kunz and Y. Brihaye, Phys. Lett. B **216**, 353 (1989).

[10] L. G. Yaffe, Phys. Rev. D **40**, 3463 (1989).

[11] Y. Brihaye and J. Kunz, Phys. Rev. D **47**, 4789 (1993).

[12] N. S. Manton, Nucl. Phys. **B135**, 319 (1978).

[13] C. Rebbi and P. Rossi, Phys. Rev. D **22**, 2010 (1980).

[14] Y. Brihaye, B. Kleihaus, and J. Kunz, Phys. Rev. D **47**, 1664 (1993).

[15] Because of additional discrete symmetries the ansatz for the sphaleron at finite mixing angle is restricted; see Eqs. (36)–(38). This leaves a single residual gauge degree of freedom U_3.

[16] Numerical analysis of the equations suggests, however, that the functions $\tilde{F}_3$, $\tilde{F}_4$, and $\tilde{F}_6$ are not regular in this parametrization [B. Kleihaus (private communication)].

[17] Another generalization, including also half-integer values of m, is given in K. Fujii, S. Otsuki, and F. Toyoda, Prog. Theor. Phys. **81**, 462 (1989).