Off-shell amplitudes in two-dimensional open string field theory

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In this article we present an explicit procedure for the regularization of tree level amplitudes involving discrete states, using open string field theory. We show that there is a natural correspondence between the discrete states and off-shell states, later acting as a regularized version of the former. A general off-shell state corresponds to several physical states. In order to obtain the well-defined S matrix elements one has to choose representatives close but not equal to the desired values of external momenta. The procedure renders finite all four-point amplitudes with an even number of (naively) divergent channels even after the regularization is removed. The rest of the amplitudes can be defined by means of such regularization.

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I. MOTIVATION AND INTRODUCTION

During the last couple of years $d \leq 1$ string models received considerable attention, Ref. [1]. The main reason for that is that they provide examples of consistent and exactly soluble string models. Recently, it has been pointed out that one can construct a nontrivial, consistent theory of strings with Dirichlet boundary conditions from a collection of two-dimensional (2D) topological gravity models, Ref. [2]. Dirichlet strings, on the other hand, have been argued to be a possible candidate for an effective QCD string theory, Ref. [3]. In a different approach toward QCD strings, Gross and Taylor showed the equivalence between 2D QCD and an effective 2D string theory without folds (discrete states), Ref. [4]. It is desirable to much better understand various 2D models and relations between them.

In this note we focus on open 2D strings. It should be remarked that there is no matrix model available in that case. The only results known until recently were due to Bershadsky and Kutasov who calculated the bulk amplitudes for tachyon-tachyon scattering in the continuum (Liouville) approach, Ref. [5]. Although simple (the only *field* in two-dimensional string theory is a massless "tachyon"), this is not a trivial theory. It possesses a large spectrum-generated W_∞ symmetry. The generators of the symmetry, discrete states (DS's), are defined only for some particular values of momenta. In the framework of string field theory, DS's appear as Becchi-Rouet-Stora-Tyutin (BRST) cohomology classes, as well as poles of the tree-level S matrix, Refs. [6,7]. In that respect, they are remnants of higher (excited) string states in 2D. However, when treated as external (asymptotic) states, presence of a sufficient number of DS's leads to tree level divergences.

A classification of four-point amplitudes involving DS's was presented in Ref. [7] (see Sec. II for more details). A four-point amplitude diverges if at least one of the kinematic invariants s, t, or u is nonpositive integer. If s is such an integer, for example, then the s channel amplitudes $\sum_{n\geq 0}\frac{A_n(t)}{s+n}$ is ill defined because the denominator blows up for the level $n=-s$. If $s>0$, on the other hand, the amplitude vanishes.

The aim of this note is to treat the divergences more carefully using field theory. We proceed in a simple way. First we generalize the concept of DS's and allow them to depart from the mass shell in order to render the tree amplitudes finite. Such a procedure gives a well-defined result. One can think of such off-shell "discrete states" as just a regularization scheme. Then we show that, upon such regularization, some divergences cancel each other. Such amplitudes are well-defined even after the regularization is removed. Our consideration may be generalized to N point amplitudes.

II. GREEN'S FUNCTIONS AND S MATRIX

The starting point in our discussion is Witten's open string field theory (OSFT) in 2D. As in any other gauge theory, to derive Green's functions in 2D OSFT one needs first to fix the gauge. We choose the so called Siegel's gauge, $b_0 A = 0$ (see Ref. [6]). Introducing the external sources J_s, one for each coefficient field, one determines the generating functional $Z(J_s)$ from which it is easy to deduce (off-shell) Green's functions:

$$Z(J_s) \propto \int \frac{1}{n!} G_n(p_1,\ldots,p_n) J_n \cdots J_1 \ , \qquad (2.1)$$

where G_n contain an overall factor $\delta^{(2)}(\sum_{i=1}^{n} p_i + Q)$, and $Q^\mu = (0, 2\sqrt{2})$ is the background charge of the matter-Liouville system. We need, however, to specify what is meant by off-shell states and how they relate to the physical states of the theory. Let us recall, first, that $H^{(1)}$ DS's can be obtained by applying the raising or lowering SU(2) charges $H_\pm = \frac{1}{2\pi i}\oint e^{\pm i\sqrt{2}x}$ to discrete tachyons $V^\pm_{s,s} = c e^{i\sqrt{2}sx} e^{(-\sqrt{2}\pm\sqrt{2}s)\varphi}$:

$$\begin{aligned} W^\pm_{s,n} &\propto (H_-)^{s-n} V^\pm_{s,s} \\ &\propto \epsilon_{i_1\cdots i_k}\alpha^{i_1}_{-j_1}\cdots\alpha^{i_k}_{-j_k}|\sqrt{2}n, -\sqrt{2}\pm\sqrt{2}s\rangle \ , \end{aligned} \qquad (2.2)$$

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where $\epsilon_{i_1 \cdots i_k} \propto \sum_{j=0}^{k} c_j \delta_{i_1,1} \cdots \delta i_{j,1}$ is the polarization tensor, and $\sum k j_k$ is the level of the state $W^{\pm}_{s,n}$. In this representation only the matter field contributes to excitations. The set of physical states can be partitioned with respect to the following equivalence relation: two states are considered equivalent if they have the same polarization structure and differ only by the values of momenta. For example, discrete and generic tachyons belong to one class, vector particles to another and so on. Denote the quotient space by P. Let us introduce, now, another set O which consists of states with the same polarization structure as physical states but with unphysical values of momenta. We refer to the elements of O as *off-shell* states. It is clear that there exists a natural bijective map $P \mapsto O$. We can visualize an element of O as an open, connected subset of $\mathcal{R}^2$, with discrete (physical) states corresponding to points cut out form the surface (Fig. 1). Note that, in general, more than one physical state maps to the same off-shell "surface."

Tree-level four-point Green's functions are given by the expression

$$G \propto \prod_{i=1}^{4} D_i \sum_{n} V_{2n1} D_n V_{4n3} \ . \tag{2.3}$$

Here, D_i represent external propagators ("legs"), the sum is over a complete set of intermediate states, and each of the states i belongs to O. To recover (on-shell) S matrix elements from the expression for Green's function (2.3) one needs to (a) cut the external legs and (b) for each external leg choose an on-shell representative. From the comment above it is clear that one off-shell Green's function corresponds to a variety of S matrix elements.

Let us consider, for example, tachyon Green's function. Contribution to the s channel reads

$$\langle \phi(p_1) \cdots \phi(p_4) \rangle \propto g^2 \prod_{i=1}^{4} \frac{1}{\mu_i^2} \left(\frac{4}{3\sqrt{3}} \right) \sum_{i=1}^{4} \mu_i^2 \frac{A_n(t,u)}{s+n} \ , \tag{2.4}$$

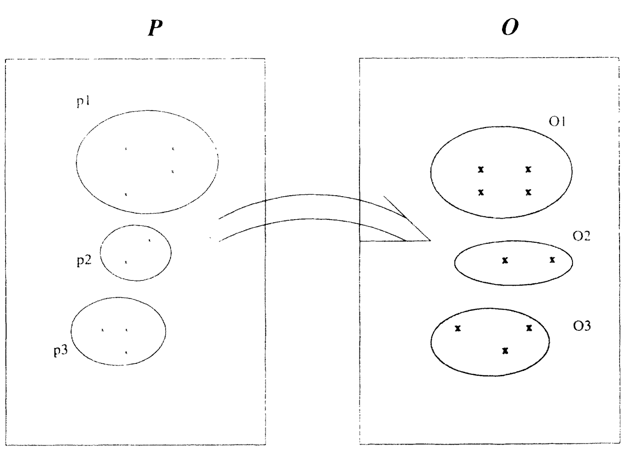

FIG. 1. Mapping from on-shell to off-shell states. Equivalence classes of discrete states (depicted by dots) are mapped to off-shell states (depicted by surfaces with punctures).

where $\mu_i^2 = \frac{1}{2}(p_i + \frac{Q}{2})^2$ is ith tachyon external leg and the first couple of residues are given by

$$\begin{aligned} A_0 &= 1 \ , \\ A_1 &= \frac{1}{2}(t-u) \ , \\ A_2 &= -\frac{1}{8} + \frac{1}{8}(t-u)^2 - \frac{5}{32} \sum_{i=1}^{4} \mu_i^2 \ . \end{aligned} \tag{2.5}$$

The total Green's function is: $G^{(\text{tot})} = G^{(s)} + G^{(t)} + G^{(\mu)}$. To obtain S matrix elements, after we calculate the residues in μ^2 we need to pick the values of tachyon (on-shell) momenta we are interested in. The same procedure can be repeated for the nontachyonic states. In this way one can reproduce the results of Ref. [7]. Let us briefly summarize them. If we denote by T a generic tachyon and by D an arbitrary DS including discrete tachyons, then (a) classes A_{TTTT} and A_{TDDD} are empty, (b) class A_{TTTD} is well defined and the amplitudes have an infinite number of intermediate physical states, and (c) class A_{TTDD} has at least one degenerate channel and it can be subdivided into three subclasses: A^{+}_{TTDD} is well defined and has a finite number of intermediate states, A^{-}_{TTDD} diverges and has an infinite number of intermediate states, while A^{deg}_{TTDD} is just a divergent number. Finally, A_{DDDD} always diverges. Let us stress, however, that being off shell, even not far from the mass shell, makes all amplitudes well defined. Moreover, as we shall see in the next section, if an amplitude is apparently divergent in two channels, these divergences can be made to cancel each other. Such amplitudes are well defined even after the regularization is removed.

III. OFF-SHELL AMPLITUDES AND REGULARIZATION

In the previous section we have defined off-shell states and discussed their relationship with physical states. We have seen that several physical states correspond to the same off-shell state. Green's functions (2.3) are well defined off-shell. In fact, none of the kinematic invariants s, t, or u degenerate because, apart from the overall momentum conservations, there is no other kinematic constraint. Cutting the external legs or, in other words, calculating the residues in μ_i^2, gives rise to the constraint $s + t + u = \sum_{i=1}^{4} \mu_i^2 \le 1$, which is still harmless. The only potentially harmful step in the procedure of S matrix calculation is the last step, namely, choosing the representatives of the external states. This is what we shall discuss next.

Let us consider a specific example, namely, the amplitude belonging to the class $A^{(\text{deg})}_{TTDD}$:

$$\langle W^{+}_{\frac{1}{2},\frac{1}{2}} W^{-}_{k_2} W^{-}_{k_3} W^{+}_{\frac{1}{2},\frac{1}{2}} \rangle = \int_0^1 \frac{dx}{x} \ . \tag{3.1}$$

Notice that (3.1) is nondynamical (momentum independent), although divergent. One would expect that it does not contribute to the *properly* defined scattering amplitude. Let us show that this is, indeed, the case. We start from the off-shell tachyon Green function (2.4). Matter and Liouville momenta are, at this point, arbitrary (subject, of course, to the conservation law). Let us, now, pick the values of momenta close to the physical values indicated in (3.1): namely,

$$
\begin{aligned}
p_1^\mu &= \left(\frac{1}{\sqrt{2}} + \epsilon_1^m, -\frac{1}{\sqrt{2}} + \epsilon_1^l\right) , \\
p_2^\mu &= (k_2 + \epsilon_2^m, -\sqrt{2} - k_2 + \epsilon_2^m) , \\
p_3^\mu &= (k_3 + \epsilon_3^m, -\sqrt{2} - k_3 + \epsilon_3^m) , \\
p_4^\mu &= \left(\frac{1}{\sqrt{2}} + \epsilon_4^m, -\frac{1}{\sqrt{2}} + \epsilon_4^l\right) .
\end{aligned}
\tag{3.2}
$$

From the momentum conservation, one has a constraint on the allowed values of ϵ_i's:

$$
\sum_{i=1}^{4} \epsilon_i^m = -\sum_{i=1}^{4} \epsilon_i^l , \tag{3.3}
$$

which is identically satisfied if we choose $\epsilon_i^m = \epsilon = -\epsilon_i^l$. Selecting a particular regularized S matrix element is equivalent to choosing sufficiently small, nonintersecting neighborhoods around those value of momenta (Fig. 2). Such neighborhoods always exist since each element of O is a Hausdorff topological space. The neighborhoods are off-shell representatives (regularizations) of the on-shell state. It follows that the choice made in (3.2) gives a particular contribution to the S matrix only if we restrict ourselves to the terms small in ϵ. The limit $\epsilon \to 0$, if exists, gives then the value of the particular on-shell amplitude. For the larger values of ϵ, on the other hand, neighborhoods start overlapping with each other. That situation corresponds to a *generic* off-shell amplitude. Below, we discard all quantities of the order $O(\epsilon)$.

Kinematic invariants, then, read

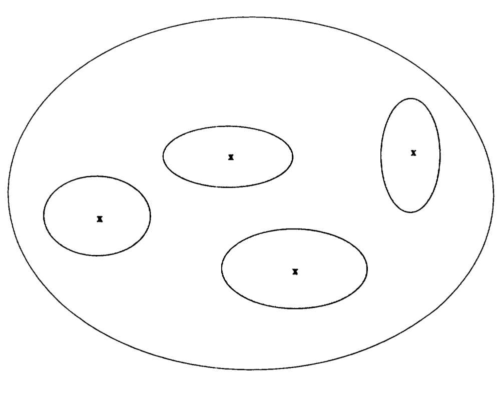

FIG. 2. A blown-up representation of an element of O. Shaded regions represent small neighborhoods around discrete states.

$$
\begin{aligned}
s &= \frac{1}{2}\left(p_1 + p_2 + \frac{Q}{2}\right)^2 \simeq 4\epsilon\left(\frac{1}{\sqrt{2}} + k_2\right) , \\
t &= \frac{1}{2}\left(p_1 + p_3 + \frac{Q}{2}\right)^2 \simeq -4\epsilon\left(\frac{1}{\sqrt{2}} + k_2\right) \simeq -s , \\
u &= \frac{1}{2}\left(p_1 + p_4 + \frac{Q}{2}\right)^2 \simeq 1 + 2\sqrt{2}\epsilon .
\end{aligned}
\tag{3.4}
$$

We see that in the limit $\epsilon \to 0, s, t \to 0$, while $u \to 1$. This means that, had we altogether neglected ϵ we would have two divergent channels, divergences arising from the $n = 0$ (tachyonic) intermediate state. Let us keep, for now, *finite* ϵ and consider the potentially divergent piece of the *total* amplitude more carefully. We have

$$
A^{(\mathrm{tot})} \simeq \frac{(\frac{27}{16})^s - (\frac{27}{16})^{-s}}{s} . \tag{3.5}
$$

We see that the residue in s vanishes, so that the amplitude has, in fact no singularities. Now we can take the limit $\epsilon \to 0$. Evidently, $A^{(\mathrm{deg})} = 0$. This result can be generalized to any four-point amplitude which (naively) diverges in *two* different channels. The crucial point here is that two divergent contributions (from two different channels) can always be made to cancel each other. In the case of an odd number of divergent channels this simple argument does not work. This is the case, for example, for $A^{(-)}_{TTDD}$. There we only have one divergent channel, so that the residue of the divergent piece is not zero and the cancellation does not occur. Similarly, some of the amplitudes A_{DDDD} have three divergent channels. (For example, for $\langle W^+_{\frac{3}{2},\frac{1}{2}}, W^+_{\frac{5}{2},\frac{1}{2}} W^-_{1,0} W^-_{1,0}\rangle$ one has $s = -9$ and $t = u = 0$.) It is easy to see that divergences of two of the channels may cancel each other, but there will be always one left. It is not a disaster, however. In that case we could keep ϵ_i finite, and have well-behaved expressions which we can *define* to be the answer we are looking for.

So, to summarize, departing from the mass shell gives a natural way to regularize the amplitudes involving discrete states since the states are not, basically, discrete any more. In the case of an even number of divergent channels divergences cancel each other, and the amplitude is well defined even after the regularization is removed. For the amplitudes with an odd number of divergent channels one can *redefine* them using the procedure above.

It is easy to see that the similar situation arises in general N-point case. As an illustration, consider five-point Green's function. It is given by

$$
G \propto g^3 \prod_{i=1}^{5} D_i V_{2K1} D_K V_{K3L} V_{5L4} , \tag{3.6}
$$

plus nonequivalent permutations. Upon chopping the external legs we are left with two intermediate propagators. Each of them, potentially, could give rise to a divergence. It is going to be the case always when a sufficient number of DS's is involved. Departing from the mass shell makes amplitudes well defined, and the procedure similar to the one developed for the four point functions may be applied.

IV. FINAL REMARKS

In this note we suggested a simple way to deal with the divergences of the tree-level amplitudes involving discrete states. Namely, we suggested to use the off-shell formulation of the 2D OSFT. We have seen there is a natural correspondence between the discrete states and off-shell states, later acting as a regularized version of the former. To obtain well-defined S matrix elements one, in general, needs to choose off-shell states close but not equal to the desired values of external momenta (somewhat analogous to the way in which in Minkowski space one replaces $m^2 \mapsto m^2 - i\epsilon$). We have shown that the procedure renders finite all four-point amplitudes with an even number of (naively) divergent channels even after the regularization is removed. The rest of the amplitudes can be defined by means of such regularization.

Discrete states are a consequence of a gauge theory in two dimensions. If the loop corrections are taken into account DS's might be "broadened." In other words, instead of a monochromatic wave one would have a wave packet. If 2D string theory has anything to do with some two dimensional condensed matter system, which is not totally unreasonable assumption (Ref. [8]), this kind of broadening of the "resonance" spectrum would be expected. It is interesting to speculate that these "dressed" states might be related to the off-shell states.

Recently, 2D closed string field theory has been formulated, Ref. [9]. Although the formulation of the theory is much more complicated than that in the open string case, there seem to be no principal difficulties in treating the divergences arising there in exactly the same way.

Finally, note that we have considered here discrete states as asymptotic string states. This is not the only possibility. As an alternative, one may consider scattering of tachyons in various discrete state backgrounds (similar project was carried out for the black hole background in Ref. [10]). Another compelling approach is to consider a field theory of a tachyon coupled to the complete set of two dimensional topological gravity states. There are indications that such an approach might shed some light on relation between string field theory and W_∞ (author would like to thank Z. Qui for discussions on this matter). These and other interesting problems shall be discussed elsewhere.

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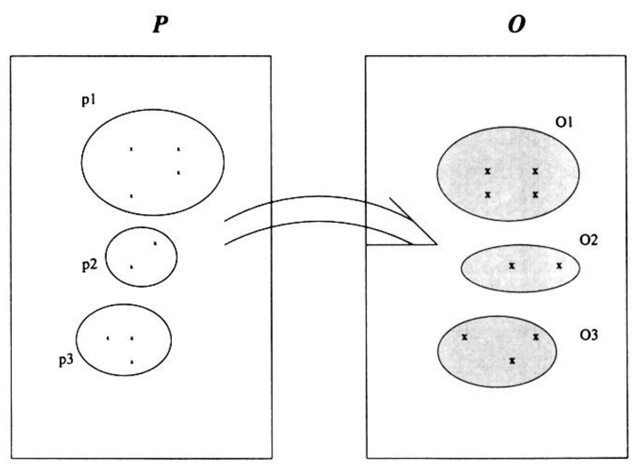

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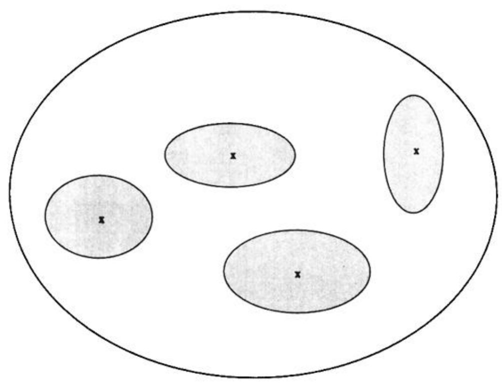

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