Geometrodynamics of Schwarzschild black holes

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The curvature coordinates T, R of a Schwarzschild spacetime are turned into canonical coordinates $T(r), \mathsf{R}(r)$ on the phase space of spherically symmetric black holes. The entire dynamical content of the Hamiltonian theory is reduced to the constraints requiring that the momenta $P_T(r), P_{\mathsf{R}}(r)$ vanish. What remains is a conjugate pair of canonical variables m and p whose values are the same on every embedding. The coordinate m is the Schwarzschild mass and the momentum p the difference of parametrization times at right and left infinities. The Dirac constraint quantization in the new representation leads to the state functional $\Psi(m; T, \mathsf{R}] = \Psi(m)$ which describes an unchanging superposition of black holes with different masses. The new canonical variables may be employed in the study of collapsing matter systems.

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I. INTRODUCTION

A. Context

General relativity was cast into canonical form by Dirac [1], and by Arnowitt, Deser, and Misner (ADM) [2]. Inside asymptotic regions, Hamiltonian dynamics is entirely generated by constraints. Their imposition as operator restrictions on the states yields the Wheeler-DeWitt equation [3,4]. DeWitt realized that by freezing all but few degrees of freedom of a cosmological model by symmetry, one can obtain exactly soluble models of quantum gravity [4]. Misner and his school turned this idea of *minisuperspace quantization* [5] into a systematic exploration of quantum cosmology [6,7].

Minisuperspace techniques were extended to *midisuperspace quantization* of infinitely dimensional models. The first system treated in this manner was the cylindrical gravitational wave [8]. It became clear that the Wheeler-DeWitt equation is often unwieldy and difficult to interpret. For cylindrical waves, the switch to an extrinsic time representation changed the Wheeler-DeWitt equation into a functional time Schrödinger equation. Gravity assumed the form of a parametrized field theory [1,9].

Things did not work that way for other infinitely dimensional systems. The most important of these is the gravitational collapse of a spherically symmetric distribution of matter. Berger, Chitre, Moncrief, and Nutku (BCMN) set this problem in the Dirac-ADM midisuperspace formalism. In their classic paper [10] they studied a spherically symmetric massless scalar field coupled to gravity. They did not succeed in finding an extrinsic time representation. Instead, they reduced the action to a privileged foliation characterized by the vanishing "radial" momentum. Their reduced Hamiltonian did not quite reproduce the field equations. This was found and corrected by Unruh [11].

The BCMN model opens a canonical route to the study of Hawking's radiation [12,13]. A standard semiclassical analysis of the Hawking effect starts with a black hole being formed by the gravitational collapse of classical matter. One studies a field that propagates on this background. The modes that disappear below the horizon are averaged out, and the thermal radiation escaping to infinity is described by a density operator. The Hawking radiation leads to the evaporation of the black hole. No general agreement has been reached on what is the final state of this process. The black hole may evaporate completely, or leave a remnant. If it evaporates completely, the question remains what happens to the information which got initially trapped below its horizon.

Midisuperspace canonical approach has two potential advantages over the standard description. First, it goes beyond the semiclassical approximation. Second, unless one encounters a Cauchy horizon, all information is registered in the canonical data on a Cauchy hypersurface. One can study what happens inside black holes and how they approach the singularity. However, to make use of this advantage, one must insist that the foliation covers all available spacetime, and that time evolution is not artificially arrested. The BCMN slicing does not meet this condition. A careful study of the BCMN model in its relation to Hawking's radiation was undertaken by Hájíček [14,16]. He generalized the model to other spherically symmetric fields, and paid special attention to the properties of the apparent horizon and choice of slicing.

One can easily see that the BCMN slicing is problematic already for primordial Schwarzschild black holes. In vacuo, the BCMN slices coincide with those of constant Killing time T. They cover only the static regions of the Kruskal diagram, and never penetrate the horizon. Canonical treatment of a complete Schwarzschild spacetime was attempted by Lund [17]. To get below the horizon, Lund used the Lamaitre slices, or the slices of constant $R < 2M$. He did not succeed in covering the whole Kruskal diagram by a single foliation, or relate the state of the Schwarzschild black hole on T = const slices to its state on the Lamaitre slices or the $2M > R$ = const slices. The best solution would have been to work in the functional time representation, but Lund presented a proof that the extrinsic time representation does not exist for vacuum Schwarzschild black holes.

Interest in primordial black holes has been revived by a surge of activity on quantization of dilatonic black holes (see recent reviews by Gidddings [18], and by Harvey and Strominger [19]). Starting from this program, Gegenberg, Kunstatter, and Louis-Martinez [20,21] discussed canonical quantization of Schwarzschild black holes within conformally invariant formulation of Einstein's theory [22]. Along different lines, Thiemann and Kastrup [23–26] discussed canonical quantization of Schwarzschild and Reisner-Nordström black holes mostly, though not entirely, within Ashtekar's canonical formalism.

Keeping this background in mind, let us state the goals and results of this paper.

B. Results

We cast the classical and quantum dynamics of primordial black holes into geometrically transparent and explicitly soluble form by choosing a natural canonical chart on the phase space. Our method amounts to finding a functional extrinsic time representation analogous to one which exists for cylindrical gravitational waves. Lund's no-go theorem is transcended because the representation does not satisfy its unnecessarily strong premises.

Geometrodynamics of Schwarzschild black holes is governed by the Dirac-ADM action restricted to spacetimes with spherical symmetry. The spatial metric $g_{ab}(x^c)$ on a symmetric hypersurface is entirely characterized by two functions $\Lambda(r)$ and $R(r)$ of a radial coordinate r. In canonical formalism, these are accompanied by the conjugate momenta $P_\Lambda(r)$ and $P_R(r)$. We choose phase space variables which have an immediate geometric meaning. The hypersurface action yields the familiar Hamiltonian and momentum constraints equivalent to those derived by BCMN [10].

Proper understanding of the canonical formalism requires a careful handling of boundary conditions. One must specify how fast the canonical data and the lapse-shift multipliers fall at infinities. The requirement that Λ can be freely varied within its falloff class mandates the addition of the ADM boundary term to the hypersurface action. Unfortunately, the introduction of this term prevents one from freely varying the lapse function at infinities. This defect is amended by parametrization, which makes the action dependent on two more variables: namely, on proper times measured by static clocks at infinities. All the variables in the parametrized action can be freely varied, and their interconnection at infinities acquires the status of natural boundary conditions. This version of the canonical action principle is vital for our treatment of black hole dynamics.

The crux of our approach is the introduction of the Killing time $T(r)$ as a canonical coordinate. The way in which $T(r)$ enters the formalism is rather delicate. To begin with, one restricts attention to the space of solutions. In a given Schwarzschild spacetime, one can specify a hypersurface by giving the familiar curvature coordinates of its points as functions of a radial label r. The Schwarzschild geometry induces on the hypersurface the canonical data which satisfy the constraints. From those data, one can locally determine the Schwarzschild mass M of the embedding spacetime, and the rate $T'(r)$ at which the Killing time changes along the hypersurface. Though $T(r)$ becomes infinite on the horizon, its change across the horizon can be consistently inferred from smooth canonical data.

Next, one forgets how the expressions for $M(r)$ and $-T'(r)$ were obtained, and turns them into definitions of two new sets of dynamical variables on the phase space. At this stage, the canonical data no longer need to satisfy the constraints, and the mass function $M(r)$ can in principle depend on r. The remarkable feature of the new variables is that they form a canonical pair: $P_M(r) = -T'(r)$. Moreover, by retaining the curvature coordinate, $R(r) \mapsto \mathsf{R}(r) = R(r)$, but modifying its conjugate momentum, $P_R(r) \mapsto P_{\mathsf{R}}(r)$, one can complete $M(r)$ and $-T'(r)$ into a new canonical chart

$$M(r),\, P_M(r) = -T'(r)\,, \quad \mathsf{R}(r) = R(r),\, P_{\mathsf{R}}(r)\,. \tag{1}$$

We exhibit the generating functional of the canonical transformation from the old to the new canonical variables. Because $T'(r)$ has an infinite jump on the horizon, the canonical transformation has there a singularity.

We can now reimpose the constraints, and evolve the variables by Hamilton equations. Under these circumstances, the mass function becomes a position-independent constant of motion. Inversely, one can show that the Hamiltonian and momentum constraints can be replaced by a much simpler set of conditions,

$$M'(r) = 0\,, \quad P_{\mathsf{R}}(r) = 0, \tag{2}$$

on the new canonical variables.

The canonical structure we have derived holds only when the lapse function in the ADM boundary action is considered as a fixed function of the label time. After parametrization, the action becomes dependent on a pair of proper times $\tau_\pm$ at infinities, and it no longer has a canonical form. It is surprising that without adding any more variables to the parametrized space $\tau_+, \tau_-, M(r), P_M(r), \mathsf{R}(r), P_{\mathsf{R}}(r)$, one can introduce on it a canonical chart

$$m, p;\; T(r), P_T(r);\; \mathsf{R}(r), P_{\mathsf{R}}(r)\,. \tag{3}$$

The final canonical variables (3) have a simple physical meaning. The canonical pair $\mathsf{R}(r), P_{\mathsf{R}}(r)$ remains unchanged. The new canonical coordinate $T(r)$ is the Killing time, and its conjugate momentum $P_T(r)$ is the mass density $-M'(r)$. The curvature coordinates T, R in spacetime are thereby turned into canonical coordinates $T(r), \mathsf{R}(r)$ on the phase space. Simultaneously, the constraints are transformed into the statement that the momenta $P_T(r)$ and $P_{\mathsf{R}}(r)$ canonically conjugate to the embedding variables $T(r)$ and $\mathsf{R}(r)$ vanish:

$$P_T(r) = 0\,, \quad P_{\mathsf{R}}(r) = 0\,. \tag{4}$$

The Hamiltonian is a linear combination of these constraints. The true ADM Hamiltonian disappeared in the transformation process to the final canonical chart (3). This phenomenon can be understood as a result of time-dependent canonical transformation.

Because the Hamiltonian weakly vanishes, the remaining canonical variables m and p are constants of motion. The canonical coordinate m is the Schwarzschild mass of the black hole. The meaning of p is more esoteric: p characterizes the difference between the parametrization times at the left and right infinities. The comparison of these two times is made possible by connecting the infinities by hypersurfaces of constant Killing time. Once set, this difference is preserved because the two parametrization clocks run at the same rate. This explains why p is a constant of motion. While it has always been suspected that the variable conjugate to the Schwarzschild mass is some sort of proper time, the correct interpretation of this quantity eluded previous investigators, likely because they did not pay enough attention to the role of parametrization, and to how the two asymptotic regions are connected with each other.

The parametrized action can be reduced to true dynamical degrees of freedom, and the dynamics generated by the unparametrized action can be compared to that we have just described. It transpires that our parametrized viewpoint corresponds to performing a time-dependent canonical transformation to "initial data."

With the new polarization (3) of the phase space, the Dirac constraint quantization of primordial black holes becomes straightforward. The state functional $\Psi(m; T, \mathsf{R}]$ of the system is reduced by the constraints to an embedding-independent function $\Psi(m)$ of the mass parameter m. Such a state function describes a superposition of primordial black holes of different masses. Once prepared, it stays the same on every hypersurface $T(r), \mathsf{R}(r)$.

The curvature coordinates are ill behaved on the horizon. However, once they are constructed from the canonical data, one can easily transform them into Kruskal coordinates by a further canonical transformation.

Primordial black holes, despite all the care needed for their proper canonical treatment, are dynamically trivial. The true interest of the new canonical variables lies in the possibilities which they open in the study of gravitational collapse of matter. These questions are now being pursued in collaboration with Hájíček and Romano.

II. SCHWARZSCHILD SOLUTION

A. Spacetime description

The recurrent theme of this paper is that canonical formalism should be guided by spacetime intuition. We thus start by summarizing what is known about spherically symmetric solutions of vacuum Einstein equations.

Any such solution is locally isometric to the Schwarzschild line element

$$ds^2 = -F(R)\,dT^2 + F^{-1}(R)\,dR^2 + R^2\,d\Omega^2 \tag{5}$$

written in the *curvature coordinates* (T, R). Here,

$$d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2 \tag{6}$$

is the line element on the unit sphere. We are using natural units in which the Newton constant of gravitation G and the speed of light c are put equal to one: $G = 1 = c$. The coefficient $F(R)$ has the form

$$F(R) = 1 - 2M/R\,, \tag{7}$$

where M is a constant. The curvature coordinate R is invariantly defined by the requirement that $4\pi R^2$ be the area of the two-spheres S^2: $T =$ const, $R =$ const which are the transitivity surfaces of the rotation group. The vector field $\partial/\partial T$ is a Killing vector field of the metric (5). It is orthogonal to the hypersurfaces $T =$ const of the *Killing time* T.

For $M = 0$ the spacetime is flat. Solutions with $M > 0$ describe black holes, solutions with $M < 0$ correspond to naked singularities. We limit ourselves to solutions with $M \geq 0$.

As $R \to \infty$, the line element (5) becomes asymptotically flat. At $R = 2M$, the solution (5) runs into a coordinate singularity. The maximal analytic extension of (5) from the region $R > 2M > 0$ across $R = 2M$ describes a primordial black hole. The complete spacetime $\mathcal{M}$ is represented by the familiar Kruskal diagram [27,28]. It is covered by four patches of curvature coordinates which meet at the horizon $F(R) = 0$. Two regions have $R > 2M$, and the Killing field $\partial/\partial T$ in them is timelike. We call them the right and left static regions, I and III. Two other regions have $R < 2M$, and the Killing field $\partial/\partial T$ in them is spacelike. We call them the past and future dynamical regions, IV and II. The past dynamical region begins and the future dynamical region ends in a true curvature singularity at $R = 0$.

The total Schwarzschild spacetime can be covered by a single patch of *Kruskal coordinates* U and V. The lines $U =$ const are radial rightgoing null geodesics $\nearrow$, the lines $V =$ const are radial leftgoing null geodesics $\nwarrow$. Both U and V grow from past to future. The horizon is transformed to the lines $U = 0$ and $V = 0$. Their intersection is the *bifurcation point.* The four regions covered by curvature coordinates are as follows: right static region I: $R > 2M$: $U < 0, V > 0$; left static region III: $R > 2M$: $U > 0, V < 0$; future dynamical region II: $R < 2M$: $U > 0, V > 0$; past dynamical region IV: $R < 2M$: $U < 0, V < 0$.

The Kruskal coordinates U and V are mapped into curvature coordinates T and R by a two-to-one transformation. Anticipating the steps we shall need later in the canonical formalism, we build this transformation in several steps. The Kruskal coordinates are dimensionless. We thus first *scale* the curvature coordinates T and R into dimensionless coordinates $\bar{T}$ and $\bar{R}$:

$$\bar{T} = \frac{T}{2M}\,, \quad \bar{R} = \frac{R}{2M}\,. \tag{8}$$

Curvature coordinates are related to Kruskal coordinates by

$$UV = \mathcal{W}(\bar{R}) := \left(1 - \bar{R}\right) \exp\left(\bar{R}\right),$$
$$\frac{V}{U} = \mathrm{sgn}\left(1 - \bar{R}\right) \exp\left(\bar{T}\right). \tag{9}$$

The T = const hypersurfaces appear in the Kruskal diagram as straight lines passing through the bifurcation point. The rightgoing branch ↗ of the horizon is labeled by $T = \infty$, the leftgoing branch ↖ by $T = -\infty$. The R = const lines are hyperbolas asymptotic to the horizon.

As $\bar{R}$ increases from $\bar{R} = 0$ (singularity) to $\bar{R} = \infty$, the function $\mathcal{W}(\bar{R})$ monotonically decreases from 1 to $-\infty$. Therefore, it has an inverse $\mathcal{R}(UV)$. This enables us to solve (9) for the curvature coordinates:

$$\bar{T} = \ln|V| - \ln|U|, \quad \bar{R} = \mathcal{R}(UV). \tag{10}$$

We see that the inversion $U \mapsto -U$, $V \mapsto -V$ leaves $\bar{T}$ and $\bar{R}$ unchanged. Two points in the Kruskal plane are labeled by the same curvature coordinates.

It is useful to describe the mapping (10) in a slightly weaker form. Introduce the *tortoise coordinate*

$$\bar{R}^* = \bar{R} + \ln|1 - \bar{R}| \tag{11}$$

and combine $\bar{T}$ and $\bar{R}^*$ into null coordinates

$$\bar{U} = \bar{T} - \bar{R}^*, \quad \bar{V} = \bar{T} + \bar{R}^*. \tag{12}$$

By multiplying and dividing the two equations (9) we obtain

$$U^2 = \exp\left(-\bar{U}\right), \quad V^2 = \exp\left(\bar{V}\right). \tag{13}$$

The Schwarzschild line element expressed in the Kruskal coordinates is everywhere regular, except at the initial and final curvature singularities at $UV = 1$.

B. Geometrodynamical description

Geometrodynamics views a given spacetime as a dynamical evolution of a three-geometry. Let us apply this viewpoint to the Schwarzschild solution (5). Take an arbitrary spherically symmetric spacelike hypersurface and let the spacetime metric induce on it the spatial geometry g. Different hypersurfaces carry different induced metrics. As we evolve the spacetime metric along a foliation which covers the entire Kruskal diagram, we build the Schwarzschild solution.

The simplest evolution is obtained along a one-parameter family of hypersurfaces T = const cutting across the regions I and III of the Kruskal diagram. The geometry induced on all of these hypersurfaces is the same: it is a wormhole geometry known under the name of the Einstein-Rosen bridge. As we change T from $-\infty$ to ∞, the geometry does not change at all. It would be more appropriate to speak in this case about geometrostatics rather than geometrodynamics. This reflects the fact that $\partial/\partial T$ is a Killing vector field, and the T = const hypersurfaces are Lie-propagated by it. This evolution has a serious defect [29]: The spacelike hypersurfaces T = const do not cover the entire Kruskal diagram, but only the static regions I and III. The progress of time is arrested at the bifurcation point through which all the hypersurfaces pass. The hypersurfaces thus do not form a foliation. Moreover, as the hypersurfaces proceed from past to future in region I, they recede from future to past in region III. The evolution does not proceed everywhere from past to future.

The dynamical regions II and IV which were not covered by the spacelike hypersurfaces T = const can be covered by another simple family of spacelike hypersurfaces, namely, R = const$< 2M$. Their geometry is again a wormhole geometry, but this time it describes a homogeneous cylinder $S^2 \times \mathbb{R}$. As R progresses from 0 to $2M$ in region IV, the cylinder opens up from the line singularity, its circumference grows larger and larger while its length per unit T shrinks, and finally degenerates into a disk of circumference $4\pi M$ as R approaches the horizon $R = 2M$. In region II, as R decreases from $2M$ to 0, the whole process is reversed. Spatial geometry is dynamical in regions IV and II. The described evolution is locally isomorphic to the dynamics of the Kantowski-Sachs universe [30]. Unfortunately, the foliations $0 < R =$ const $< 2M$ are asymptotically null at infinities, they approach the horizon for $R \to 2M$, and again, they do not cover the whole Kruskal diagram. What they miss are exactly the static regions.

These shortcomings set our task. We want to study the spatial geometry on a hypersurface which is spacelike and cuts the Kruskal diagram all the way through: it starts at left infinity, goes through the static region III, crosses the horizon into a dynamical region, traverses it until it again reaches the horizon, crosses the horizon to the static region I, and continues to right infinity. At infinities, such a hypersurface should be asymptotically spacelike, approaching some static hypersurface $T = T_-$ = const at left infinity, and in general some other static hypersurface, $T = T_+$ = const, at right infinity. One can cover the whole Kruskal diagram by a foliation of such hypersurfaces. Even better, one can admit all of them at once, and work in *many-fingered time formalism.*

III. CANONICAL FORMALISM FOR SPHERICALLY SYMMETRIC SPACETIMES

The geometrodynamical approach does not start from the known Schwarzschild solution: it generates the Schwarzschild solution by evolving a spherically symmetric geometry. The evolution is governed by the Dirac-ADM action. In this section, we introduce the Dirac-ADM action, and carefully discuss the necessary boundary conditions.

A. Hypersurface Lagrangian L_Σ

Take a spherically symmetric three-dimensional Riemannian space (Σ, g) and adapt the coordinates x^a of its points $x \in \Sigma$ to the symmetry: $x^a = (r, \theta, \phi)$. The line element $d\sigma$ on Σ is completely characterized by two functions $\Lambda(r)$ and $R(r)$ of the radial label r:

$$d\sigma^2 = \Lambda^2(r)\,dr^2 + R^2(r)\,d\Omega^2 . \qquad (14)$$

Again, $d\Omega$ is the line element on the unit sphere.

Primordial black holes have the topology $\Sigma = \mathbb{R} \times S^2$, and $r \in \mathbb{R}$ thus ranges from $-\infty$ to ∞. The coefficients $\Lambda(r)$ and $R(r)$ cannot vanish because the line element must be regular. We take both of them to be positive, $\Lambda(r) > 0$ and $R(r) > 0$. Then, $R(r)$ is the curvature radius of the two-sphere $r =$ const, and $d\sigma = \Lambda(r)dr$ is the radial line element oriented from the left infinity to right infinity.

Under transformations of r, $R(r)$ behaves as a scalar, and $\Lambda(r)$ as a scalar density. This will simplify the momentum constraint. Keeping this goal in mind, we have avoided the usual exponential form of the metric coefficients. It is important to keep track of the density character of the fundamental canonical variables. We shall always denote those canonical coordinates which are scalars by Latin letters, and those which are scalar densities by Greek letters.

The line element (14) leads to the curvature scalar

$$\begin{aligned}\mathbf{R}[g] = &-4\Lambda^{-2}R^{-1}R'' + 4\Lambda^{-3}R^{-1}\Lambda' R' \\ &-2\Lambda^{-2}R^{-2}R'^2 + 2R^{-2} . \end{aligned}\qquad (15)$$

Let us foliate a spherically symmetric spacetime $\mathcal{M}$ by spherically symmetric leaves Σ, and label the leaves by a time parameter $t \in \mathbb{R}$. The metric coefficients Λ and R then depend not only on r, but also on t. The leaves of the foliation are related by the familiar lapse function N and the shift vector N^a. Because of the spherical symmetry, only the radial component N^r of the shift vector survives, and both $N(t,r)$ and $N^r(t,r)$ depend solely on the t, r variables.

The extrinsic curvature K_{ab} of the leaves is given by the rate of change $\dot{g}_{ab}$ of the metric with the label time t:

$$K_{ab} = \frac{1}{2N}\big(- \dot{g}_{ab} + N_{(a|b)}\big) . \qquad (16)$$

For the spherically symmetric line element (14), K_{ab} is diagonal, with

$$K_{rr} = -N^{-1}\Lambda\left(\dot{\Lambda} - (\Lambda N^r)'\right) , \qquad (17)$$

$$K_{\theta\theta} = -N^{-1}R\big(\dot{R} - R'N^r\big) , \quad K_{\phi\phi} = \sin^2\theta\, K_{\theta\theta} . \qquad (18)$$

The vacuum dynamics of the metric field follows from the ADM action

$$S_\Sigma[g, N, N^a] = \int dt \int_\Sigma d^3x\, L_\Sigma \qquad (19)$$

whose Lagrangian L_Σ is

$$L_\Sigma = (16\pi)^{-1}\, N|g|^{1/2}(K^{ab}K_{ab} - K^2 + \mathbf{R}[g]) . \qquad (20)$$

(In natural units $G = 1 = c$, the Einstein constant $\kappa = 8\pi G/c^4$ reduces to 8π.) In a spherically symmetric spacetime (14), (17), (18),

$$N|g|^{1/2}\,(K^{ab}K_{ab} - K^2) = -2N^{-1}\sin\theta\left(2\big(- \dot{\Lambda} + (\Lambda N^r)'\big)\big(- \dot{R} + R'N^r\big)R + \big(- \dot{R} + R'N^r\big)^2\Lambda\right) . \qquad (21)$$

Integration over θ and ϕ gives the ADM action of a Schwarzschild black hole:

$$\begin{aligned}S_\Sigma[R, \Lambda;\, N, N^r] = \int dt \int_{-\infty}^{\infty} dr \Bigg[&- N^{-1}\left(R\big(- \dot{\Lambda} + (\Lambda N^r)'\big)\big(- \dot{R} + R'N^r\big) + \frac{1}{2}\Lambda\big(- \dot{R} + R'N^r\big)^2\right) \\ &+ N\left(-\Lambda^{-1}RR'' + \Lambda^{-2}RR'\Lambda' - \frac{1}{2}\Lambda^{-1}R'^2 + \frac{1}{2}\Lambda\right)\Bigg] . \end{aligned}\qquad (22)$$

We shall discuss the appropriate boundary terms after passing to the Hamiltonian formalism.

B. Canonical form of the action

By differentiating the ADM action with respect to the velocities $\dot{\Lambda}$ and $\dot{R}$ we obtain the momenta

$$P_\Lambda = -N^{-1}\,R(\dot{R} - R'N^r) , \qquad (23)$$

$$P_R = -N^{-1}\left(\Lambda\big(\dot{R} - R'N^r\big) + R\big(\dot{\Lambda} - (\Lambda N^r)'\big)\right) . \qquad (24)$$

Throughout this paper, we denote those canonical coordinates which are spatial scalars by Latin letters, and those which are spatial densities by Greek letters. The conjugate momenta always carry complementary weights. Therefore, the momentum P_R conjugate to the scalar R is a density, while the momentum P_Λ conjugate to the density Λ is a scalar.

Equations (23) and (24) can be inverted for the velocities:

$$\dot{\Lambda} = -NR^{-2}\big(RP_R - \Lambda P_\Lambda\big) + \big(\Lambda N^r\big)' , \qquad (25)$$

$$\dot{R} = -N\,R^{-1}P_\Lambda + R'N^r . \qquad (26)$$

They allow us to write the extrinsic curvature as a function of the canonical momenta:

$$K_{rr} = \Lambda R^{-2}\big(RP_R - \Lambda P_\Lambda\big) , \quad K_{\theta\theta} = P_\Lambda . \qquad (27)$$

By symmetry, the curvature of a normal section of Σ attains its extremal values K_1, K_2, K_3 (called *principal curvatures*) for those sections which are either tangential $\|$ or normal $\perp$ to the two-spheres $r =$ const. The last equation enables us to express these principal curvatures $K_1 = K_\|$ and $K_2 = K_3 = K_\perp$ in terms of the momenta:

$$K_\parallel = K^r_r = \Lambda^{-1}R^{-2}\left(RP_R - \Lambda P_\Lambda\right),$$
$$K_\perp = K^\theta_\theta = K^\phi_\phi = R^{-2}P_\Lambda. \qquad (28)$$

Inversely,

$$P_\Lambda = R^2 K_\perp\,,\quad P_R = R\Lambda\left(K_\parallel + K_\perp\right). \qquad (29)$$

This endows the canonical momenta with an invariant geometric meaning. Note that P_Λ is proportional to $K_\perp$, but P_R is proportional to the *sum* of $K_\parallel$ and $K_\perp$, not to $K_\parallel$ itself. This sum also differs from the *mean curvature*

$$K := K_1 + K_2 + K_3 = K^a_a = K_\parallel + 2K_\perp. \qquad (30)$$

The action (22) can be cast into the canonical form by the Legendre dual transformation

$$S_\Sigma[\Lambda, R, P_\Lambda, P_R;\, N, N^r]$$
$$= \int dt \int_{-\infty}^{\infty} dr\,\left(P_\Lambda \dot\Lambda + P_R \dot R - NH - N^r H_r\right). \quad (31)$$

By this process, we obtain the super-Hamiltonian

$$H := -R^{-1}P_R P_\Lambda + \frac{1}{2}R^{-2}\Lambda P_\Lambda^2 + \Lambda^{-1}RR'' - \Lambda^{-2}RR'\Lambda' + \frac{1}{2}\Lambda^{-1}R'^2 - \frac{1}{2}\Lambda \qquad (32)$$

and supermomentum

$$H_r := P_R R' - \Lambda P_\Lambda'. \qquad (33)$$

The form of H_r is dictated by the requirement that it generate Diff$\mathbb{R}$ of the scalars R and P_Λ, and of the scalar densities P_R and Λ. The minus sign in (33) is due to the fact that the momentum P_Λ is a scalar, and the coordinate Λ a scalar density, rather than the other way around.

The expressions (32) and (33) can be obtained from those derived by BCMN [10] by a point transformation.

C. Falloff of the canonical variables

So far, we have paid no attention to the behavior of the canonical variables Λ, R and P_Λ, P_R at infinity. The importance of the falloff conditions was pointed out by Regge and Teitelboim [31,32], and their form refined by many authors. We shall follow the treatment of Beig and O'Murchadha [33].

Primordial black holes have two spatial infinities rather than just one. We shall formulate the falloff conditions at right infinity, and then state what the corresponding conditions are at left infinity.

Let x^a be a global system of coordinates on Σ which is asymptotically Cartesian. Such a system is related to the spherical system of coordinates r, θ, ϕ by the standard flat space formulae. At $r \to \infty$, the metric g_{ab} and the conjugate momentum p^{ab} are required to fall off as

$$g_{ab}(x^c) = \delta_{ab} + r^{-1}h_{ab}(n^c) + O^\infty(r^{-(1+\epsilon)}), \qquad (34)$$
$$p^{ab}(x^c) = r^{-2}k^{ab}(n^c) + O^\infty(r^{-(2+\epsilon)}). \qquad (35)$$

Here, $n^a := x^a/r$, and $f(x^a) = O^\infty(r^{-n})$ means that f falls off like r^{-n}, $f_{,a}$ like $r^{-(n+1)}$, and so on for all higher spatial derivatives. The leading term in $g_{ab} - \delta_{ab}$ is of the order r^{-1}, and the leading term in p^{ab} is one order higher, r^{-2}. The coefficients h_{ab} and k^{ab} are required to be smooth functions on S^2, and to have opposite parities: $h_{ab}(n^c)$ is to be even, and $k^{ab}(n^c)$ is to be odd, i.e.,

$$h_{ab}(-n^c) = h_{ab}(n^c)\,,\quad k^{ab}(-n^c) = -k^{ab}(n^c). \qquad (36)$$

Together with the canonical data, the lapse function and the shift vector are assumed to behave as

$$N(x^c) = N_+(n^c) + O^\infty(r^{-\epsilon}), \qquad (37)$$
$$N^a(x^c) = N^a_+(n^c) + O^\infty(r^{-\epsilon}) \qquad (38)$$

at infinity.

Because $dr = n_a dx^a$ and $d\Omega^2 = r^{-2}(\delta_{ab} - n_a n_b)\, dx^a dx^b$, the spherically symmetric metric (14) can easily be transformed into Cartesian coordinates:

$$g_{ab} = \Lambda^2 n_a n_b + \left(\frac{R}{r}\right)^2 \left(\delta_{ab} - n_a n_b\right), \qquad (39)$$

$$g^{ab} = \Lambda^{-2}n^a n^b + \left(\frac{r}{R}\right)^2 \left(\delta^{ab} - n^a n^b\right). \qquad (40)$$

The general falloff conditions (34) then determine the behavior of the metric coefficients R and Λ at infinity:

$$\Lambda(t,r) = 1 + M_+(t)r^{-1} + O^\infty(r^{-(1+\epsilon)}) \qquad (41)$$

and

$$R(t,r) = r + \rho_+(t) + O^\infty(r^{-\epsilon}). \qquad (42)$$

Here, because of spherical symmetry, $M_+(t)$ and $\rho_+(t)$ cannot depend on the angles n^c, but they can still depend on t. Of course, M_+ is the Schwarzschild mass as observed at right infinity. The general falloff conditions allow R to differ by the amount $\rho_+(t)$ from r at infinity . Because we want r to coincide with R at infinity, we impose stronger falloff conditions on R by putting $\rho_+ = 0$:

$$R(r) = r + O^\infty(r^{-\epsilon}). \qquad (43)$$

Consistency then demands that the shift vector must also asymptotically vanish, $N^r_+ = 0$:

$$N^r(r) = O^\infty(r^{-\epsilon}). \qquad (44)$$

Unlike N^r, the lapse function cannot vanish at infinity. If it did, the time there would stand still. Instead, $N(t,r)$ assumes at infinity an angle-independent value $N_+(t)$:

$$N(r) = N_+(t) + O^\infty(r^{-\epsilon}). \qquad (45)$$

Our next task is to determine the falloff of the canonical momenta. The second fundamental form $K_{ab}dx^a dx^b$ can be transformed into Cartesian coordinates by the same procedure as the metric. By using (27) we get

$$K_{rr}dr^2 + K_{\theta\theta}d\theta^2 + K_{\phi\phi}d\phi^2 = \Lambda R^{-2}\left(RP_R - \Lambda P_\Lambda\right) dr^2 + P_\Lambda\, d\Omega^2 = \left(\Lambda R^{-2}(RP_R - \Lambda P_\Lambda)n_a n_b + r^{-2}P_\Lambda\left(\delta_{ab} - n_a n_b\right)\right)\, dx^a dx^b\,. \tag{46}$$

From here we read the extrinsic curvature and, by referring back to the metric (40), find the canonical momentum

$$p^{ab} := |g|^{1/2}\left(Kg^{ab} - K^{ab}\right) = 2r^{-2}\Lambda^{-1}P_\Lambda n^a n^b + R^{-1}P_R\left(\delta^{ab} - n^a n^b\right)\,. \tag{47}$$

This expression for p^{ab} is even in the angular variables. This means that k^{ab} is even. The requirement that k^{ab} be odd implies that k^{ab} must vanish, i.e.

$$P_\Lambda = O^\infty(r^{-\epsilon})\,,\quad P_R = O^\infty(r^{-(1+\epsilon)})\,. \tag{48}$$

The same considerations apply at $r \to -\infty$. Taken together, they imply the falloff conditions

$$\Lambda(t,r) = 1 + M_\pm(t)|r|^{-1} + O^\infty(|r|^{-(1+\epsilon)})\,, \tag{49}$$

$$R(t,r) = |r| + O^\infty(|r|^{-\epsilon})\,, \tag{50}$$

$$P_\Lambda(t,r) = O^\infty(|r|^{-\epsilon})\,, \tag{51}$$

$$P_R(t,r) = O^\infty(|r|^{-(1+\epsilon)}) \tag{52}$$

for the canonical variables, and the falloff conditions

$$N(t,r) = N_\pm(t) + O^\infty(|r|^{-\epsilon})\,, \tag{53}$$

$$N^r(t,r) = O^\infty(|r|^{-\epsilon}) \tag{54}$$

for the Lagrange multipliers.

Let us note again that $R(r)$ approaches $|r|$ at the rate $O^\infty(|r|^{-\epsilon})$ as $r \to \pm\infty$. After we reconstruct the Killing time $T(r)$ from the canonical data, we shall prove a similar result for the approach of the $t=$ const foliation to the $T=$ const foliation:

$$T(t,r) = T_\pm(t) + O^\infty(|r|^{-\epsilon})\,. \tag{55}$$

The falloffs of the canonical data ensure that the Liouville form

$$\int_{-\infty}^{\infty} dr\,\left(P_\Lambda\dot\Lambda + P_R\dot R\right) \tag{56}$$

is well defined. They also imply that the super-Hamiltonian and supermomentum fall off as

$$H = O^\infty(|r|^{-(1+\epsilon)})\,,\quad H_r = O^\infty(|r|^{-(1+\epsilon)})\,. \tag{57}$$

Equations (53), (54), and (57) ensure that the Hamiltonian is well defined. The canonical action S_Σ thus has a good meaning.

The falloff conditions for spherically symmetric *vacuum* spacetimes may easily be strengthened. The only necessary condition on how fast the canonical variables fall down is that the Schwarzschild solution itself can be made to fall down that fast by an appropriate choice of the hypersurface and its parametrization r at infinities. By choosing the hypersurface to coincide with a $T=t$ hypersurface outside a spherical tube $R=R_0=$const, and by labeling it there by the curvature coordinate, $R=|r|$, we achieve that

$$\Lambda(t,r) = 1 + M_\pm(t)|r|^{-1}\,,\quad R(t,r) = |r|\,,$$
$$P_\Lambda(t,r) = 0\,,\quad P_R(t,r) = 0 \tag{58}$$

for the Schwarzschild solution. We are thus free to require that our phase space variables satisfy (59) everywhere outside a compact region.

D. Boundary terms

Arnowitt, Deser, and Misner complemented the hypersurface action S_Σ by a boundary action $S_{\partial\Sigma}$ at infinity:

$$S = S_\Sigma + S_{\partial\Sigma}\,. \tag{59}$$

A primordial black hole has two infinities, and hence there are two boundary contributions

$$S_{\partial\Sigma} = -\int dt\,\left(N_+(t)E_+(t) + N_-(t)E_-(t)\right)\,. \tag{60}$$

Each of them is the product of the lapse function with the ADM energy

$$E_\pm = (16\pi)^{-1}\int_{S^2_\pm} dS^a\,\delta^{bc}\left(g_{ab,c} - g_{bc,a}\right) \tag{61}$$

which is given by an integral over a two-sphere $S^2_\pm$ at infinity. In the asymptotically Cartesian coordinates

$$dS^a = r^2 n^a \sin\theta\, d\theta d\phi\,,\quad |r|_{,a} = n_a\,,$$
$$n_{a,b} = |r|^{-1}(\delta_{ab} - n_a n_b)\,, \tag{62}$$

and the asymptotic form (34) of the metric yields

$$E_\pm = \lim_{r\to\pm\infty}\frac{1}{2}|r|\left(\Lambda^2 - 1\right) = M_\pm(t)\,. \tag{63}$$

The ADM energy of a black hole is its Schwarzschild mass.

The total action of a primordial black hole takes the form

$$S[\Lambda, P_\Lambda, R, P_R;\, N, N^r] = \int dt \int_{-\infty}^{\infty} dr\,\left(P_\Lambda\dot\Lambda + P_R\dot R - NH - N^r H_r\right) - \int dt\,(N_+M_+ + N_-M_-)\,. \tag{64}$$

Variations of the canonical variables should preserve the prescribed falloffs. In particular, the leading term in the variation of Λ is given by the variation of the Schwarzschild mass:

$$\delta\Lambda_{\pm} = \delta M_{\pm}\,|r|^{-1} + O^{\infty}(|r|^{-2})\,. \tag{65}$$

As emphasized by Regge and Teitelboim [31], without the ADM boundary action this particular variation of the action would lead to an inconsistency. On the other hand, when the ADM boundary action is included, another inconsistency would arise if one allowed the variation of the lapse function at infinities. It is therefore important to treat $N_{\pm}(t)$ as *prescribed functions* of t.

To see why this is so, let us identify those terms in the Hamiltonian density $NH + N^r H_r$ whose variations lead to boundary terms. The variation of P_R does not yield any boundary term because there are no derivatives of P_R anywhere in the action. The variables P_Λ and R also do not cause any trouble, because the falloff conditions ensure that the boundary terms brought in by the variation of P_Λ and R safely vanish. The sole troublemaker is the derivative term $NRR'\Lambda^{-2}\Lambda'$ in $-NH$. Its variation with respect to Λ yields the boundary term

$$N_+(t)\delta M_+(t) + N_-(t)\delta M_-(t)\,. \tag{66}$$

If there were no ADM boundary action, the variation of the hypersurface action S_Σ with respect to Λ would lead to the conclusion that $N_{\pm}(t) = 0$; i.e., it would freeze the evolution at infinities. The ADM boundary action is designed so that its variation with respect to $M_{\pm}(t)$ exactly cancels the boundary term (66), and the unwanted conclusion does not follow.

The ADM boundary action leads, however, to an inconvenient caveat. Without it, the lapse and the shift functions in the hypersurface action S_Σ can be freely varied. When the boundary action is included, the variation of the total action $S = S_\Sigma + S_{\partial\Sigma}$ yields

$$\delta_N S = -\int dt\,\big(M_+(t)\delta N_+(t) + M_-(t)\delta N_-\big)\,. \tag{67}$$

If we allowed $N(t,r)$ to be varied at infinities, we would get an unwanted "natural boundary condition" $M_+ = 0 = M_-$. This would exclude black hole solutions and leave only a flat spacetime. Therefore, in the variational principle (64) we must demand that the values $N_{\pm}(t)$ of the lapse function at infinities be some *prescribed* functions of t which cannot be varied. This means that the lapse function N in our variational principle has *fixed ends*.

E. Parametrization at infinities

The necessity of fixing the lapse function at infinities can be removed by *parametrization.* The lapse function is the rate of change of the proper time τ with respect to the label time t in the direction normal to the foliation. Because $N^r_{\pm}(t) = 0$, we can write

$$N_{\pm}(t) = \pm\dot{\tau}_{\pm}(t)\,, \tag{68}$$

where $\tau_{\pm}(t)$ is the proper time measured on standard clocks moving along the $r =$ const worldlines at infinities. By convention, we let the proper time at the left infinity decrease from the past to the future, to match the behavior of the Killing time T in the Kruskal diagram. This introduces the minus sign in (68) at $-\infty$.

We now replace the lapse function in the ADM boundary action by the derivatives of $\tau_{\pm}$, (68), and treat $\tau_{\pm}(t)$ as additional variables:

$$S[\Lambda, P_\Lambda, R, P_R\,;\, N, N^r;\, \tau_+, \tau_-] = \int dt \int_{-\infty}^{\infty} dr\,\big(P_\Lambda\dot{\Lambda} + P_R\dot{R} - NH - N^r H_r\big) - \int dt\,\big(M_+\dot{\tau}_+ - M_-\dot{\tau}_-\big)\,. \tag{69}$$

This rearrangement of the action is called *the parametrization at infinities.* Notice that $N_{\pm}$ still appears in the hypersurface part S_Σ of the action.

The variables $\tau_{\pm}(t)$ in the action (69) can be freely varied. The result of their variation is a valid equation, namely, the mass conservation

$$\dot{M}_+(t) = 0 = \dot{M}_-(t)\,. \tag{70}$$

The lapse function at infinities $N_{\pm}(t)$ can also be freely varied, because it now occurs only in the hypersurface action, and the super-Hamiltonian asymptotically vanishes.

The only remaining question is what happens under the variation of Λ. As before, the variation of the hypersurface action S_Σ gives the boundary term (66). On the other hand, the variation of the parametrized ADM boundary action in (69) now yields

$$-\dot{\tau}_+(t)\delta M_+(t) + \dot{\tau}_-(t)\delta M_-(t)\,. \tag{71}$$

Before parametrization, the variation of Λ at infinities, i.e., the variation of $M_{\pm}(t)$, produced merely an identity. After parametrization, the situation is different: the variation of $M_{\pm}(t)$ relates $\tau_{\pm}(t)$ to $N_{\pm}(t)$ by (68). These equations thereby follow as *natural boundary conditions* from the parametrized action principle (69).

As we proceed, we shall at first pay little attention to the ADM boundary action. We shall meet it again in Sec. VII.

IV. RECONSTRUCTING THE MASS AND TIME FROM THE CANONICAL DATA

A. Reconstruction program

In canonical gravity, we know the intrinsic metric and extrinsic curvature of a hypersurface, but we do not have any *a priori* knowledge about how the hypersurface is located in spacetime. Suppose that we are given the canonical data $\Lambda, R, P_\Lambda, P_R$ on a spherically symmetric hypersurface cutting across a Schwarzschild black hole. Can we tell from those data the mass of the black hole? And can we infer from them how the hypersurface is drawn in the Kruskal diagram?

Disregarding some subtleties concerning the anchoring of the hypersurface at infinities, the answer to both of these questions is *yes*. To arrive at the answers, we start from the knowledge that the hypersurface must ultimately be embedded in a spacetime endowed with the line element (5). Let the hypersurface be a leaf of a foliation:

$$T = T(t,r)\,, \quad R = R(t,r)\,. \tag{72}$$

The line element (5) induces an intrinsic metric and extrinsic curvature on the hypersurface. By comparing these quantities with the canonical data, we connect the spacetime formalism with the canonical formalism. This connection enables us to identify the Schwarzschild mass M and the embedding (72) from the canonical data.

We can expect discontinuities at the horizon where different patches of the curvature coordinates T and R meet each other. Indeed, T becomes infinite at the horizon. We shall see, however, that the transition of T across the horizon is well under our control, and can be predicted from smooth canonical data. We can then easily pass from the curvature coordinates T, R to the Kruskal coordinates U, V which are continuous across the horizon. The direct reconstruction of the Kruskal coordinates would be much more cumbersome.

To implement our program, we substitute the foliation (72) into the Schwarzschild line element (5) and get

$$\begin{aligned} ds^2 = &-\left(F\dot{T}^2 - F^{-1}\dot{R}^2\right) dt^2 \\ &+2\left(- FT'\dot{T} + F^{-1}R'\dot{R}\right) dtdr \\ &+\left(- FT'^2 + F^{-1}R'^2\right) dr^2 + R^2 d\Omega^2. \end{aligned} \tag{73}$$

By comparing this result with the ADM form of the line element

$$\begin{aligned} ds^2 = &-\left(N^2 - \Lambda^2(N^r)^2\right) dt^2 + 2\Lambda^2 N^r dtdr \\ &+\Lambda^2 dr^2 + R^2 d\Omega^2\,, \end{aligned} \tag{74}$$

we obtain a set of three equations:

$$\Lambda^2 = -FT'^2 + F^{-1}R'^2\,, \tag{75}$$

$$\Lambda^2 N^r = -FT'\dot{T} + F^{-1}R'\dot{R}\,, \tag{76}$$

$$N^2 - \Lambda^2(N^r)^2 = F\dot{T}^2 - F^{-1}\dot{R}^2. \tag{77}$$

The first two equations can be solved for N^r:

$$N^r = \frac{-FT'\dot{T} + F^{-1}R'\dot{R}}{-FT'^2 + F^{-1}R'^2}\,. \tag{78}$$

This solution N^r, together with Λ of Eq. (75), can be substituted into the remaining equation (77), which can then be solved for the lapse:

$$N = \frac{R'\dot{T} - T'\dot{R}}{\sqrt{-FT'^2 + F^{-1}R'^2}}\,. \tag{79}$$

The lapse function N should be positive. The transition from (77) to (79) requires taking a square root. We must check that the square root we have taken is positive.

First of all, the denominator of (79) is real and positive: it is equal to Λ. We shall check that the numerator of (79) is positive separately in each region of the Kruskal diagram. Because the lapse function is a spatial scalar, we first choose in each region an appropriate radial label r : $r = R$ in region I, $r = T$ in region II, $r = -R$ in region III, and $r = -T$ in region IV. Under these choices, the numerator of (79) becomes $\dot{T}$ in region I, $-\dot{R}$ in region II, $-\dot{T}$ in region III, and $\dot{R}$ in region IV. With the label time going to the future, all these expressions are positive.

We now substitute the expressions (78) and (79) for N^r and N into (23) and calculate P_Λ. The time derivatives $\dot{T}$ and $\dot{R}$ dutifully drop out, and we get the relation

$$-T' = R^{-1}F^{-1}\Lambda P_\Lambda\,. \tag{80}$$

When we substitute this T' back into (75), we can calculate F as a function of the canonical data:

$$F = \left(\frac{R'}{\Lambda}\right)^2 - \left(\frac{P_\Lambda}{R}\right)^2. \tag{81}$$

Taken together, the last two equations express T' in terms of the canonical data. Moreover, because we know F in terms of M and R, (7), we can also determine the Schwarzschild mass

$$M = \frac{1}{2}R^{-1}P_\Lambda^2 - \frac{1}{2}\Lambda^{-2}RR'^2 + \frac{1}{2}R\,. \tag{82}$$

Equations (80) – (82) accomplish our goal. Equation (82) enables us to read the mass of the Schwarzschild black hole from the canonical data on any small piece of a spacelike hypersurface. It does not matter whether that piece is close to or far away from infinities, or even whether it lies inside or outside the horizon. Equations (80) and (81) determine the *difference* of the Killing times $T(r_1)$ and $T(r_2)$ between any two points, r_1 and r_2, of the hypersurface. To determine $T(r)$ itself, we need to know T at one point of the hypersurface, say, at the right infinity. Equations (80) – (82) are the key to our treatment of the Schwarzschild black holes. We shall now explore their consequences.

B. Across the horizon

Our assertion that Eqs. (80) and (81) determine the difference of the Killing times between any two points of the hypersurface requires a caveat. On the horizon, the coefficient $F(r)$ vanishes, and the time gradient (80) becomes infinite. We must show that we can propagate our knowledge of time across the horizon.

A spacelike hypersurface must intersect both the leftgoing and the rightgoing branches of the horizon. Unless it passes straight through the bifurcation point, it has two intersections with the horizon. Inside the horizon, the hypersurface lies either entirely within the future dynamical region, or entirely within the past dynamical region.

How does one recognize from the smooth canonical data where the hypersurfaces crosses the horizon? One looks at the points where the combination (81) vanishes. The function (81) can be written as a product of two factors:

$$F = F_+ \times F_- \,, \quad F_\pm := \frac{R'}{\Lambda} \pm \frac{P_\Lambda}{R} \,. \tag{83}$$

At the horizon, at least one of the two factors must vanish.

To find what branches of the horizon are described by the factors $F_\pm$, we must determine the signs of several quantities at the intersection of the hypersurface with the horizon. This is easily done by inspecting the Kruskal diagram. We chose our radial label to grow from left infinity to right infinity. Therefore, inside the future dynamical region $T(r)$ increases with r, $T'(r) > 0$, and inside the past dynamical region it decreases with r, $T'(r) < 0$. As the hypersurface is entering a dynamical region from the left static region, $R(r)$ is falling, $R'(r) < 0$, and as it is exiting the dynamical region into the right static region, $R(r)$ is growing, $R'(r) > 0$. If the hypersurface passes through the bifurcation point, both $R'(r) = 0$ and $T'(r) = 0$.

In the dynamical regions, $F < 0$, and Eq. (80) tells us that $T'(r)$ and $P_\Lambda(r)$ have the same sign. Therefore, $P_\Lambda(r)$ is positive in the future dynamical region, and negative in the past dynamical region. From continuity, $P_\Lambda(r) > 0$ at intersections with the future horizon $\vee$, negative at intersections with the past horizon $\wedge$, and zero when the hypersurface crosses the bifurcation point. We already know that $R'(r) < 0$ when the hypersurface dives from the left static region through the $>$ part of the horizon into a dynamical region, and $R'(r) > 0$ when it reemerges through the $<$ part of the horizon into the right static region. Putting these facts together, we see that the equation $F_+(r) = 0$ defines the leftgoing branch of the horizon, and $F_-(r) = 0$ defines the rightgoing branch. When both $F_+(r)$ and $F_-(r)$ simultaneously vanish, the hypersurface goes through the bifurcation point.

We can now return to the problem of determining the passage of time across the horizon. To be definite, let the hypersurface cross the horizon from the future dynamical region into the right static region at a point $r_0 : R'_0 > 0, P_{\Lambda 0} > 0$. Because $R'_0 > 0$, we can choose R as a radial coordinate on the hypersurface in the vicinity of the crossing. The crossing condition $F_-(r_0) = 0$ then implies $(\Lambda P_\Lambda)_0 = R_0$, and Eq. (80) reduces to

$$\frac{dT(R)}{dR} = -\frac{1}{2} R_0 \Lambda_0^2 \, \frac{1}{R - R_0} \tag{84}$$

for R close to R_0. From here we determine the transition of $T(R)$ across the horizon:

$$T(R) = -\frac{1}{2} R_0 \Lambda_0^2 \, \ln |R - R_0| + \text{const} \,. \tag{85}$$

As expected, T becomes infinite at the horizon, but in a well-determined rate. The value of the "constant" is determined by matching $T(R)$ at one side of the horizon to its given value.

If the hypersurface passes through the bifurcation point, $T(r)$ changes continuously and remains finite. We conclude that we can determine the passage of $T(r)$ through the horizon from the canonical data.

V. MASS FUNCTION AND TIME GRADIENT AS CANONICAL VARIABLES

Equations (80) – (82) were obtained from the known form of the Schwarzschild solution, i.e., by implicitly using the Einstein equations. Let us now forget their humble origin, and promote the expressions for $M(r)$ and $-T'(r)$ to *definitions* of two sets of dynamical variables on our phase space. Note that $M(r)$ is a local functional of the canonical data, and as such it depends on r. Indeed, prior to imposing the constraints and the Hamilton equations on the data, $M(r)$ does not need to be constant.

We can interpret the function $M(r)$ as the mass content of the wormhole to the left of the two-sphere labeled by r. The other function $-T'(r)$ tells us the rate at which the Killing time $T(r)$ falls with r. A remarkable feature of these two functions is that they form a pair of canonically conjugate variables. Anticipating the outcome of the proof we are going to present, we denote the dynamical variable $-T'(r)$ as $P_M(r)$. Our density notation still applies: by its construction (82), $M(r)$ is a spatial scalar, while

$$P_M = R^{-1} F^{-1} \Lambda P_\Lambda \tag{86}$$

is a scalar density.

Because the expressions for $M(r)$ and $P_M(r)$ do not contain P_R, they have vanishing Poisson brackets with $R(r)$. Unfortunately, their Poisson brackets with $P_R(r)$ do not vanish. We thus cannot complement the variables $M(r)$, $P_M(r)$ by the canonical pair $R(r)$, $P_R(r)$, and get thereby a new canonical chart on the phase space.

Obviously, we need to modify the momentum $P_R(r)$ in such a way that the new momentum, $P_{\mathsf{R}}(r)$, will commute with $M(r)$ and $P_M(r)$, but still remains conjugate to

$$\mathsf{R} = R \,. \tag{87}$$

The only way of doing this is to add to $P_R(r)$ a dynamical variable $\Theta(r)$ that does not depend on $P_R(r)$:

$$P_{\mathsf{R}}(r) = P_R(r) + \Theta(r;\, R, \Lambda, P_\Lambda] \,. \tag{88}$$

To guess the correct Θ is tricky. Our guiding principle is that the variables $M(r), \mathsf{R};\, P_M(r), P_{\mathsf{R}}(r)$ should form a canonical chart whose canonical coordinates are spatial scalars, and momenta are scalar densities. This determines the form of the supermomentum by the requirement that $H_r(r)$ generate Diff$\mathbb{R}$. The same requirement had already fixed the form of the supermomentum in the original canonical variables. These considerations show that

$$P_R R' - \Lambda P'_\Lambda = P_M M' + P_{\mathsf{R}} \mathsf{R}' \,. \tag{89}$$

We already know how the new canonical variables M, P_M, and R depend on the original canonical variables. Therefore, by substituting expressions (81) and (82) and (86) and (87) into (89) we are able to determine Θ. This gives us the missing transformation equation for P_{R}:

$$P_{\mathsf{R}} = P_R - \frac{1}{2}R^{-1}\Lambda P_\Lambda - \frac{1}{2}R^{-1}F^{-1}\Lambda P_\Lambda - R^{-1}\Lambda^{-2}F^{-1}\big((\Lambda P_\Lambda)'(RR') - (\Lambda P_\Lambda)(RR')'\big) . \tag{90}$$

By inspecting the form of the constraints, we see that P_{R} can be expressed as their linear combination:

$$P_{\mathsf{R}} = F^{-1}\big(R^{-1}P_\Lambda H + R'\Lambda^{-2}H_r\big) . \tag{91}$$

Our task is now clear: We must prove that the transition

$$\Lambda(r), P_\Lambda(r);\ R(r), P_R(r) \ \mapsto\ M(r), P_M(r);\ \mathsf{R}(r), P_{\mathsf{R}}(r) \tag{92}$$

given by (81) and (82), (86) and (87), and (90) is a canonical transformation. We prove this by showing that the difference of the Liouville forms is an exact form:

$$\int_{-\infty}^{\infty} dr\, \big(P_\Lambda(r)\delta\Lambda(r) + P_R(r)\delta R(r)\big) - \int_{-\infty}^{\infty} dr\, \big(P_M(r)\delta M(r) + P_{\mathsf{R}}(r)\delta \mathsf{R}(r)\big) = \delta\omega[\Lambda, P_\Lambda, R, P_R] . \tag{93}$$

The difference of the integrands is evaluated by straightforward rearrangements:

$$P_\Lambda\delta\Lambda + P_R\delta R - P_M\delta M - P_{\mathsf{R}}\delta\mathsf{R} = \delta\left(\Lambda P_\Lambda + \frac{1}{2}RR'\ln\left|\frac{RR' - \Lambda P_\Lambda}{RR' + \Lambda P_\Lambda}\right|\right) + \left(\frac{1}{2}R\delta R\ln\left|\frac{RR' + \Lambda P_\Lambda}{RR' - \Lambda P_\Lambda}\right|\right) . \tag{94}$$

To prove (93), we integrate (94) in r and argue that the boundary terms

$$\frac{1}{2}R\delta R\ln\left|\frac{RR' + \Lambda P_\Lambda}{RR' - \Lambda P_\Lambda}\right| \tag{95}$$

vanish.

At $r \to \pm\infty$, the falloff conditions imply that $\Lambda \to 1$, $R \to |r|$, $R' \to \pm 1$, $P_\Lambda = O(|r|^{-\epsilon})$, and $\delta R = O(|r|^{-\epsilon})$. The boundary term is of the order

$$\frac{1}{2}\left[R\delta R\ln\left|1 + \frac{2\Lambda P_\Lambda}{RR'}\right|\right]_{-\infty}^{\infty} \approx \left[\delta R\,|R'^{-1}\Lambda P_\Lambda|\,\right]_{-\infty}^{\infty} = O(|r|^{-\epsilon}) \tag{96}$$

and hence vanishes at infinities.

There are also boundary terms at the horizon, where $R\delta R$ is finite, but the logarithm becomes infinite. However, because of the absolute value within the logarithm, the infinite boundary term inside the horizon matches the infinite boundary term outside the horizon, and they can be considered as canceling each other when the integral of the derivative term is interpreted through its principal value. This proves (93) and identifies ω:

$$\omega[R, \Lambda, P_\Lambda] = \int_{-\infty}^{\infty} dr\, \left(\Lambda P_\Lambda + \frac{1}{2}RR'\ln\left|\frac{RR' - \Lambda P_\Lambda}{RR' + \Lambda P_\Lambda}\right|\right) . \tag{97}$$

The functional (97) is well defined. The falloff of the canonical variables at infinities implies

$$\frac{1}{2}RR'\ln\left|\frac{RR' - \Lambda P_\Lambda}{RR' + \Lambda P_\Lambda}\right| \approx -\Lambda P_\Lambda + O(|r|^{-(1+\epsilon)}) . \tag{98}$$

The integrand of (97) thus falls faster than $|r|^{-1}$, which avoids the logarithmic singularity. Close to the horizon $r = r_0$, the integrand of (97) behaves as $\ln|r - r_0|$, and hence the integral from a given r to r_0 stays finite.

Equations (93) and (97) lead to the generating functional of the canonical transformation (92). The generating functional $\Omega[P_\Lambda, P_R;\ M, \mathsf{R}]$ emerges when we introduce the *old momenta* and *new coordinates* as a new coordinate chart on the phase space, rewrite (93) in the form

$$-\int_{-\infty}^{\infty} dr\big(\Lambda(r)\delta P_\Lambda(r) + R(r)\delta R(r)\big) - \int_{-\infty}^{\infty} dr\, \big(P_M(r)\delta M(r) + P_{\mathsf{R}}(r)\delta\mathsf{R}(r)\big) = \delta\Omega[P_\Lambda, P_R;\ M, \mathsf{R}] , \tag{99}$$

in the new chart. This is done by calculating $\Lambda > 0$ from the mass equation (82),

$$\Lambda = \frac{|RR'|}{\sqrt{R(R-2M) + {P_\Lambda}^2}} , \tag{101}$$

and substituting it back into (100). The result can be written in the form

and express

$$\Omega := -\int_{-\infty}^{\infty} dr\, \big(\Lambda(r)P_\Lambda(r) + R(r)P_R(r)\big) + \omega[\Lambda, P_\Lambda, R, P_R] = \int_{-\infty}^{\infty} dr\left(-RP_R + \frac{1}{2}RR'\ln\left|\frac{RR' - \Lambda P_\Lambda}{RR' + \Lambda P_\Lambda}\right|\right) \tag{100}$$

$$\Omega[P_\Lambda, P_R;\, M, \mathsf{R}] = -\int_{-\infty}^{\infty} dr\, \mathsf{R}(r) P_R(r) \;+ \int_{-\infty}^{\infty} dr\, \frac{1}{2}\mathsf{R}\mathsf{R}' \ln\left|\frac{\sqrt{\mathsf{R}(\mathsf{R}-2M)+P_\Lambda{}^2}-P_\Lambda}{\sqrt{\mathsf{R}(\mathsf{R}-2M)+P_\Lambda{}^2}+P_\Lambda}\right| . \tag{102}$$

By comparing the coefficients of the independent variations $\delta P_\Lambda, \delta P_R, \delta M, \delta\mathsf{R}$ in (99), we generate the canonical transformation (92) by Ω:

$$\begin{aligned} \Lambda(r) &= -\,\delta\Omega/\delta P_\Lambda(r)\,, \quad P_M(r) = -\,\delta\Omega/\delta M(r)\,, \\ R(r) &= -\,\delta\Omega/\delta P_R(r)\,, \quad P_{\mathsf{R}}(r) = -\,\delta\Omega/\delta \mathsf{R}(r)\,. \end{aligned} \tag{103}$$

When resolved with respect to the new canonical variables, these equations give our old transformation equations (82), (86), (87), and (90).

One should perhaps note that both integrands of (102) fall at infinities at a slow rate, as $O(|r|^{-\epsilon})$, and hence the generating functional Ω is singular. This happens because we added to the well-defined functional ω an ill-defined term

$$-\int_{-\infty}^{\infty} dr\, \left(P_\Lambda \Lambda + P_R R\right) . \tag{104}$$

The simplest way out of this difficulty is to let the momenta fall off faster than $O(|r|^{-\epsilon})$. We have seen this can always be done for the Schwarzschild black hole, (59).

From the transformation formulae (81) and (82), (86) and (87), (90), and the *weak* falloff conditions (49)–(52) we easily deduce the falloff of the new canonical variables:

$$M(t,r) = M_\pm(t) + O^\infty(|r|^{-\epsilon})\,, \tag{105}$$

$$\mathsf{R}(t,r) = |r| + O^\infty(|r|^{-\epsilon})\,, \tag{106}$$

$$P_M(t,r) = O^\infty(|r|^{-(1+\epsilon)})\,, \tag{107}$$

$$P_{\mathsf{R}}(t,r) = O^\infty(|r|^{-(1+\epsilon)})\,. \tag{108}$$

Recalling that $P_M(r) = -T'(r)$, we can integrate (107) with respect to r and prove the earlier statement (55) about the behavior of the foliation at infinities.

The old canonical variables are continuous (and sufficiently differentiable) functions of r even across the horizon. The transformation equations imply that the new canonical coordinates $M(r)$ and $\mathsf{R}(r)$ are also continuous across the horizon, but this cannot be said about their conjugate momenta. Equations (86) and (91) indicate that P_M and P_{R} are both proportional to F^{-1}. While the coefficients of F^{-1} are continuous, F goes to zero on the horizon, and generically changes its sign. As a result, P_M and P_{R} become infinite on the horizon, and generically suffer an infinite jump.

This means that when the canonical data are such that F vanishes for some r (which, in particular, always happens for the Schwarzschild solution), our canonical transformation becomes singular. In other words, ω is not a differentiable functional of the old canonical variables. One must use the new canonical variables with caution.

Except at the horizon, the canonical transformation (92) can be inverted for the old variables:

$$\Lambda = \left(\mathsf{F}^{-1}\mathsf{R}'^2 - \mathsf{F}P_M{}^2\right)^{1/2}, \tag{109}$$

$$P_\Lambda = \mathsf{R}\mathsf{F}P_M \left(\mathsf{F}^{-1}\mathsf{R}'^2 - \mathsf{F}P_M{}^2\right)^{-1/2}, \tag{110}$$

$$R = \mathsf{R}\,, \tag{111}$$

$$\begin{aligned} P_R = P_{\mathsf{R}} &+ \frac{1}{2}P_M \\ &+ R^{-1}\mathsf{F}^{-1}\left(\mathsf{F}^{-1}\mathsf{R}'^2 - \mathsf{F}P_M{}^2\right)^{-1}\left((\mathsf{R}\mathsf{F}P_M)'(\mathsf{R}\mathsf{R}')\right. \\ &\left. -(\mathsf{R}\mathsf{F}P_M)(\mathsf{R}\mathsf{R}')'\right) + \frac{1}{2}\mathsf{F}P_M\,. \end{aligned} \tag{112}$$

In these equations, F is an abbreviation for

$$\mathsf{F} = 1 - 2M\mathsf{R}^{-1}\,. \tag{113}$$

VI. $M(t,r)$ AS A CONSTANT OF MOTION

Guided by the spacetime form of the Schwarzschild solution, we have introduced the Schwarzschild mass $M(r)$ as a dynamical variable on our phase space. We shall now prove that if the canonical data satisfy the constraints, the mass function $M(r)$ does not depend on r, and if they also satisfy the Hamilton equations, $M(r)$ is a constant of motion.

Both statements follow by straightforward algebra. By differentiating the definition (82) of $M(r)$ with respect to r, we find that $M'(r)$ is a linear combination of the constraints (32) and (33):

$$M' = -\Lambda^{-1}\left(R'\,H + R^{-1}P_\Lambda\, H_r\right). \tag{114}$$

Because $M(r)$ is a spatial scalar,

$$\left\{M(r), H_r(r')\right\} = M'(r)\,\delta(r,r')\,, \tag{115}$$

Eq. (114) can be translated into the statement that the Poisson bracket of $M(r)$ with the supermomentum weakly vanishes.

Equations (114) and (91) express M' and P_{R} as linear combinations of the constraints. Inversely, we can express the constraints in terms of the new canonical variables. We already know that

$$H_r = P_{\mathsf{R}}\mathsf{R}' + P_M M'\,. \tag{116}$$

The super-Hamiltonian H is then calculated from (114). By using (109) and (110) we obtain

$$H = -\frac{\mathsf{F}^{-1}M'\mathsf{R}' + \mathsf{F}P_M P_{\mathsf{R}}}{\left(\mathsf{F}^{-1}\mathsf{R}'^2 - \mathsf{F}P_M{}^2\right)^{1/2}}\,. \tag{117}$$

These expressions are useful for showing the closure of the Poisson brackets:

$$\left\{M(r), H(r')\right\} = -\Lambda^{-3}R'\,H_r\,\delta(r,r')\,. \tag{118}$$

The same calculation in terms of the old variables is much more cumbersome. One should note that the right-hand side of (118) does not contain the super-Hamiltonian, only the supermomentum. Because $H(r)$ generates the time evolution, (118) means that $M(r)$ is a constant of motion.

From Eqs. (116) and (117) we can conclude that the Hamiltonian and momentum constraints

$$H(r) = 0\,, \quad H_r(r) = 0 \tag{119}$$

are entirely equivalent to a new set of constraints,

$$M'(r) = 0\,, \quad P_{\mathsf{R}}(r) = 0, \tag{120}$$

except on the horizon.

On the horizon, we need to be more circumspect. Let the old canonical variables satisfy the constraints (119). We know that such canonical data correspond to the Schwarzschild solution, and hence there are at most two values of r for which $F = 0$. From (114) and (91) we conclude that $M'(r) = 0$ everywhere, and $P_R(r) = 0$ except at the horizon points. If we insist that $P_R(r)$ be a continuous function of r when the data satisfy the constraints, we conclude that $P_R(r) = 0$ everywhere.

Inversely, let us impose the constraints (120) on the new canonical variables. Except at the horizon points, Eqs. (116) and (117) imply the old constraints (119). At the horizon points, $F(r) = 0$, and $P_M(r)$ becomes infinite in such a way that $F(r)P_M(r)$ stays finite [cf. (86)]. We can again argue from continuity that the new constraints imply the old constraints even at the horizon points. In this sense, the constraint systems (119) and (120) are equivalent everywhere.

VII. THE TALE OF THREE ACTIONS

Written in terms of the new canonical variables M, P_M, R, and P_{R}, the hypersurface action becomes

$$S_\Sigma[M, P_M, \mathsf{R}, P_{\mathsf{R}};\, N, N^r] = \int dt \int_{-\infty}^{\infty} dr \left(P_M(r)\dot{M}(r) + P_{\mathsf{R}}(r)\dot{\mathsf{R}}(r) - N(r)H(r) - N^r H_r(r) \right). \tag{121}$$

The super-Hamiltonian H and supermomentum H_r are now functions (117) and (116) of the new variables. The variation of (121) with respect to N and N^r imposes the Hamiltonian and momentum constraints (119). We have found that these constraints are equivalent to a new set of constraints, (120), which are simple functions of the new variables. The action (121) is equivalent to a new action

$$\begin{aligned} S_\Sigma[M, P_M, \mathsf{R}, P_{\mathsf{R}};\, N^M, N^{\mathsf{R}}] = & \int dt \int_{-\infty}^{\infty} dr \left(P_M(r)\dot{M}(r) + P_{\mathsf{R}}(r)\dot{\mathsf{R}}(r) \right) \\ & - \int dt \int_{-\infty}^{\infty} dr \left(N^M(r)M'(r) + N^{\mathsf{R}}(r)P_{\mathsf{R}}(r) \right), \end{aligned} \tag{122}$$

in which the new constraints, rather than the old ones, are adjoined to the Liouville form. This is done by a new set $N^M(r)$ and $N^{\mathsf{R}}(r)$ of Lagrange multipliers. The falloff conditions (105) – (108) imply that the super-Hamiltonian almost coincides with M' at infinities:

$$H(r) = \mp M'(r) + O^\infty(|r|^{-(2+\epsilon)})\,. \tag{123}$$

The asymptotic values of the multipliers N and N^M are thus related by

$$N^M_\pm(t) = \mp N_\pm(t)\,. \tag{124}$$

The hypersurface action must again be complemented by the ADM boundary action (60). In the new variables, the ADM energy (63) is the value of the mass function $M(r)$ at infinity, $E_\pm = M_\pm$. The boundary action, like the hypersurface action (122), is again a very simple function of the new variables:

$$S_{\partial\Sigma} = -\int dt\, (N_+M_+ + N_-M_-)\,. \tag{125}$$

The total action is the sum

$$\begin{aligned} & S[M, P_M, \mathsf{R}, P_{\mathsf{R}};\, N^M, N^{\mathsf{R}}] \\ & \quad = S_\Sigma[M, P_M, \mathsf{R}, P_{\mathsf{R}};\, N^M, N^{\mathsf{R}}] + S_{\partial\Sigma}[M;\, N^M]\,. \end{aligned} \tag{126}$$

It is transparent how the boundary action cancels the boundary term obtained by varying $M(r)$ in the hypersurface action.

The lapse functions $N_\pm(t)$ at infinities must be treated as prescribed functions of the time parameter t. After the constraints are imposed, the boundary part of the total action survives as a true, t-dependent, Hamiltonian of the black hole. We shall discuss this reduction process in the next section.

The ends $N_\pm(t)$ of $N(t,r)$ are freed by parametrizing the action at infinities:

$$S_{\partial\Sigma}[M;\, \tau_+, \tau_-] = -\int dt\, (M_+\dot{\tau}_+ - M_-\dot{\tau}_-)\,. \tag{127}$$

The total action

$$\begin{aligned} & S[M, P_M, \mathsf{R}, P_{\mathsf{R}};\, \tau_+, \tau_-;\, N^M, N^{\mathsf{R}}] \\ & \quad = S_\Sigma[M, P_M, \mathsf{R}, P_{\mathsf{R}};\, N^M, N^{\mathsf{R}}] + S_{\partial\Sigma}[M;\, \tau_+, \tau_-] \end{aligned} \tag{128}$$

depends now on two additional variables $\tau_\pm(t)$, which can also be freely varied. It is no longer a canonical action, because τ_+ and τ_- do not come with their conjugate momenta.

There are two entirely different ways in which the action (128) can be brought into canonical form. The first one is standard, the second one rather unexpected. Let us explain the standard method first.

One can say that the action (128) has a mixed Hamiltonian-Lagrangian form, and that $\tau_\pm(t)$ is a pair of Lagrangian configuration variables. The purely Hamiltonian form should be reached by the Legendre dual transformation which complements $\tau_\pm$ by the conjugate momenta $\pi_\pm$. However, because the action is linear in the velocities $\dot\tau_\pm$, as soon as one starts implementing this program, one obtains two new constraints

$$\begin{aligned} C_+ &:= \pi_+ + M_+ = 0\,, \\ C_- &:= -\pi_- + M_- = 0\,. \end{aligned} \tag{129}$$

They have vanishing Poisson brackets among themselves, and with the hypersurface constraints $M'(r) = 0 = P_{\mathsf{R}}(r)$. The resulting constraint system is thus first class. The new constraints $C_\pm$ must be adjoined to the canonical action by Lagrange multipliers $N_\pm$:

$$\begin{aligned} &S\big[M, P_M, \mathsf{R}, P_{\mathsf{R}};\, \tau_+, \pi_+, \tau_-, \pi_-;\, N^M, N^{\mathsf{R}}, N_+, N_-\big] \\ &\qquad = \int dt \int_\infty^\infty dr\, \Big(P_M(r)\dot M(r) + P_{\mathsf{R}}(r)\dot{\mathsf{R}}(r) - N^M(r)M'(r) - N^{\mathsf{R}}(r)P_{\mathsf{R}}(r) \Big) \\ &\qquad\quad + \int dt\, \Big(\pi_+\dot\tau_+ + \pi_-\dot\tau_- - N_+C_+ - N_-C_-\Big)\,. \end{aligned} \tag{130}$$

The variation of (130) with respect to the momenta $\pi_\pm$ leads back to Eq. (68). The new multipliers $N_\pm$ thus are what the symbols suggest: the lapse function at infinities.

The price we paid for the canonical form (130) was a couple of new variables and a couple of new constraints. It is gratifying to learn that one can get the same product for free: The mixed variables $M, P_M;\, \tau_+, \tau_-$ in the action (128) can simply be transformed into a canonical chart. It is even more gratifying that the new canonical variables have a clean geometric meaning: they turn out to be the Killing time $T(r)$ and the mass density $P_T(r)$ along the hypersurface, complemented by a canonical pair of constants of motion.

To see how this comes about, notice that the action (128) is linear in the time derivatives $\dot M(r), \dot\tau_\pm$. The homogeneous part of (128) thus defines a one-form

$$\Theta := \int_{-\infty}^\infty dr\, P_M(r)\delta M(r) - \big(M_+\delta\tau_+ - M_-\delta\tau_-\big) \tag{131}$$

on $\big(M(r), P_M(r);\, \tau_+, \tau_-\big)$.

We shall now cast (131) into a Liouville form. First, we replace the mass function $M(r)$ by the mass at left infinity m, and by the mass density $\Gamma(r)$:

$$m = M_-\,, \quad \Gamma(r) = M'(r)\,. \tag{132}$$

Inversely,

$$M(r) = m + \int_{-\infty}^r dr'\, \Gamma(r')\,. \tag{133}$$

By introducing (133) into (131) we get

$$\begin{aligned} \Theta = &\left((\tau_+ - \tau_-) + \int_{-\infty}^\infty dr\, P_M(r)\right)\delta m \\ &+ \int_{-\infty}^\infty dr\, \left(\tau_+\delta\Gamma(r) + P_M(r)\int_{-\infty}^r dr'\, \delta\Gamma(r')\right) \\ &+ \delta\big(M_-\tau_- - M_+\tau_+\big)\,. \end{aligned} \tag{134}$$

To rearrange (134), we write the identity

$$\left(\int_\infty^r dr'\, P_M(r') \int_{-\infty}^r dr'\, \delta\Gamma(r')\right)' = P_M(r)\int_{-\infty}^r dr'\, \delta\Gamma(r') + \delta\Gamma(r)\int_\infty^r dr'\, P_M(r')\,, \tag{135}$$

which we then integrate from $r = -\infty$ to $r = \infty$:

$$\begin{aligned} &\int_{-\infty}^\infty dr\, P_M(r)\int_{-\infty}^r dr'\, \delta\Gamma(r') \\ &\qquad = -\int_{-\infty}^\infty dr\, \delta\Gamma(r)\int_\infty^r dr'\, P_M(r')\,. \end{aligned} \tag{136}$$

We immediately see that

$$\begin{aligned} \Theta = &\left((\tau_+ - \tau_-) + \int_{-\infty}^\infty dr'\, P_M(r')\right)\delta m \\ &+ \int_{-\infty}^\infty dr\, \left(\tau_+ - \int_\infty^r dr'\, P_M(r')\right)\delta\Gamma(r) \\ &+ \delta\big(M_-\tau_- - M_+\tau_+\big)\,. \end{aligned} \tag{137}$$

This shows that

$$m = M_- , \tag{138}$$

$$p = (\tau_+ - \tau_-) + \int_{-\infty}^{\infty} dr'\, P_M(r') \tag{139}$$

and

$$\Gamma(r) = M'(r) , \tag{140}$$

$$P_\Gamma(r) = \tau_+ - \int_{\infty}^{r} dr'\, P_M(r') \tag{141}$$

is a canonical chart: Indeed,

$$\Theta = p\delta m + \int_{-\infty}^{\infty} dr\, P_\Gamma(r)\delta\Gamma(r) \tag{142}$$

$$+ \delta(M_-\tau_- - M_+\tau_+) \tag{143}$$

differs from the Liouville form only by an exact form.

When passing from $P_M(r) = -T'(r)$ to $T(r)$ we had to fix a constant of integration. The Killing time (141) is fixed by requiring that it matches the proper time τ_+ on the parametrization clock at right infinity. By construction, $T(r)$ is a spatial scalar and $\Gamma(r)$ a scalar density. We would like to change $T(r)$ into a canonical coordinate, and $\Gamma(r)$ into a canonical momentum. This is done by an elementary canonical transformation (in the sense of Carathéodory [34])

$$T(r) = P_\Gamma(r) = \tau_+ - \int_{\infty}^{r} dr'\, P_M(r') , \tag{144}$$

$$P_T(r) = -\Gamma(r) = -M'(r) \tag{145}$$

which sends Θ into

$$\Theta = p\delta m + \int_{-\infty}^{\infty} dr\, P_T(r)\delta T(r) + \delta\omega , \tag{146}$$

with

$$\omega = (M_-\tau_- - M_+\tau_+) + \int_{-\infty}^{\infty} dr\, P_\Gamma(r)\Gamma(r)$$

$$= M_-(\tau_- - \tau_+) - \int_{-\infty}^{\infty} dr \int_{\infty}^{r} dr'\, P_M(r')M'(r') . \tag{147}$$

The last equations, (146) and (147), show that the transformation

$$\tau_+, \tau_-;\ M(r), P_M(r) \;\mapsto\; m, p;\ T(r), P_T(r) \tag{148}$$

given by Eqs. (138) and (139) and (144) and (145) constructs a canonical chart from the originally mixed variables.

The transformation (148) casts the parametrized action (122), (127), and (128) to an extremely simple canonical form

$$S[m,p;\, T, P_T, \mathsf{R}, P_\mathsf{R};\, N^T, N^\mathsf{R}] = \int dt \left(p\dot{m} + \int_{-\infty}^{\infty} dr\, \big(P_T(r)\dot{T}(r) + P_\mathsf{R}(r)\dot{\mathsf{R}}(r) \big) \right) - \int dt \int_{-\infty}^{\infty} dr\, \big(N^T(r)P_T(r) + N^\mathsf{R}(r)P_\mathsf{R}(r) \big) . \tag{149}$$

[We gave the multiplier $-N^M(r)$ a new name $N^T(r)$.] In the transition from (128) to (149) we have discarded the total time derivative $\dot{\omega}$. Such a procedure does not change equations of motion, and it is used throughout classical mechanics. When applied to canonical action, it generates canonical transformations. Here we have used it for bringing the action to canonical form.

Because the multipliers $N^T(r)$ and $N^\mathsf{R}(r)$ are freely variable, the action (149) enforces the constraints

$$P_T(r) = 0 , \quad P_\mathsf{R}(r) = 0 . \tag{150}$$

The boundary action disappeared from (149) and the Hamiltonian took the form

$$H[N^T] + H[N^\mathsf{R}] := \int_{-\infty}^{\infty} dr\, N^T(r)P_T(r) + \int_{-\infty}^{\infty} dr\, N^\mathsf{R}(r)P_\mathsf{R}(r) . \tag{151}$$

It is a linear combination of constraints, and as such it weakly vanishes.

It is well known that the smeared super-Hamiltonian

$$H[N] = \int_{-\infty}^{\infty} dr\, N(r)H(r) \tag{152}$$

in the Dirac-ADM action (31) generates the change of the canonical data when the hypersurface is displaced by the proper time $N(r)$ in the normal direction. Similarly, the smeared supermomentum

$$H_r[N^r] = \int_{-\infty}^{\infty} dr\, N^r(r)H_r(r) \tag{153}$$

generates the change of the data when the hypersurface is shifted by the tangential vector $N^r(r)$.

The Hamiltonian (151) generates a different type of displacement. The Hamilton equations

$$\begin{aligned} \dot{T}(t,r) &= \{T(t,r),\, H[N^T]\} = N^T(t,r) , \\ \dot{\mathsf{R}}(t,r) &= \{\mathsf{R}(t,r),\, H[N^T]\} = 0 \end{aligned} \tag{154}$$

reveal how $H[N^T]$ displaces the hypersurface in the Kruskal diagram. It shifts it along the lines of constant R by the amount $N^T(t,r)$ of Killing time which differs from one line to another. Similarly, $H[N^\mathsf{R}]$ displaces the hypersurface along the lines of a constant T in such a way that the curvature coordinate R changes by the amount $N^\mathsf{R}(t,r)$. The Hamiltonian (151) thus generates spacetime diffeomorphisms in (T, R). The elaboration of this general point can be found in Isham and Kuchař [35].

The new constraints (150) have a very simple form: a number of canonical momenta is set equal to zero. Locally, any system of first-class constraints can be brought to such a form [36], but it is usually impossible to find explicitly the necessary transformation. Our result demonstrates that this is feasible for Schwarzschild black holes, and that it can be done globally. The momenta which are required to vanish are conjugate to the embedding variables $T(r)$ and $\mathsf{R}(r)$ which locate the hypersurface in the Kruskal diagram.

There is a pair of canonically conjugate variables in the action (149), namely, $m(t)$ and $p(t)$, which is not subject to any constraints. However, there is no nonvanishing Hamiltonian in (149) which would evolve these variables. Both m and p are thus constants of motion. The meaning of m as the mass at infinity is clear, and so is its conservation. The significance of p is at first puzzling. The momentum p was introduced by the transformation (139). Because of (144), p can be interpreted as

$$p(t) = T_-(t) - \tau_-(t)\,. \tag{155}$$

But should not the Killing time coincide with the parametrization time also at left infinity, and should not thus p simply vanish? Are we not missing a constraint?

The answer to this question is *no*. The times $\tau_\pm$ are introduced by the parametrization process and their *origins* are entirely independent. The left infinity does not know what the right infinity does. When we shift the origins by *different amounts* $\alpha_- =$ const and $\alpha_+ =$ const at the left and right infinities,

$$\tau_-(t) \mapsto \tau_-(t) + \alpha_-\,, \quad \tau_+(t) \mapsto \tau_+(t) + \alpha_+\,, \tag{156}$$

the parametrized action is unchanged.

The origin of the Killing time $T(r)$ is also arbitrary. We have chosen it so that $T(r)$ matches τ_+ at *right infinity*. Once defined this way, $T(r)$ can be used to propagate the choice of the origin from right infinity to left infinity. [After the constraints are imposed, and the Schwarzschild solution is found, this propagation amounts to drawing the straight line across the Kruskal diagram, from the $\tau_+(t) = 0$ point at right infinity, through the bifurcation point of the horizon, and all the way up to the left infinity.] There is, however, no reason why the parametrization clock at left infinity should have been set to zero at this propagated origin. The variable p tells us the difference between the origins of the parametrization times at the right and left infinities. More precisely, p is the value of the Killing time $T(r)$ (which is matched to the parametrization time at right infinity) at the origin of the parametrization time at left infinity. Once set, the parametrization clock and the Killing time clock run at the same pace, both of them measuring intervals of proper time. Therefore, it does not matter *when* we read their difference. This is the reason why $p(t)$ of Eq. (155) is a constant of motion.

To summarize, we have arrived at three different canonical actions describing the same physical system, namely, primordial black holes. The *unparametrized canonical action* (126) has a nonvanishing Hamiltonian. The two *parametrized canonical actions* that follow have only constraints. The first of these, (130), has more variables and more constraints than are really necessary. Also, it does not disentangle the variables which are constrained to vanish from those that survive as true dynamical degrees of freedom. The last of our actions, (149), sticks to the original number of variables and constraints, and at the same time clearly identifies the true degrees of freedom. We believe it provides the simplest canonical framework for studying Schwarzschild black holes.

VIII. REDUCED CANONICAL THEORY

Each canonical action we have introduced predicts a Hamiltonian evolution. To compare these evolutions, we first reduce the actions to the same set of true degrees of freedom. The action is reduced by solving the constraints and substituting the solutions back into the action.

Let us start with the unparametrized action (126). The constraint $M'(r) = 0$ tells us that only the homogeneous mode of $M(r)$ survives:

$$M(t,r) = \mathbf{m}(t)\,. \tag{157}$$

By substituting (157) and $P_{\mathsf{R}}(r) = 0$ back into (126) we obtain

$$S[\mathbf{m},\mathbf{p}] = \int dt \left(\mathbf{p}\dot{\mathbf{m}} - \big(N_+(t) + N_-(t)\big)\mathbf{m}\right). \tag{158}$$

The form of the reduced action enabled us to identify

$$\mathbf{p} := \int_{-\infty}^{\infty} dr\, P_M(r) \tag{159}$$

as the momentum canonically conjugate to $\mathbf{m}$. The reduced action has one degree of freedom $\mathbf{m}$ and a true time-dependent Hamiltonian

$$\mathbf{h}(t,\mathbf{m},\mathbf{p}) = \mathcal{N}(t)\mathbf{m}\,, \quad \mathcal{N}(t) := N_+(t) + N_-(t)\,. \tag{160}$$

This Hamiltonian is proportional to $\mathbf{m}$, with a coefficient $\mathcal{N}(t)$ which is a prescribed function of t. The Hamilton equations of motion

$$\dot{\mathbf{m}} = \partial\mathbf{h}(t,\mathbf{m},\mathbf{p})/\partial\mathbf{p} = 0\,,$$
$$\dot{\mathbf{p}} = -\partial\mathbf{h}(t,\mathbf{m},\mathbf{p})/\partial\mathbf{m} = -\mathcal{N}(t) \tag{161}$$

indicate that $\mathbf{m}(t)$ is a constant of motion, but $\mathbf{p}(t)$ changes in time. This is consistent with (159) which identifies $\mathbf{p}$ with $-(T_+ - T_-)$. The difference of the Killing times between the left and the right infinities stays the same only if we evolve the hypersurface by the lapse function which has opposite values at $\pm\infty$, i.e., by $\mathcal{N} = 0$.

Next, reduce the parametrized action (130) by solving both the hypersurface constraints and the additional constraints (129). We get

$$S[\mathbf{m},\mathbf{p},\mathcal{T}] = \int dt\,\big(\mathbf{p}\dot{\mathbf{m}} - \dot{\mathcal{T}}\mathbf{m}\big)\,, \tag{162}$$

with

$$\mathcal{T} := \tau_+ - \tau_-\,. \tag{163}$$

The action (162) can be obtained from the action (158) by putting

$$\mathcal{N}(t) = \dot{\mathcal{T}}(t)\,. \tag{164}$$

With this replacement, the Hamilton equations of the two actions are the same, (161). Unlike $\mathcal{N}(t)$, $\mathcal{T}(t)$ in action (162) can be varied. By varying $\mathcal{T}(t)$, we obtain once more the conservation of $\mathbf{m}(t)$.

The reduction of our last action, (149), by the constraints $P_T(r) = 0 = P_{\mathsf{R}}(r)$ is trivial. We obtain

$$S[m,p] = \int dt\, p\dot{m}\,. \tag{165}$$

We have already observed that the Hamiltonian $h(m,p)$ vanishes, and that both m and p are constants of motion.

How are the actions (158) and (162) with a nonvanishing Hamiltonian (160) related to the action (165)? By a time-dependent canonical transformation. Let $\mathcal{T}(t)$ be a primitive function of $\mathcal{N}(t)$, as in (164). Treat $\mathcal{T}(t)$ as a prescribed function of t. Take the function

$$\Omega(t,\mathbf{m},p) = \mathbf{m}\big(p - \mathcal{T}(t)\big) \tag{166}$$

of the old coordinate $\mathbf{m}$ and the new momentum p, and let it generate a canonical transformation from $\mathbf{m}$ and $\mathbf{p}$ to m and p:

$$\mathbf{p} = \partial\Omega(t,\mathbf{m},p)/\partial\mathbf{m} = p - \mathcal{T}(t)\,,$$
$$m = \partial\Omega(t,\mathbf{m},p)/\partial p = \mathbf{m}\,. \tag{167}$$

Time-dependent canonical transformations change the Hamiltonian:

$$h = \mathbf{h} + \partial\Omega(t,\mathbf{m},p)/\partial t = \dot{\mathcal{T}}(t)\mathbf{m} - \mathbf{m}\dot{\mathcal{T}}(t) = 0\,. \tag{168}$$

Our particular generating function (166) turns our particular Hamiltonian (160) to zero.

Equation (167) reproduces the definition (139) of the momentum p. The Hamilton equations (161) ensure that the new momentum p of equation (167) does not change in time. The same conclusion follows from the new Hamiltonian (168). The three actions generate the same dynamics.

The canonical transformation (167) takes the canonical momentum $\mathbf{p}$ at t and transforms it to the value which it has at an instant when $\mathcal{T}(t)$ happens to vanish. The Schwarzschild mass has the same value for any t. One can thus view (167) as a transformation to "initial data."

Similarly, one can view the transition from the unparametrized action (126) to our final action (149) as a time-dependent canonical transformation prior to the reduction.

IX. QUANTUM BLACK HOLES

The canonical action (149) is a good starting point for the Dirac constraint quantization. The new configuration space is covered by the coordinates $T(r), \mathsf{R}(r)$, and m. The first two coordinates locate the hypersurface, the third one, m, is the single degree of freedom of a primordial Schwarzschild black hole. The constraints are

$$P_T(r) = 0\,, \quad P_{\mathsf{R}}(r) = 0\,, \tag{169}$$

and the Hamiltonian of the system vanishes: $h = 0$.

The state of the black hole on a hypersurface $T(r), \mathsf{R}(r)$ at the label time t should be described by a state functional $\Psi\big(m,t;\, T,\mathsf{R}\big]$ over the configuration space. The momenta are represented by the operators

$$\hat{p} = -i\,\partial/\partial m\,, \quad \hat{P}_T(r) = -i\,\delta/\delta T(r)\,,$$
$$\hat{P}_{\mathsf{R}}(r) = -i\,\delta/\delta \mathsf{R}(r)\,. \tag{170}$$

The Dirac rules call for imposing the constraints as operator restrictions on the state functional:

$$\hat{P}_T(r)\,\Psi\big(m,t;\, T,\mathsf{R}\big] = 0\,, \quad \hat{P}_{\mathsf{R}}(r)\,\Psi\big(m,t;\, T,\mathsf{R}\big] = 0\,. \tag{171}$$

Equations (171) imply that the state cannot depend on the embedding variables:

$$\Psi = \Psi(m,t)\,. \tag{172}$$

The state must still satisfy the Schrödinger equation

$$i\,\dot{\Psi}(m,t) = \hat{h}\,\Psi(m,t)\,. \tag{173}$$

Because $\hat{h} = 0$, (173) ensures that Ψ does not depend on t:

$$\Psi(m,t) = \Psi(m)\,. \tag{174}$$

The state function (174) describes a superposition of primordial black holes of different masses. There is not much for it to do: once prepared, it stays the same on every hypersurface $T(r), \mathsf{R}(r)$ and for all t.

Let us compare this description of states with that one which follows from the unparametrized canonical action (126). Let us choose the $\boldsymbol{\Psi}\big(t; M, \mathsf{R}\big]$ representation. As before, the $P_{\mathsf{R}}(r) = 0$ constraint implies that $\boldsymbol{\Psi}$ does not depend on $\mathsf{R}(r)$. The $M'(r) = 0$ constraint translates into the statement that $\boldsymbol{\Psi}$ is an eigenfunction of $\hat{M}(r)$ with a constant eigenvalue $M(r) = \mathbf{m}$:

$$\hat{M}'(r)\,\boldsymbol{\Psi} = \big(\hat{M}(r)\boldsymbol{\Psi}\big)' = 0 \implies \hat{M}(r)\,\boldsymbol{\Psi} = \mathbf{m}\,\boldsymbol{\Psi}\,. \tag{175}$$

In the $M(r)$ representation

$$\boldsymbol{\Psi}\big[M;\, t\big) = \boldsymbol{\psi}(\mathbf{m},t)\,\delta\big(M(r) - \mathbf{m}\big)\,, \tag{176}$$

and we can continue working with the coefficient $\boldsymbol{\psi}(\mathbf{m},t)$. This coefficient must satisfy the Schrödinger equation with the reduced Hamiltonian (160):

$$i\,\dot{\boldsymbol{\psi}}(\mathbf{m},t) = \mathcal{N}(t)\mathbf{m}\,\boldsymbol{\psi}(\mathbf{m},t)\,. \tag{177}$$

Its solution is

$$\boldsymbol{\psi}(\mathbf{m},t) = \boldsymbol{\phi}(\mathbf{m})\,\exp\big(-i\mathbf{m}\mathcal{T}(t)\big)\,, \tag{178}$$

where $\mathcal{T}(t)$ is a primitive function to $\mathcal{N}(t)$, as in (164). Unlike (174), the state function (178) oscillates in the $\mathcal{T}$ time.

We have seen that the classical transition from the action (126) to the action (149) is achieved by a time-dependent canonical transformation (166) – (168). We want to show that the ensuing quantum theories are connected by a time-dependent unitary transformation

$$\hat{\mathbf{W}}(t) = \exp\big(- i\mathcal{T}(t)\hat{\mathbf{m}}\big) . \tag{179}$$

Let us start in the *Heisenberg picture.* The fundamental Heisenberg operators $\hat{\mathbf{m}}(t)$ and $\hat{\mathbf{p}}(t)$ of the unparametrized theory depend on time, and they satisfy the Heisenberg equations of motion

$$\frac{d\hat{\mathbf{m}}(t)}{dt} = \frac{1}{i}\left[\hat{\mathbf{m}}(t), \hat{\mathbf{h}}(t)\right] , \quad \frac{d\hat{\mathbf{p}}(t)}{dt} = \frac{1}{i}\left[\hat{\mathbf{p}}(t), \hat{\mathbf{h}}(t)\right] , \tag{180}$$

with the Heisenberg Hamiltonian $\hat{\mathbf{h}}(t) = \mathbf{h}\big(t, \hat{\mathbf{m}}(t), \hat{\mathbf{p}}(t)\big)$. The Heisenberg states $|\boldsymbol{\phi}_0\rangle$ refer to $t = 0$ and do not depend on time.

Change now the fundamental Heisenberg operators $\hat{\mathbf{m}}(t)$ and $\hat{\mathbf{p}}(t)$ into new fundamental Heisenberg operators $\hat{m}(t)$ and $\hat{p}(t)$ by a general time-dependent unitary operator $\hat{\mathbf{W}}(t) = \mathbf{W}\big(t, \hat{\mathbf{m}}(t), \hat{\mathbf{p}}(t)\big)$:

$$\hat{m}(t) := \hat{\mathbf{W}}(t)\, \hat{\mathbf{m}}(t)\, \hat{\mathbf{W}}^{-1}(t) , \tag{181}$$

$$\hat{p}(t) := \hat{\mathbf{W}}(t)\, \hat{\mathbf{p}}(t)\, \hat{\mathbf{W}}^{-1}(t) . \tag{182}$$

It is easy to show that the new Heisenberg operators form a conjugate pair, and that they satisfy the Heisenberg equations of motion

$$\frac{d\hat{m}(t)}{dt} = \frac{1}{i}\left[\hat{m}(t), \hat{h}(t)\right] , \quad \frac{d\hat{p}(t)}{dt} = \frac{1}{i}\left[\hat{p}(t), \hat{h}(t)\right] \tag{183}$$

with the new Heisenberg Hamiltonian

$$\hat{h}(t) = \frac{1}{i}\frac{\partial \hat{\mathbf{W}}(t)}{\partial t}\, \hat{\mathbf{W}}^{-1}(t) + \hat{\mathbf{W}}(t)\, \hat{\mathbf{h}}(t)\, \hat{\mathbf{W}}^{-1}(t) . \tag{184}$$

In the *Schrödinger picture*, the fundamental operators $\hat{\mathbf{m}}$, $\hat{\mathbf{p}}$ and $\hat{m}$, $\hat{p}$ become time independent, while the state $|\boldsymbol{\phi}_0\rangle$ is evolved by the respective Hamilton operators:

$$|\boldsymbol{\psi}(t)\rangle = \mathsf{T}\exp\left(-i\int_0^t dt\, \hat{\mathbf{h}}(t)\right) |\boldsymbol{\phi}_0\rangle , \tag{185}$$

$$|\Psi(t)\rangle = \mathsf{T}\exp\left(-i\int_0^t dt\, \hat{h}(t)\right) |\boldsymbol{\phi}_0\rangle . \tag{186}$$

Here, T stands for the time ordering. We have two Schrödinger states, $|\boldsymbol{\psi}(t)\rangle$ and $|\Psi(t)\rangle$, corresponding to two alternative descriptions, $\mathbf{m}, \mathbf{p}, \mathbf{h}(t)$ and $m, p, h(t)$, of the same quantum system. These states are related by

$$|\Psi(t)\rangle = \mathsf{T}\exp\left(-i\int_0^t dt\, \hat{h}(t)\right) \mathsf{T}\exp\left(i\int_0^t dt\, \hat{\mathbf{h}}(t)\right) |\boldsymbol{\psi}(t)\rangle . \tag{187}$$

Apply this general scheme to our simple system. The unitary operator (179) yields the Heisenberg fundamental operators (181) and (182),

$$\hat{m}(t) = \hat{\mathbf{m}}(t) , \tag{188}$$

$$\hat{p}(t) = \hat{\mathbf{p}}(t) + \mathcal{T}(t) , \tag{189}$$

which are related exactly as their classical counterparts (167). [For simplicity, we assume that $\mathcal{T}(t = 0) = 0$.] The new Heisenberg Hamilton operator (184) vanishes like the classical Hamiltonian (168). The Heisenberg equations of motion (183) then guarantee that the Heisenberg operators $\hat{p}(t)$ and $\hat{m}(t)$ are operator constants of motion. By (189), the eigenvalues of the operators $\hat{\mathbf{m}}$ and $\hat{m}$ are the same, $\mathbf{m} = m$.

The same situation can be described in the Schrödinger picture. Equation (187) relates the states. In the m representation,

$$\Psi(m) = \Psi(m,t) = \exp\,\big(i\mathcal{T}(t)\mathbf{m}\big)\, \boldsymbol{\psi}(\mathbf{m}, t) = \boldsymbol{\phi}(\mathbf{m}) . \tag{190}$$

This clarifies the relation between states (174) and (178).

The last of our three actions, (130), has two additional constraints (129). The states now depend on two more configuration variables $\tau_\pm$. The hypersurface constraints reduce the states to the form $\psi(\mathbf{m}, \tau_+, \tau_-, t)$. The Hamiltonian of the action (130) vanishes, and the Schrödinger equation implies that ψ cannot depend on t. The reduced state function must still satisfy the (reduced) constraints (129):

$$\hat{C}_\pm\, \psi(\mathbf{m}, \tau_+, \tau_-) = 0$$

$$\iff \quad \big(\mp\, i\partial/\partial\tau_\pm + \mathbf{m}\big)\psi(\mathbf{m}, \tau_+, \tau_-) = 0 . \tag{191}$$

These can be viewed as two Schrödinger equations in the proper times $\tau_\pm$. Their solution is the state function

$$\psi(\mathbf{m}, \tau_+, \tau_-) = \phi(\mathbf{m}) \exp\big(- i\, \mathbf{m}(\tau_+ - \tau_-)\big) . \tag{192}$$

This is the same state as (178), but now written in terms of the proper times $\tau_\pm$ rather than the label time t. Though they describe it in slightly different ways, our three actions lead to the same quantum dynamics.

Because primordial black holes have only one degree of freedom m which is a constant of motion, their states are rather simple. Still, there are some interesting questions to ask. The state function (174) does not change in time. However, one can construct significant hypersurface-dependent operators, such as the intrinsic and extrinsic geometry of an embedding $T(r), \mathsf{R}(r)$, and ask what their expectation values are. We defer this conceptual exercise to a later paper.

X. INCLUSION OF SOURCES

Schwarzschild black holes are empty vessels and what can happen to them is rather limited. True dynamics requires filling them with matter. This was the intent of the original BCMN model. Matter can propagate on the wormhole topology, or it can close the wormhole and change the topology of Σ into $\mathbb{R}^3$. The latter case is physically more interesting. The following discussion assumes that the wormhole is closed.

We must now ask whether what we have done in the vacuum can be repeated in the presence of matter.

Introduce a massless scalar field propagating in the spacetime $(\mathcal{M}, \gamma)$:

$$S^{\phi}[\phi;\gamma] = -(8\pi)^{-1} \int d^4X\, |\gamma|^{1/2}\, \gamma^{\alpha\beta} \phi_{,\alpha}\phi_{,\beta}\,. \quad (193)$$

After the ADM decomposition and midisuperspace reduction by spherical symmetry, the Lagrangian action takes the form

$$S^{\phi}[\phi;\, \Lambda, R;\, N, N^r] = \frac{1}{2}\int dt \int_0^{\infty} dr\, \left(N^{-1}\Lambda R^2 (\dot\phi - N^r\phi')^2 - N\,\Lambda^{-1} R^2 \phi'^2\right)\,. \quad (194)$$

By introducing the momentum

$$\pi = \partial L^{\phi}/\partial\dot\phi = N^{-1}\Lambda R^2(\dot\phi - N^r\phi')\,, \quad (195)$$

we cast the action (194) into canonical form by the Legendre dual transformation:

$$S^{\phi}[\phi,\pi;\, \Lambda, R;\, N, N^r]$$

$$= \int dt \int_0^{\infty} dr\, \left(\pi\dot\phi - NH^{\phi} - N^r H^{\phi}_r\right)\,. \quad (196)$$

In this process, we obtain the energy density

$$H^{\phi} = \frac{1}{2}\,\Lambda^{-1}(R^{-2}\pi^2 + R^2\phi'^2) \quad (197)$$

and momentum density

$$H^{\phi}_r = \pi\phi' \quad (198)$$

of the scalar field.

To couple the scalar field to gravity, we add the field action (196) to the gravitational action (31). The variation of the total action with respect to N and N^r leads to Hamiltonian and momentum constraints on the extended phase space:

$$H + H^{\phi} = 0\,, \quad H_r + H^{\phi}_r = 0\,. \quad (199)$$

Again, up to a point transformation, this is the result obtained by BCMN [10].

We arrived at the functional time formalism for the Schwarzschild black hole by transforming the original geometric variables into new canonical variables (92), and then into (148). Do the charms work in the presence of sources?

The message of Sec. IV is that (92) is a canonical transformation on the geometric phase space irrespective of any constraints or dynamics. Therefore, we can introduce the new canonical variables exactly as in the vacuum spacetime. [The transformation (148) needs to be modified, to accommodate the changed topology of Σ.]

One cannot, however, expect that this immediately simplifies the constraints. The underlying physical reason is that the matter field curves the spacetime in which it propagates, and propagates in the spacetime which it curves. This is reflected by the presence of the metric variable Λ in the field energy (197) in the Hamiltonian constraint (199). This variable is a function (109) of the new canonical variables. The Hamiltonian constraint is no longer equivalent to a simple equation $P_T(r) = -M'(r) = 0$, but it provides an implicit information about how the energy density $M'(r)$ depends on sources. The structure of this equation is presently being investigated by Romano in collaboration with the author. An analogous study of scalar fields coupled to a cylindrical gravitational wave by Braham [37] shows that resolution of such equations is feasible.

XI. KRUSKAL COORDINATES AS PHASE SPACE VARIABLES

Our main device has been the reconstruction of the curvature coordinates T and R from the canonical data. This turned the curvature coordinates *in spacetime* into canonical coordinates *in phase space.*

Unfortunately, the Killing time T becomes infinite on the horizon, and the canonical transformation which leads to it has a corresponding singularity. From the spacetime picture one knows that the entire Schwarzschild solution can be covered by a single patch of spacetime coordinates, the Kruskal coordinates, which are well behaved on the horizon. It is natural to ask whether these coordinates, rather than the curvature coordinates, can be interpreted as canonical coordinates.

The direct reconstruction of Kruskal coordinates from the canonical data is cumbersome. It is better to reach them via the curvature coordinates. Effectively, we are asked to reexpress the spacetime transformation (8) – (13) as a point transformation on the phase space.

Because the spacetime transformation involves exponentials, it is first necessary to turn the curvature coordinates into dimensionless quantities. This is done by *scal-*

ing on the phase space $(m, T(r), \mathsf{R}(r); p, P_T(r), P_{\mathsf{R}}(r))$. The desired configuration space operation mimics (8):

$$\bar{T}(r) = \frac{T(r)}{2m}, \quad \bar{\mathsf{R}}(r) = \frac{\mathsf{R}(r)}{2m}, \quad \bar{m} = m. \tag{200}$$

The scaled curvature coordinates $\bar{T}$ and $\bar{\mathsf{R}}$ are dimensionless, while $\bar{m}$ keeps the dimension of length.

It is important that the scaling (200) be done with the Schwarzschild mass m at infinity rather than with the mass function $M(r)$. Although these two variables coincide on the constraint surface, the coordinates $\bar{T}(r)$ and $\bar{\mathsf{R}}(r)$ scaled with the mass function would not have strongly vanishing Poisson brackets, and hence could not be used as canonical coordinates on the phase space.

The configuration transformation (200) can be completed into a point transformation on the phase space:

$$P_T(r) = \frac{P_{\bar{T}}(r)}{2\bar{m}}, \tag{201}$$

$$P_{\mathsf{R}}(r) = \frac{P_{\bar{\mathsf{R}}}(r)}{2\bar{m}}, \tag{202}$$

$$p = \bar{p} - \frac{1}{\bar{m}} \int_{-\infty}^{\infty} dr \left(P_{\bar{T}}(r)\bar{T}(r) + P_{\bar{\mathsf{R}}}(r)\bar{\mathsf{R}}(r) \right). \tag{203}$$

Inversely,

$$P_{\bar{T}}(r) = 2mP_{\bar{T}}(r), \tag{204}$$

$$P_{\bar{\mathsf{R}}}(r) = 2mP_{\mathsf{R}}(r), \tag{205}$$

$$\bar{p} = p + \frac{1}{m} \int_{-\infty}^{\infty} dr \left(P_T(r)T(r) + P_{\mathsf{R}}(r)\mathsf{R}(r) \right). \tag{206}$$

From here, we can figure out the dimensions of the momenta. The unscaled coordinates T and R have the dimension of length, while the conjugate momenta P_T and P_{R} are dimensionless. The scaling reverts these dimensions: The scaled coordinates $\bar{T}$ and $\bar{\mathsf{R}}$ are dimensionless, while the scaled momenta $P_{\bar{T}}$ and $P_{\bar{\mathsf{R}}}$ have the dimension of length. Scaling does not change the dimension of the discrete variables: m and $\bar{m}$, and p and $\bar{p}$ all have the dimension of length.

Because the transformation from curvature coordinates to Kruskal coordinates is double-valued, it is better to write the transformation *from* Kruskal coordinates *to* curvature coordinates. The configuration part of this transformation follows the pattern of (10):

$$\bar{T}(r) = \ln|V(r)| - \ln|U(r)|, \quad \bar{\mathsf{R}}(r) = \mathcal{R}\big(U(r)V(r)\big). \tag{207}$$

By differentiating (13) with respect to $\bar{T}$ and $\bar{R}$ we obtain the Jacobi matrix

$$U_{,\bar{T}} = -\frac{1}{2}U, \quad U_{,\bar{R}} = \frac{1}{2}UF^{-1}(\bar{R}), \tag{208}$$

$$V_{,\bar{T}} = \frac{1}{2}V, \quad V_{,\bar{R}} = \frac{1}{2}VF^{-1}(\bar{R}). \tag{209}$$

We took into account that

$$\frac{d\bar{R}^*(\bar{R})}{d\bar{R}} = F^{-1}(\bar{R}) = \left(1 - \bar{R}^{-1}\right)^{-1}. \tag{210}$$

The completion of (207) into a point transformation is straightforward:

$$P_{\bar{T}}(r) = V_{,\bar{T}}(r)P_V(r) + U_{,\bar{T}}(r)P_U(r) = \frac{1}{2}\Big(V(r)P_V(r) - U(r)P_U(r)\Big), \tag{211}$$

$$P_{\bar{\mathsf{R}}}(r) = V_{,\bar{R}}(r)P_V(r) + U_{,\bar{R}}(r)P_U(r) = F^{-1}\Big(\mathcal{R}\big(U(r)V(r)\big)\Big)\,\frac{1}{2}\Big(V(r)P_V(r) + U(r)P_U(r)\Big). \tag{212}$$

We assume that the Kruskal variables $U(r), P_U(r), V(r), P_V(r)$ are continuous. The transformation equations (207), (211), and (212) reveal that the curvature variables $\bar{\mathsf{R}}(r)$ and $P_{\bar{T}}(r)$ will also be continuous, whereas $\bar{T}(r)$ and $P_{\bar{\mathsf{R}}}(r)$ become infinite when $U(r) = 0$ or $V(r) = 0$.

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