Scattering of a tachyon by a two-dimensional black hole

J. Y. Kim, H. W. Lee, and Y. S. Myung
Department of Physics, Inje University, Kimhae 621-749, Korea
(Received 23 February 1994)

We study the scattering process of a black hole in two-dimensional string theory at the tree level. Unlike the higher-dimensional black holes, the linearized equations of motion lead to exactly solvable one-dimensional Schrödinger-type equations for the unit of $M = 1$ (M is the black hole mass). Considering the asymptotic form of the solutions, the exact S matrices are derived for the combined graviton-dilaton mode as well as the tachyon mode. We confirm that only the tachyon mode plays an important role in extracting any information for the black hole.

PACS number(s): 04.70.Bw, 11.25.Db, 11.55.Ds

I. INTRODUCTION

String theory in two spacetime dimensions has classical solutions which are analogues of black holes and provides a simplified context in which to study black hole physics [1,2]. Stringy black hole solutions may play a role of toy model for a $d = 4$ real black hole. In all stringy black holes, the dilaton and tachyon field play crucial roles [3].

An easy way of understanding the attributes of a physical system is to find out how it reacts to external perturbations and, in the first instant, to infinitesimal perturbations. In the case of a black hole, this is the only method available to us because there is no other way in which an external observer can explore the other side of the horizon. The reaction of an object to an infinitesimal perturbation is determined by the enumeration of the so-called normal modes of oscillation. For the black hole, this enumeration reduces to finding how a black hole reacts to incident waves of different sorts. The solution of this latter problem bears on the stability of a black hole and the determination of the quasinormal modes [4,5]. From general considerations, one may expect that a fraction of the energy in the incident waves will be irreversibly absorbed by the black hole, while the remaining fraction will be scattered (or reflected) back to infinity. In other words, it would appear that it may be possible to visualize the black hole as presenting an effective potential barrier (or well) to the oncoming waves [6].

In this paper we study the scattering process of the black hole in two-dimensional string theory [7]. We find a potential barrier for the tachyonic mode (t), a potential well for one graviton-dilaton mode ($h+\varphi$) and no potential for the other graviton-dilaton mode ($h-\varphi$). Also we find exact solutions for the unit of $M = 1$ (M is the black hole mass). From the asymptotic form of these solutions, we obtain the transmission and reflection coefficients for graviton-dilaton ($h+\varphi$) and tachyon (t) modes. It is found that the transmission amplitude for graviton-dilaton mode ($h+\varphi$) is a pure phase and thus there is no reflection. This means that this mode propagates freely from $+\infty$ to $-\infty$. However, the tachyon mode (t) plays the crucial role in getting any information from the black hole.

II. FORMALISM

Let us start with the σ-model action of $d = 2$ critical string theory for a graviton ($g_{\mu\nu}$), dilaton (Φ), and tachyon (T) ($\mu, \nu = 0, 1$) given by [2,8]

$$S_\sigma = \frac{1}{8\pi\alpha'} \int d^2x\sqrt{G}[g_{\mu\nu}\nabla x^\mu \nabla x^\nu + \alpha' R\Phi + 2T] \ . \quad (1)$$

The conformal invariance requires the β-function equations

$$R_{\mu\nu} + \nabla_\mu\nabla_\nu\Phi + \nabla_\mu T \nabla_\nu T = 0 \ , \quad (2)$$

$$R + (\nabla\Phi)^2 + 2\nabla^2\Phi + (\nabla T)^2 - 2T^2 - 8 = 0 \ , \quad (3)$$

$$\nabla^2 T + \nabla\Phi\nabla T + 2T = 0 \ . \quad (4)$$

These equations can also be derived from the $d = 2$ target space effective action [9] with substitutions

$$-P\Phi_{\rm PST} \to \Phi, \quad T_{\rm PST} \to \sqrt{2}T, \quad -R_{\rm PST} \to R \ . \quad (5)$$

Here we choose the tachyon potential term $-T^2$, which determines the tachyon mass, and neglect all higher-order terms. In this sense we consider string theory at the tree level. The classical equations of motion for Φ and T lead to (3) and (4), respectively. However, the equation of motion for $g_{\mu\nu}$ is given by

$$\nabla^2\Phi g_{\mu\nu} - \nabla_\mu\nabla_\nu\Phi - \nabla_\mu T\nabla_\nu T + \tfrac{1}{2}g_{\mu\nu}[(\nabla\Phi)^2 + (\nabla T)^2 - 2T^2 - 8] = 0 \ . \quad (6)$$

Contracting (6) with $g_{\mu\nu}$ leads to the dilaton equation as

$$\nabla^2\Phi + (\nabla\Phi)^2 - 2T^2 - 8 = 0 \ . \quad (7)$$

Substituting (7) into (3), one finds the contracted version of Einstein's equation,

$$R + \nabla^2\Phi + (\nabla T)^2 = 0 \ , \quad (8)$$

0556-2821/94/50(6)/3942(7)/$06.00

which can also be obtained from (2). Note that the tachyon source terms in the equations for the graviton (2) and dilaton (7) are at least quadratic in T, while the tachyon equation itself is linear in T to leading order. This means that one can consistently view a weak tachyon field as a small perturbation and thus ignore its back reaction on the background geometry to leading order [9].

Let us look for the static background solutions of the graviton-dilaton sector in the absence of the tachyon. Substituting the ansatz

$$\overline{\Phi} = 2Q\phi, \quad \overline{T} = 0, \quad \overline{g}_{\mu\nu} = \begin{pmatrix} -f & 0 \\ 0 & f^{-1} \end{pmatrix},$$

into (7) and (8), one finds

$$2Qf' + 4Q^2 f - 8 = 0 , \tag{9}$$

$$f'' + 2Qf' = 0 , \tag{10}$$

where a prime denotes the differentiation with respect to ϕ. From (9) and (10), one finds a one-parameter family of physically static geometries [2,3]:

$$f = 1 - Me^{-2Q\phi}, \quad Q = \sqrt{2} ,$$

where the parameter M (> 0) is proportional to the mass of the black hole. All these solutions approach the linear dilaton vacuum in the asymptotically flat region ($\phi \to +\infty$). The horizon of the black hole occurs at $\phi_{\rm EH} = (1/2Q)\ln M$. Here, for simplicity we choose the unit of $M = 1$ and $\phi_{\rm EH} = 0$. As a consequence of the fact that $\phi_{\rm EH} = 0$ defines a null surface, the space interior to $\phi = 0$ is incommunicable to the space outside.

To study the scattering problem specifically, we introduce the small perturbed fields $h_{\mu\nu}(\phi,\tau)$, $\varphi(\phi,\tau)$, and $t(\phi,\tau)$ as [5–10]

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu} = \overline{g}_{\mu\nu}[1 - h(\phi,\tau)],$$

$$\Phi = \overline{\Phi} + \varphi(\phi,\tau) ,$$

$$T = \overline{T} + \tilde{t} \equiv \exp\left(-\frac{\overline{\Phi}}{2}\right)[0 + t(\phi,\tau)] .$$

To obtain the linearized equations governing the perturbations, we linearize (2), (4), and (7) as

$$\delta R_{ab}(h) + \overline{\nabla}_a \overline{\nabla}_b \varphi - \delta\Gamma^c_{ab}(h)\overline{\nabla}_c \overline{\Phi} = 0 , \tag{11}$$

$$\overline{\nabla}_a \overline{\nabla}^a \varphi - \overline{g}^{ab}\delta\Gamma^c_{ab}\overline{\nabla}_c\overline{\Phi} - h^{ab}\overline{\nabla}_a\overline{\nabla}_b\overline{\Phi} + 2\overline{g}^{ab}\overline{\nabla}_a\overline{\Phi}\,\overline{\nabla}_b\varphi - h^{ab}\overline{\nabla}_a\overline{\Phi}\,\overline{\nabla}_b\overline{\Phi} = 0 , \tag{12}$$

$$\overline{\nabla}_a \overline{\nabla}^a \tilde{t} + \overline{g}^{ab}\overline{\nabla}_a\bar{\Phi}\overline{\nabla}_b\tilde{t} + 2\tilde{t} = 0 , \tag{13}$$

where

$$\delta R_{ab}(h) = \tfrac{1}{2}\overline{\nabla}_a\overline{\nabla}_b h^c{}_c + \tfrac{1}{2}\overline{\nabla}^c\overline{\nabla}_c h_{ab} - \tfrac{1}{2}\overline{\nabla}^c\overline{\nabla}_b h_{ca} - \tfrac{1}{2}\overline{\nabla}^c\overline{\nabla}_a h_{bc} ,$$

$$\delta\Gamma^c_{ab}(h) = \tfrac{1}{2}\overline{g}^{cd}(\overline{\nabla}_b h_{ad} + \overline{\nabla}_a h_{bd} - \overline{\nabla}_d h_{ab}) .$$

Here a,b,c,d denote the orthonormal indices and the overbar indicates a background object.

First, let us consider the graviton-dilaton modes. Equation (11) implies

$$\overline{\nabla}^2_\phi h_{\tau\tau} + \overline{\nabla}^2_\tau h_{\phi\phi} + 2\overline{g}_{\phi\phi}\overline{\nabla}^2_\tau\varphi - 2Q\overline{g}_{\tau\tau}\overline{\nabla}_\phi h = 0 , \tag{14}$$

$$\overline{\nabla}^2_\phi h_{\tau\tau} + \overline{\nabla}^2_\tau h_{\phi\phi} + 2\overline{g}_{\tau\tau}\overline{\nabla}^2_\phi\varphi + 2Q\overline{g}_{\tau\tau}\overline{\nabla}_\phi h = 0 , \tag{15}$$

$$\overline{\nabla}_\phi\overline{\nabla}_\tau\varphi + Q\overline{\nabla}_\tau h = 0 . \tag{16}$$

From (14) + (15), one finds

$$\left(f^2\frac{\partial^2}{\partial\phi^2} + f'f\frac{\partial}{\partial\phi} - \frac{\partial^2}{\partial\tau^2}\right)(h - \varphi) = 0 . \tag{17}$$

Since the region interior to the horizon ($\phi < 0$) is of no relevance to our consideration, let us introduce the new coordinate (ϕ^*)

$$\phi^* \equiv \phi + \frac{1}{2Q}\ln(1 - Me^{-2Q\phi}) .$$

Note that ϕ^* ranges from $-\infty$ to $+\infty$, while ϕ ranges from the event horizon of the black hole ($\phi_{\rm EH} = 0$) to $+\infty$. Defining $H \equiv h - \varphi$, (17) can be rewritten as

$$\left(\frac{\partial^2}{\partial\phi^{*2}} - \frac{\partial^2}{\partial\tau^2}\right)H = 0 . \tag{18}$$

Here we take the trial solution of the form

$$H(\phi^*,\tau) = I(\phi^*)e^{-ik\tau} , \tag{19}$$

and substitute this into (18), then we have the free-field equation

$$\left(\frac{d^2}{d\phi^{*2}} + k^2\right)I(\phi^*) = 0 . \tag{20}$$

The only allowed solution for $I(\phi^*)$ is

$$I_\infty(\phi^*) = C\exp(-ik\phi^*) \quad (\phi^* \to \infty) ,$$

$$I_{\rm EH}(\phi^*) = C\exp(-ik\phi^*) \quad (\phi^* \to -\infty) ,$$

where C is the undertermined amplitude. This means that there is no scattering of $H(= h - \varphi)$ mode by the black hole. That is, the combined graviton-dilaton mode H does not feel the presence of black hole.

Next, let us derive the Schrödinger-type equation for the other graviton-dilaton mode ($h + \varphi$). From (16), we obtain the relation

$$Qh = -\left(\frac{\partial}{\partial\phi} - \frac{1}{2}\frac{f'}{f}\right)\varphi + U(\phi). \tag{21}$$

Here, $U(\phi)$ is a function of ϕ only, and we set $U(\phi) = 0$ for simplicity. Also, from (12), one has

$$\left(f^2\frac{\partial^2}{\partial\phi^2} - f'f\frac{\partial}{\partial\phi} + 2Qf' - \frac{\partial^2}{\partial\tau^2}\right)\varphi = 0\ . \tag{22}$$

From (17) + (22)×2 together with (21), one finds the equation for the $(h + \varphi) \equiv J$ mode:

$$\left(\frac{\partial^2}{\partial\phi^{*2}} + 4Qf'f - \frac{\partial^2}{\partial\tau^2}\right)J = 0\ . \tag{23}$$

Substituting $J(\phi^*, \tau) = K(\phi^*)e^{-ik\tau}$ into (23), we obtain

$$\left(\frac{d^2}{d\phi^{*2}} + (k^2 - V_J)\right)K = 0\ , \tag{24}$$

where the potential (well) is given by

$$V_J = \frac{-8Q^2\exp(2Q\phi^*)}{[M + \exp(2Q\phi^*)]^2}\ .$$

We will consider the solution of (24) for $M = 1$ because in this case we can find an exact solution. With $M = 1$ and $Q = \sqrt{2}$, the potential V_J becomes (Fig. 1)

$$V_J = \frac{-16\exp(2\sqrt{2}\phi^*)}{[1 + \exp(2\sqrt{2}\phi^*)]^2} = \frac{V_{0J}}{(\cosh\sqrt{2}\phi^*)^2}\ , \tag{25}$$

where the depth of potential well is given by $V_{0J} = -4$. Inserting (25) into (24), we have

$$\left(\frac{d^2}{d\phi^{*2}} + k^2 + \frac{4}{(\cosh\sqrt{2}\phi^*)^2}\right)K = 0\ . \tag{26}$$

Now we derive the Schrödinger-type equation for the tachyon mode. Considering (13) together with $t(\phi^*, \tau) = L(\phi^*)e^{-ik\tau}$, we derive the equation

$$\left(\frac{d^2}{d\phi^{*2}} + (k^2 - V_T)\right)L = 0\ , \tag{27}$$

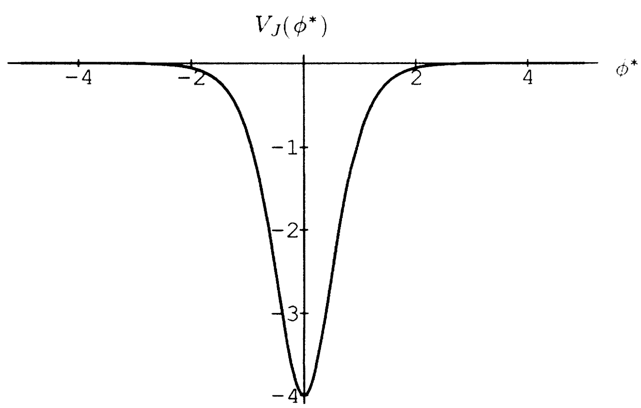

FIG. 1. Potential well of graviton-dilaton mode $(h + \varphi)$ surrounding the $d = 2$ black hole in starred coordinate (ϕ^*).

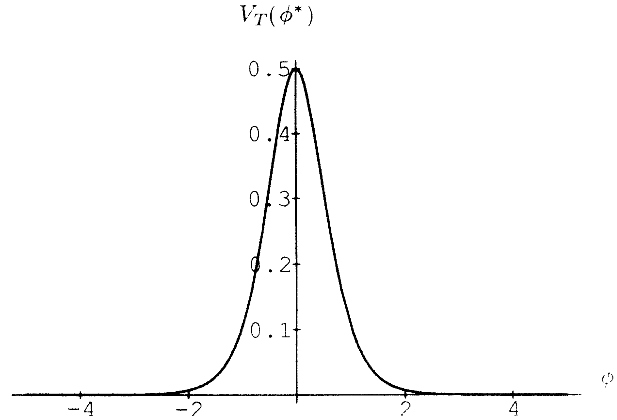

FIG. 2. Potential barrier of tachyonic mode (t) surrounding the $d = 2$ black hole in starred coordinate (ϕ^*).

where the tachyon potential (barrier) is given by

$$V_T = \frac{Q^2M\exp(2Q\phi^*)}{[M + \exp(2Q\phi^*)]^2}\ ,$$

and, with $M = 1, Q = \sqrt{2}$ (Fig. 2),

$$V_T = \frac{2\exp(2\sqrt{2}\phi^*)}{[1 + \exp(2\sqrt{2}\phi^*)]^2} = \frac{1}{2(\cosh\sqrt{2}\phi^*)^2}\ . \tag{28}$$

Substituting (28) into (27), we have the Schrödinger-type equation

$$\left(\frac{d^2}{d\phi^{*2}} + k^2 - \frac{1}{2(\cosh\sqrt{2}\phi^*)^2}\right)L = 0\ . \tag{29}$$

Comparing (29) with Eq. (26), the corresponding height of potential barrier is given by $V_{0T} = \frac{1}{2}$.

III. SOLUTION

We wish to find the general solution of the Schrödinger-type equation

$$\left(\frac{d^2}{d\phi^{*2}} + k^2 - \frac{V_0}{(\cosh\sqrt{2}\phi^*)^2}\right)\Psi = 0\ . \tag{30}$$

After solving this equation for arbitrary V_0, one can simply obtain the result for graviton-dilaton $(h + \varphi)$ with $V_0 = -4$ and for tachyon (t) with $V_0 = \frac{1}{2}$. In order to solve this Schrödinger-type equation (30), we make the substitution

$$\Psi = (\cosh\sqrt{2}\phi^*)^{-2\lambda}u\ ,$$

$$\lambda = \tfrac{1}{4}(\sqrt{1 - 2V_0} - 1)\ ,$$

then the equation for μ takes the form

$$\frac{d^2u}{d\phi^{*2}} - 4\sqrt{2}\lambda\tanh\sqrt{2}\phi^*\frac{du}{d\phi^*} + 8(\lambda^2 + \tfrac{1}{8}k^2)u = 0\ . \tag{31}$$

We consider only the scattering for $k^2 > 0$, since there is no scattering for $k^2 < 0$. If we introduce a new independent variable

$$z = -(\sinh\sqrt{2}\phi^*)^2 ,$$

then the equation for u reduces to the hypergeometric equation

$$z(1-z)\frac{d^2u}{dz^2} + [\tfrac{1}{2} - (1-2\lambda)z]\frac{du}{dz} - (\lambda^2 + \tfrac{1}{8}k^2)u = 0 . \quad (32)$$

The parameters α, β, γ which occur in the general form of the hypergeometric equation,

$$z(1-z)\frac{d^2u}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{du}{dz} - \alpha\beta u = 0 , \quad (33)$$

take in our case the values

$$\gamma = \tfrac{1}{2} \quad \alpha = -\lambda + \frac{ik}{2\sqrt{2}}, \quad \beta = -\lambda - \frac{ik}{2\sqrt{2}} .$$

The two exact solutions of Eq. (32) which lead, respectively, the even and the odd wave functions of u are of the form

$$u_1 = F\left(-\lambda + \frac{ik}{2\sqrt{2}}, -\lambda - \frac{ik}{2\sqrt{2}}, \frac{1}{2}; z\right) , \quad (34)$$

$$u_2 = \sqrt{z}F\left(-\lambda + \frac{ik}{2\sqrt{2}} + \frac{1}{2}, -\lambda - \frac{ik}{2\sqrt{2}} + \frac{1}{2}, \frac{3}{2}; z\right) . \quad (35)$$

The general form of the wave function is

$$\Phi = C_1(\cosh\sqrt{2}\phi^*)^{-2\lambda}F\left(-\lambda + \frac{ik}{2\sqrt{2}}, -\lambda - \frac{ik}{2\sqrt{2}}, \frac{1}{2}; z\right)$$
$$+C_2(\cosh\sqrt{2}\phi^*)^{-2\lambda}\sqrt{z}F\left(-\lambda + \frac{ik}{2\sqrt{2}} + \frac{1}{2}, -\lambda - \frac{ik}{2\sqrt{2}} + \frac{1}{2}, \frac{3}{2}; z\right) . \quad (36)$$

Here the coefficients C_1 and C_2 will be determined from the condition that as $\phi^* \to -\infty$ the wave function has the asymptotic form

$$\Phi \sim \exp(-ik\phi^*) ,$$

since the incident wave is coming from $\phi = +\infty$ to $\phi^* = -\infty$.

To find the asymptotic form of expression (36), we use the relation

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-z)^{-\alpha}F\left(\alpha, \alpha+1-\gamma, \alpha+1-\beta; \frac{1}{z}\right)$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z)^{-\beta}F\left(\beta, \beta+1-\gamma, \beta+1-\alpha; \frac{1}{z}\right) . \quad (37)$$

Then we derive the asympotic expressions

$$\Psi(\phi^* \to +\infty) \sim (C_1A_1 + C_2A_2)(\tfrac{1}{2})^{-ik/\sqrt{2}}\exp(-ik\phi^*)$$
$$+(C_1B_1 + C_2B_2)(\tfrac{1}{2})^{ik/\sqrt{2}}\exp(ik\phi^*) , \quad (38)$$

$$\Psi(\phi^* \to -\infty) \sim (C_1A_1 - C_2A_2)(\tfrac{1}{2})^{-ik/\sqrt{2}}\exp(ik\phi^*)$$
$$+(C_1B_1 - C_2B_2)(\tfrac{1}{2})^{ik/\sqrt{2}}\exp(-ik\phi^*) . \quad (39)$$

Here for the sake of simplicity we have introduced the notation

$$A_1 = \frac{\Gamma(1/2)\Gamma(-ik/\sqrt{2})}{\Gamma(-\lambda - ik/2\sqrt{2})\Gamma(\lambda + 1/2 - ik/2\sqrt{2})} ,$$

$$A_2 = \frac{\Gamma(3/2)\Gamma(-ik/\sqrt{2})}{\Gamma(-\lambda + 1/2 - ik/2\sqrt{2})\Gamma(\lambda + 1 - ik/\sqrt{2})} ,$$

$$B_1 = \frac{\Gamma(1/2)\Gamma(ik/\sqrt{2})}{\Gamma(-\lambda + ik/2\sqrt{2})\Gamma(\lambda + 1/2 + ik/2\sqrt{2})} ,$$

$$B_2 = \frac{\Gamma(3/2)\Gamma(ik/\sqrt{2})}{\Gamma(-\lambda + 1/2 + ik/2\sqrt{2})\Gamma(\lambda + 1 + ik/2\sqrt{2})} .$$

The first term in (38) is the incoming plane wave, the second is the reflected outgoing wave (remember that the incident wave is coming from $+\infty$ to $-\infty$). The requirement that at $\phi^* \to -\infty$ there is only a left-moving wave implies that the first term in (39) should vanish

$$C_1A_1 - C_2A_2 = 0 , \quad (40)$$

while the second term is the transmitted wave. The transmission amplitude will then be of the form

$$T = \frac{\text{transmitted}}{\text{incoming}} = \frac{(C_1B_1 - C_2B_2)\left(\tfrac{1}{2}\right)^{ik/\sqrt{2}}}{(C_1A_1 + C_2A_2)\left(\tfrac{1}{2}\right)^{-ik/\sqrt{2}}} . \quad (41)$$

Substituting (40) into (41) and performing some transformation using the relation for the Γ function,

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}\ ,$$

we have

$$T=\left(\tfrac{1}{4}\right)^{-ik/\sqrt{2}}e^{i\delta}\frac{\sinh(\pi k/\sqrt{2})}{\sinh(\pi k/\sqrt{2})+i\cos\pi(2\lambda+1/2)}\ , \quad (42)$$

where $e^{i\delta}$ is defined as

$$e^{i\delta}\equiv\frac{\Gamma(ik/\sqrt{2})\Gamma(-\lambda-ik/2\sqrt{2})\Gamma(\lambda+1/2-ik/2\sqrt{2})}{\Gamma(-ik/\sqrt{2})\Gamma(-\lambda+ik/2\sqrt{2})\Gamma(\lambda+1/2+ik/2\sqrt{2})}\ .$$

Note that $|e^{i\delta}|=1$, since $|\Gamma(z)|=|\Gamma(z^*)|$. From this, we find the transmission coefficient $\mathcal{T}=|T|^2$:

$$\mathcal{T}=\frac{[\sinh(\pi k/\sqrt{2})]^2}{[\sinh(\pi k/\sqrt{2})]^2+[\cos\pi(2\lambda+1/2)]^2}\ . \quad (43)$$

Similarly one can derive the reflection amplitude (R) and coefficient ($\mathcal{R}=|R|^2$):

$$R=\left(\tfrac{1}{4}\right)^{-ik/\sqrt{2}}e^{i\delta}\frac{\cos\pi(2\lambda+1/2)}{\cos\pi(2\lambda+1/2)-i\sinh(\pi k/\sqrt{2})}\ , \quad (44)$$

$$\mathcal{R}=\frac{[\cos\pi(2\lambda+1/2)]^2}{\sinh(\pi k/\sqrt{2})]^2+[\cos\pi(2\lambda+1/2)]^2}\ . \quad (45)$$

We apply this general form of solution to the following two cases.

A. Graviton-dilton mode

From the graviton-dilaton ($K=h+\varphi$) scattering, inserting $V_0=-4$ or equivalently $\lambda=\frac{1}{2}$, one finds the exact wave function as

$$K=C_1(\cosh\sqrt{2}\phi^*)^{-1}F\left(-\frac{1}{2}+\frac{ik}{2\sqrt{2}},-\frac{1}{2}-\frac{ik}{2\sqrt{2}},\frac{1}{2};z\right)+C_2(\cosh\sqrt{2}\phi^*)^{-1}\sqrt{z}F\left(\frac{ik}{2\sqrt{2}},-\frac{ik}{2\sqrt{2}},\frac{3}{2};z\right)\ . \quad (46)$$

The transmission amplitude and coefficient are

$$T_K=\left(\tfrac{1}{4}\right)^{-ik/\sqrt{2}}\exp(i\delta_K)\ , \quad (47)$$

where

$$\exp(i\delta_K)=\frac{\Gamma(ik/\sqrt{2})\Gamma(-1/2-ik/2\sqrt{2})\Gamma(1-ik/2\sqrt{2})}{\Gamma(-ik/\sqrt{2})\Gamma(-1/2+ik/2\sqrt{2})\Gamma(1-ik/2\sqrt{2})}\ ,$$

and

$$\mathcal{T}_K=|T_K|^2=1\ . \quad (48)$$

Note that the transmission amplitude (T_K) for $K=h+\varphi$ in (47) is a pure phase. This means that there is no reflection, i.e., $R_K=\mathcal{R}_K=0$. Even though the potential well has arisen from the black hole, the graviton-dilaton mode $K=h+\varphi$ propagates freely from $+\infty$ to $-\infty$. This corresponds to a transmission resonance.

B. Tachyon mode

For the tachyon (t) scattering, inserting $V_0=\frac{1}{2}$ or $\lambda=-\frac{1}{4}$, we have

$$L=C_1(\cosh\sqrt{2}\phi^*)^{1/2}F\left(\frac{1}{4}+\frac{ik}{2\sqrt{2}},\frac{1}{4}-\frac{ik}{2\sqrt{2}},\frac{1}{2};z\right)+C_2(\cosh\sqrt{2}\phi^*)^{1/2}\sqrt{z}\times F\left(\frac{3}{4}+\frac{ik}{2\sqrt{2}},\frac{3}{4}-\frac{ik}{2\sqrt{2}},\frac{2}{3};z\right)\ , \quad (49)$$

$$T_t=\left(\tfrac{1}{4}\right)^{-ik/\sqrt{2}}\exp(i\delta_t)\frac{\sinh(\pi k/\sqrt{2})}{\sinh(\pi k/\sqrt{2})+i}\ , \quad (50)$$

$$\mathcal{T}_t=\frac{[\sinh(\pi k/\sqrt{2})]^2}{1+[\sinh(\pi k/\sqrt{2})]^2}\ , \quad (51)$$

$$R_t=\left(\tfrac{1}{4}\right)^{-ik/\sqrt{2}}\exp(i\delta_t)\frac{1}{1-i\sinh(\pi k/\sqrt{2})}\ , \quad (52)$$

$$\mathcal{R}_t=\frac{1}{1+[\sinh(\pi k/\sqrt{2})]^2}\ , \quad (53)$$

where

$$\exp(i\delta_t)=\frac{\Gamma(ik/\sqrt{2})[\Gamma(1/4-ik/2\sqrt{2})]^2}{\Gamma(-ik/\sqrt{2})[\Gamma(1/4+ik/2\sqrt{2})]^2}\ .$$

As might be expected, one finds that $\mathcal{T}_t+\mathcal{R}_t=1$.

IV. DISCUSSION

At a first glance, it seems that black holes in two-dimensional theory is very different from that in four-dimensional theory. In the case of Schwarzschild black hole, there exist two kinds of Schrödinger-type equations from the graviton ($g_{\mu\nu}$). One is the Zerilli equation which arose in the even-parity perturbation [4–11]:

$$\frac{d^2\Psi_e}{dr^{*2}}+(k^2-V_Z)\Psi_e=0\ . \quad (54)$$

Here units are used in which $G=c=2M_4=1$, so that the horizon is at $r^*=-\infty$ ($r=1$). r^* is related to the Schwarzschild radial coordinate r by

$$\frac{d}{dr^*}=\frac{r-1}{r}\frac{d}{dr}\ .$$

In these units, the Zerilli potential V_Z is given by

$$V_Z=\frac{2(n+1)r^3+3r^2+9r/2n+9/4n^2}{r^4(r+3/2n)^2}(r-1)\ ,$$

where the parameter n, in terms of the multipole index $l \geq 2$ of the perturbation, is $n \equiv (l-1)(l+2)/2$. The other is the Regge-Wheeler (RW) equation

$$\frac{d^2\Psi_o}{dr^{*2}} + (k^2 - V_{\rm RW})\Psi_o = 0 \; , \qquad (55)$$

which differs only in the details of the potential

$$V_{\rm RW} = \frac{2(n+1)r - 3}{r^4}(r-1) \; .$$

The RW equation arose in the study of odd-parity perturbations in the same formalism. Chandrasekhar showed explicitly the connection between the Zerilli and RW equations. The existence of different descriptions of the perturbations of the black hole led Chandrasekhar to consider the general question of the relationship between two potentials which are equivalent in the sense of producing the same physical consequences [more specifically, having the same reflection ($\mathcal{R}$) and transmission ($\mathcal{T}$) coefficients]. Further, according to Anderson's formalism of intertwining operators for any potential [12], there are equivalent potentials. In fact, there exist an infinite number of equivalent potentials. In Schwarzschild black holes, for example, the Zerilli and RW potentials are only two of an infinite set of possible potentials. Of course, all these potentials belong to potential barriers (not potential wells). However, in the case of two-dimensional black hole, we found a potential well as well as a potential barrier. The graviton-dilaton mode $(h+\varphi)$ with potential well is trivial in the sense of scattering process, while the tachyonic mode with potential barrier is very important to study the two-dimensional black hole.

In order to understand this apparent discrepancy between two-dimensional and four-dimensional black holes, let us consider the counting of degrees of freedom. From d-dimensional theory, a symmetric traceless tensor field $h_{\mu\nu}$ has $d(d+1)/2 - 1$ independent components. d of which are eliminated by the gauge condition that specifies $\partial_\mu h^\mu{}_\nu = 0$ [13]. In addition, $(d-1)$ are eliminated by our freedom to make further gauge transformations $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ with $\partial_\mu \xi^\mu = 0$ and $\partial^2 \xi^\mu = 0$. Hence, the number of degrees of freedom for the gravitational field in d dimensions is

$$\tfrac{1}{2}d(d+1) - 1 - d - (d-1) = \tfrac{1}{2}d(d-3) \; .$$

for $d = 4$, we obtain two propagating physical gravitons (Zerilli and RW potentials). However, this is -1 for $d = 2$. This means that, in two dimensions, the contribution of graviton is equal and opposite to that of a spinless particle (dilaton). Thus, in a viewpoint of $d = 4$ theory, general relativity is not much of a theory in two dimensions. In this connection, the Lagrangian $\sqrt{g}R$ is a total derivative for $d = 2$, and consequently the left-hand side of Einstein equation $(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)$ vanishes identically. This is the reason why one usually chooses either light-cone gauge or conformal gauge (as in our case) in two dimensions. In analyzing two-dimensional stringy black hole, we start with one graviton, one dilaton, and one tachyon. These, in turn, give rise to one graviton dilaton $(h-\varphi)$, the other $(h+\varphi)$ and tachyonic modes. However, two modes $(h-\varphi, h+\varphi)$ are trivial in the scattering process, while the tachyonic mode is very important to extract the informations of two-dimensional black hole. In this sense, we regard the tachyonic mode as a truly propagating degree of freedom in the black hole background. If the above view of extracting the physical degrees of freedom is consistent with that of $d = 4$ general relativity, two graviton-dilaton modes are trivial gauge artefacts. The reason comes from the fact that in our theory gauge fixing is lacking from the beginning, in view of $d = 4$ gravity theory. Therefore, the net physical degrees of freedom for two-dimensional stringy black hole should be given by

$$-1(\text{graviton}) + 1(\text{dilaton}) + 1(\text{tachyon}) = 1(\text{tachyon}) \; .$$

Finally, we note that S-matrix problem of the Schwarzschild black hole (i.e., the knowledge of both $\mathcal{R}$ and $\mathcal{T}$) is incomplete. In considering this problem, it is customary to restrict oneself to the potential $V(x)$ which satisfies the requirement [6]

$$\int_{-\infty}^{\infty} (1+|x|)V(x)dx$$

is bounded. The $d = 4$ potentials V_Z and $V_{\rm RW}$ do not satisfy the above requirement. On the other hand, the $d = 2$ tachyon potential V_T satisfies the requirement that

$$\int_{-\infty}^{\infty} (1+|\phi^*|)V_T(\phi^*)d\phi^* = \frac{1}{\sqrt{2}}\left(1 + \frac{\ln 2}{\sqrt{2}}\right)$$

is finite (bounded). This implies that one can solve the S-matrix problem completely in a two-dimensional black hole.

ACKNOWLEDGMENTS

This work was supported in part by Inje Research and Scholarship Foundation and the Basic Science Institute Program, Ministry of Education, 1994, Project No. BSRI-94-2413.

[1] E. Witten, Phys. Rev. D **44**, 314 (1991).
[2] G. Mandal, A. Sengupta, and S. R. Wadia, Mod. Phys. Lett. A **6**, 1685 (1991).
[3] S. P. de Alwis and J. Lykken, Phys. Lett. B **269**, 464 (1991); A. Dhar, G. Mandal, and S. R. Wadia, Mod. Phys. Lett. A **7**, 3703 (1992); S. R. Das, *ibid.* **8**, 69 (1993).
[4] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1403 (1957); C. V. Vishveshwara, Phys. Rev. D **1**, 2870 (1970); F. J. Zerilli, Phys. Rev. Lett. **24**, 737 (1970).
[5] J. Y. Kim, H. W. Lee, and Y. S. Myung, Phys. Lett. B

328, 291 (1994).
[6] S. Chandrasekhar, in *Space and Geometry*, edited by R. A. Matzner and L. C. Shepley (Texas University Press, Austin, 1982).
[7] E. Verlinde and H. Verlinde, Nucl. Phys. **B406**, 43 (1993); M. Li, Mod. Phys. Lett. A **8**, 2481 (1993); K. Demeterfi and I. R. Klebanov, Phys. Rev. Lett. **71**, 3409 (1993); A. Strominger, Phys. Rev. D **48**, 5769 (1993).
[8] S. K. Rama, Phys. Rev. Lett. **70**, 3186 (1993).
[9] A. Peet, L. Susskind, and L. Thorlacius, Phys. Rev. D **48**, 2415 (1993).
[10] C. J. Ahn, O. J. Kwon, Y. J. Park, K. Y. Kim, and Y. D. Kim, Phys. Rev. D **47**, 1699 (1993).
[11] A. Anderson and R. H. Price, Phys. Rev. D **43**, 3147 (1991).
[12] A. Anderson, Phys. Rev. D **37**, 546 (1988); A. Anderson and R. Camporesi, Commun. Math. Phys. **130**, 61 (1991).
[13] S. Weinberg, in *General Relativity*, edited by S. W. Hawking and W. Israel (Cambridge University Press, London, 1979).