

Quantum states of string-inspired lineal gravity

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(Received 7 March 1994)

We construct quantum states for a (1+1)-dimensional gravity-matter model that is also a gauge theory based on the centrally extended Poincaré group. Explicit formulas are found, which exhibit interesting structures. For example, wave functionals are gauge invariant except for a gauge noninvariant phase factor that is the Kirillov-Kostant one-form on the (co)adjoint orbit of the group. However, no evidence for gravity-matter forces is found.

PACS number(s): 04.60.Kz, 04.60.Ds, 11.15.Tk

I. INTRODUCTION

The string-inspired model for lineal gravity [1,2] has been studied in the last few years with the aim of gleaning useful information about black hole physics. Even though many papers have been published, the quantum mechanical theory has not been solved; only semiclassical analyses of uncertain validity have been carried out. Of course the obstacle to a complete quantal solution is the intractability of quantum gravity, which persists even when the world has been dimensionally reduced to one, lineal dimension.

In this paper we report new results in our approach to the problem of quantizing string-inspired lineal gravity, once it has been reformulated as a gauge theory [2] of the extended Poincaré group [3,4]. Specifically when point particles are coupled to the gravitational degrees of freedom, the quantum states can be constructed, and we present explicit wave functionals for the one- and many-particle cases. The pure-gravity wave functionals, which had been previously found [5,6], are also discussed.

The rationale for a gauge theoretical formulation of gravity theory is the hope that familiar techniques for quantizing gauge theories can be successfully employed, thereby circumventing apparently intractable problems of quantum gravity (diffeomorphism constraints, Wheeler-DeWitt equation, etc.). Our success with the point particle problem encourages optimism.

However, another reason should be put forward in favor of the gauge theoretical formulation. When the string-inspired model was first proposed, the gravity action was taken to be

$$\bar{I}_G = \frac{1}{4\pi G} \int d^2x \sqrt{-\bar{g}} e^{-2\varphi} (\bar{R} + 4\bar{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \lambda), \quad (1.1)$$

where φ is the “dilaton” field and λ a cosmological constant. (Over bars are used to distinguish the above from a subsequent, redefined expression; see below.) Matter is

coupled only to $\bar{g}_{\mu\nu}$ in a conformally invariant manner, so the trace of the energy-momentum tensor is given solely by its quantum anomaly, proportional to the scalar curvature \bar{R} , which according to the dynamics implied by (1.1) is a nontrivial quantity. Since (1+1)-dimensional semiclassical Hawking radiation is governed by the trace anomaly, the above results would indicate that black hole phenomena, Hawking radiation, etc. arise in this model. Indeed a “black hole” classical solution to the equations has been identified [1,2].

Subsequently, it was also realized that a redefinition of variables

$$\bar{g}_{\mu\nu} = e^{2\varphi} g_{\mu\nu}, \quad (1.2a)$$

$$\eta = e^{-2\varphi}, \quad (1.2b)$$

transforms (1.1) into a much simpler expression [3]:

$$I_G = \frac{1}{4\pi G} \int d^2x \sqrt{-g} (\eta R - \lambda). \quad (1.3)$$

Moreover, since (1.2a) describes a conformal redefinition of the metric and since the matter fields are coupled conformally, the form of the matter action does not change with the redefinition (1.2) except that $g_{\mu\nu}$ replaces $\bar{g}_{\mu\nu}$. But the dynamics implied by (1.3) leads to vanishing R , so there is no trace anomaly and no black hole effects, at least semiclassically.

If one concludes that the conformal trace anomaly interferes with field redefinition as in (1.2), invalidating the equivalence theorem, so “that there is not a unique quantization of dilaton gravity” [7], the theory loses all predictive power, even as the formalism loses descriptive ability. But it may be that the above observations on the semiclassical theory are inconclusive. In this context, one should take note of the published claim that even in the original formulation (1.1) there is no trace anomaly, because any anomaly can be compensated by a shift in the dilaton field [8]. [The freedom of shifting the dilaton field is especially evident in (1.3), where it is recognized that translating η by a constant changes the action only

by the topological term $\propto \int d^2x \sqrt{-g} R$.] Moreover, in a recent calculation the "black hole" mass vanishes [9]. [In Ref. [9] mass/energy is given a gauge theoretical definition and agrees with the Arnowitt-Deser-Misner value.]

We feel that the gauge principle provides unambiguous direction on how to proceed through this maze, since gauge invariance resolves quantum field theoretic ambiguities. As we shall see, the quantum states that we construct support the claim that there is no gravity-matter interaction.

In Sec. II we review the model (1.3) in its gauge theoretical formulation and describe classical solutions. The manner in which geometry of space-time and the trajectory of a particle are encoded in a gauge theory is noteworthy for its subtlety. Section III is devoted to the formal quantum gauge theory. Section IV contains a discussion of the pure gravity quantum states; particle states are constructed in Section V. Concluding remarks comprise the last Section VI.

II. GAUGE THEORY FOR LINEAL GRAVITY AND ITS CLASSICAL SOLUTION

The model that we consider is based on the four-parameter extended Poincaré group, in 1+1 dimensions, whose Lie algebra is

$$\begin{aligned} [P_a, P_b] &= \epsilon_{ab} I, \\ [P_a, J] &= \epsilon_a{}^b P_b, \\ [P_a, I] &= [J, I] = 0. \end{aligned} \quad (2.1)$$

The central element I modifies the conventional algebra of translation generators P_a , while the (Lorentz) rotation generator J satisfies conventional commutators. Indices $(a, b) = (0, 1)$ label a (1+1)-dimensional Minkowski tangent space, with metric tensor $h_{ab} = \text{diag}(1, -1)$, which is used to raise and lower tangent-space indices. The antisymmetric symbol ϵ^{ab} is normalized by $\epsilon^{01} = 1$. Although the group is not semisimple, there exists an invariant, nonsingular bilinear form $P_a P^a - 2IJ$, which defines a metric tensor on the four-dimensional space of the adjoint representation. This metric tensor is used to move indices, so that a four-component contravariant vector V^A [$A = (a, 2, 3) = (0, 1, 2, 3)$] (transforming with the adjoint representation) is related to a covariant vector V_A (transforming with the coadjoint representation) by $V_a = h_{ab} V^b$, $V_2 = -V^3$, $V_3 = -V^2$. Thus an invariant inner product is defined by

$$\begin{aligned} \langle W, V \rangle &\equiv W_A V^A = W_a V^a - W_2 V_3 - W_3 V_2 \\ &= W^a V_a - W^3 V^2 - W^2 V^3. \end{aligned} \quad (2.2)$$

The gauge theory involves a gauge connection, which is an element of the Lie algebra,

$$A_\mu = e_\mu^a P_a + \omega_\mu J + a_\mu I \quad (2.3)$$

and into which are collected the zweibein e_μ^a , the spin connection ω_μ , and a fourth potential a_μ associated with the center I . In the usual way, one constructs from (2.1)

and (2.3) the gauge curvature:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \\ &= F_{\mu\nu}^a P_a + F_{\mu\nu}^2 J + F_{\mu\nu}^3 I, \end{aligned} \quad (2.4a)$$

$$\frac{1}{2} \epsilon^{\mu\nu} F_{\mu\nu} = \epsilon^{\mu\nu} \left\{ (D_\mu e_\nu)^a P_a + \partial_\mu \omega_\nu J + (\partial_\mu a_\nu + \frac{1}{2} e_\mu^a \epsilon_{ab} \epsilon_\nu^b) I \right\}, \quad (2.4b)$$

$$(D_\mu e_\nu)^a \equiv \partial_\mu e_\nu^a + \epsilon^a{}_b \omega_\mu e_\nu^b. \quad (2.5)$$

These quantities transform with the adjoint representation, whose properties may be determined from the structure constants of the Lie algebra (2.1). Given a group element U , then

$$A_\mu \rightarrow A_\mu^U = U^{-1} A_\mu U + U^{-1} \partial_\mu U, \quad (2.6)$$

$$F_{\mu\nu} \rightarrow F_{\mu\nu}^U = U^{-1} F_{\mu\nu} U. \quad (2.7)$$

When U is parametrized as

$$U = e^{\theta^a P_a} e^{\alpha J} e^{\beta I} \quad (2.8)$$

with local parameters $(\theta^a, \alpha, \beta)$, the transformation (2.6) in component form reads

$$e_\mu^a \rightarrow (e^U)_\mu^a = (\Lambda^{-1})^a{}_b (e_\mu^b + \epsilon^b{}_c \theta^c \omega_\mu + \partial_\mu \theta^b), \quad (2.9a)$$

$$\omega_\mu \rightarrow (\omega^U)_\mu = \omega_\mu + \partial_\mu \alpha, \quad (2.9b)$$

$$\begin{aligned} a_\mu \rightarrow (a^U)_\mu &= a_\mu - \theta^a \epsilon_{ab} e_\mu^b - \frac{1}{2} \theta^a \theta_a \omega_\mu + \partial_\mu \beta \\ &\quad + \frac{1}{2} \partial_\mu \theta^a \epsilon_{ab} \theta^b, \end{aligned} \quad (2.9c)$$

where $\Lambda^a{}_b$ is the Lorentz transformation matrix:

$$\Lambda^a{}_b = \delta^a{}_b \cosh \alpha + \epsilon^a{}_b \sinh \alpha. \quad (2.10)$$

To construct an invariant Lagrange density and action, we introduce a quartet of Lagrange multiplier fields,

$$\eta_A = (\eta_a, \eta_2, \eta_3) = (\eta_a, -\eta^3, -\eta^2), \quad (2.11)$$

transforming in the coadjoint representation:

$$\eta_a \rightarrow (\eta^U)_a = (\eta_b - \eta_3 \epsilon_{bc} \theta^c) \Lambda^b{}_a, \quad (2.12a)$$

$$\eta_2 \rightarrow (\eta^U)_2 = \eta_2 - \eta_a \epsilon^a{}_b \theta^b - \frac{1}{2} \eta_3 \theta^a \theta_a, \quad (2.12b)$$

$$\eta_3 \rightarrow (\eta^U)_3 = \eta_3. \quad (2.12c)$$

We then form an invariant by contracting η_A with $\epsilon^{\mu\nu} F_{\mu\nu}^A$, and take, for the action,

$$\begin{aligned} I_g &= \frac{1}{4\pi G} \int d^2x \frac{1}{2} \epsilon^{\mu\nu} (\eta_a F_{\mu\nu}^a + \eta_2 F_{\mu\nu}^2 + \eta_3 F_{\mu\nu}^3) \\ &= \frac{1}{4\pi G} \int d^2x \epsilon^{\mu\nu} [\eta_a (D_\mu e_\nu)^a + \eta_2 \partial_\mu \omega_\nu \\ &\quad + \eta_3 (\partial_\mu a_\nu + \frac{1}{2} e_\mu^a \epsilon_{ab} e_\nu^b)]. \end{aligned} \quad (2.13)$$

Since the Lagrange density involves the gauge invariant inner product $\langle \eta, F_{\mu\nu} \rangle$, the action is manifestly gauge invariant. One can show that I_g is equivalent to I_G and \bar{I}_G [3,4].

A gauge invariant point particle action requires introducing an additional variable, the ‘‘Poincaré coordinate’’ q^a . A first order action for a particle reads [4,10]

$$I_p = \int d\tau \left\{ p_a (D_\tau q)^a - \frac{1}{2} N (p^a p_a + m^2) \right\}, \quad (2.14)$$

$$(D_\tau q)^a \equiv \dot{q}^a + \epsilon^a_b (q^b \omega_\mu - e^b_\mu) \dot{X}^\mu. \quad (2.15)$$

The particle dynamical variables are p_a, q^a , and X^μ , each a function of τ , which is an affine parameter—(2.14) is τ -reparametrization invariant—and the overdot denotes τ differentiation. The gravitational variables ω_μ, a_μ , and e^a_μ in (2.14) and (2.15) are evaluated on the particle trajectory $X^\mu(\tau)$. The mass-shell constraint is enforced by the Lagrange multiplier $N(\tau)$. The gauge theoretical formalism also accommodates in a very natural manner various nonminimal gravity-matter interactions mediated by a velocity-dependent interaction with the potentials $(e^a_\mu, \omega_\mu, a_\mu)$ [4]. But we do not consider these elaborations here (see however the discussion in Sec. VI and in the Appendix).

When the transformation law for the gravitational variables (2.9) and (2.12) is supplemented with one for q^a and p_a ,

$$q^a \rightarrow (q^U)^a = (\Lambda^{-1})^a_b (q^b + \epsilon^b_c \theta^c), \quad (2.16)$$

$$p_a \rightarrow (p^U)_a = p_b \Lambda^b_a, \quad (2.17)$$

where the local gauge parameters (θ^a, α) are evaluated on the particle trajectory $X^\mu(\tau)$, one finds that the Lagrangian in (2.14) is gauge invariant.

[The transformation law (2.16) indicates that q^a comprise the first two components of a contravariant four-vector q^A , transforming in the adjoint representation [i.e., like (2.9) without the derivative terms] with $q^2 = -q_3 = 1$ and $q^3 = -q_2 = \frac{1}{2} q^a q_a$, so that q^A 's squared ‘‘length’’ vanishes $\langle q, q \rangle = 0$. (The third component of any contravariant vector, equivalently the fourth component of a covariant vector, is itself always gauge invariant.) Similarly, from (2.17) we conclude that p_a comprise the first two components of a covariant four-vector p_A transforming in the coadjoint representation [i.e., like (2.12)], with vanishing fourth component $p_3 = p^2 = 0$, so the squared ‘‘length’’ of p_A , $\langle p, p \rangle$, is given by $p_a p^a$ and is constrained by N to be $-m^2$. A manifestly covariant formalism and many more details about the extended Poincaré group, its properties and representations are given in Ref. [4].]

It is important to notice from (2.12) and (2.16) that a gauge transformation may be used to set η_a to zero [$\theta^a(x) = \epsilon^{ab} \eta_b(x) / \eta_3(x)$] or q^a to zero [$\theta^a(X(\tau)) = -\epsilon^a_b q^b(\tau)$]. In particular, in the gauge $q^a = 0$, the matter action (2.14) reduces to the conventional matter-gravity action. (To recognize this, one should also replace p_a by $p_b \epsilon^b_a$.) Thus, we appreciate that the Poincaré coordinate is analogous to the Higgs field in conventional gauge theoretic symmetry breaking: its presence ensures gauge invariance, while a special gauge, the unitary gauge (analogous to $q^a = 0$), exposes physical content.

Equations of motion that follow upon varying the Lagrange multiplier multiplet $\eta_A = (\eta_a, \eta_2, \eta_3)$ in I_g require a vanishing $F_{\mu\nu}$:

$$F_{\mu\nu} = 0. \quad (2.18)$$

Varying the gravitational variables $A^A_\mu = (e^a_\mu, \omega_\mu, a_\mu)$ in $I_g + I_p$ leads to an equation for the Lagrange multiplier multiplet:

$$\partial_\mu \eta + [A_\mu, \eta] = 4\pi G \epsilon_{\mu\nu} J^\nu, \quad (2.19)$$

where

$$\eta = \eta^a P_a - \eta_3 J - \eta_2 I \quad (2.20)$$

and the matter current J^μ is given by

$$J^\mu = \int d\tau j \dot{X}^\mu(\tau) \delta^2[x - X(\tau)], \quad (2.21a)$$

$$j \equiv j^a P_a + j^2 J + j^3 I \\ = -\epsilon^{ab} p_b P_a - q^a \epsilon_a^b p_b I \quad (j^2 = 0 = j_3). \quad (2.21b)$$

Varying p_a in I_p gives

$$(D_\tau q)^a = N p^a \quad (2.22)$$

with p_a satisfying the constraint

$$p_a p^a = -m^2. \quad (2.23)$$

Varying q_a in I_p leaves, with the help of (2.22),

$$\dot{p}_a = -\epsilon_a^b p_b \omega_\mu \dot{X}^\mu. \quad (2.24)$$

Finally, the variation with respect to X^μ does not produce an equation independent of the above.

A classical solution to the system is obtained by setting

$$A_\mu = 0 \quad (2.25)$$

in order to satisfy (2.18), and the general solution is a gauge transformation of (2.25). With vanishing A_μ , (2.24) becomes $\dot{p}_a = 0$ and is solved by a constant, which we choose to write as $\hat{p}_b \epsilon^b_a$, so that \hat{p}_a is timelike when it is normalized by (2.23):

$$p_a = \hat{p}_b \epsilon^b_a, \quad (2.26a)$$

$$\hat{p}_a \hat{p}^a = m^2. \quad (2.26b)$$

Equation (2.22) reduces to $\dot{q}^a = N p^a$ and is solved, using (2.26a), by

$$q^a = \hat{p}_b \epsilon^{ba} \int^\tau d\tau' N(\tau') + \hat{q}^a, \quad (2.27)$$

where \hat{q}^a are integration constants. Finally the equations for η are solved after choosing the parametrization

$$X^0(\tau) = \tau, \quad (2.28)$$

by

$$\eta = 2\pi G \epsilon [\sigma - X(t)] j + \hat{\eta} \quad (2.29)$$

since j given in (2.21b) is a constant by virtue of (2.26a) and (2.27). Here once again the $\hat{\eta}$ are integration constants and $t = x^0$, $\sigma = x^1$, $X = X^1$. Note that η_3 is gauge

invariant and so is the squared "length," $\langle \eta - \hat{\eta}, \eta - \hat{\eta} \rangle = (2\pi Gm)^2$.

The solution as it stands does not define a geometry, because A_μ and therefore e_μ^a vanish. Also the particle trajectory $X(t)$ is unspecified. Finally we observe that although a parameterization τ for the particle trajectory has been fixed in (2.28), $N(\tau)$ remains undetermined in (2.27). Thus we must answer the question of how physical information is coded in the above solution. The answer is subtle.

The general solution of the equations of motion is obtained from (2.25)–(2.29) by a gauge transformation with arbitrary parameters $\theta^a(x)$, $\alpha(x)$, and $\beta(x)$. Thus the geometrical gravitational variables now become, from (2.9) and (2.25),

$$e_\mu^a(x) = (\Lambda^{-1})^a_b \partial_\mu \theta^b(x), \quad (2.30a)$$

$$\omega_\mu(x) = \partial_\mu \alpha(x), \quad (2.30b)$$

$$a_\mu(x) = \partial_\mu \beta(x) + \frac{1}{2} \partial_\mu \theta^a(x) \epsilon_{ab} \theta^b(x). \quad (2.30c)$$

According to (2.17) and (2.26a) the momentum takes the form

$$p_a(\tau) = \hat{p}_c \epsilon^c_b \Lambda^b_a \quad (2.31)$$

while the Poincaré coordinate becomes, from (2.16) and (2.27),

$$q^a(\tau) = (\Lambda^{-1})^a_b [\hat{p}_c \epsilon^{cb} \int^\tau d\tau' N(\tau') + \hat{q}^b + \epsilon^b_c \theta^c(X(\tau))]. \quad (2.32)$$

Lastly the Lagrange multiplier multiplet η reads, according to (2.12) and (2.29),

$$\eta_a(x) = \hat{p}_a 2\pi G \epsilon [\sigma - X(t)] + [\hat{\eta}_a - \hat{\eta}_3 \epsilon_{ab} \theta^b(x)], \quad (2.33a)$$

$$\eta_2(x) = -\hat{p}_a [\hat{q}^a + \epsilon^a_b \theta^b(x)] 2\pi G \epsilon [\sigma - X(t)] + [\hat{\eta}_2 - \hat{\eta}_a \epsilon^a_b \theta^b(x) - \frac{1}{2} \hat{\eta}_3 \theta^a(x) \theta_a(x)], \quad (2.33b)$$

$$\eta_3(x) = \hat{\eta}_3. \quad (2.33c)$$

A geometrical solution is exposed in the unitary gauge $q^a = 0$ and also requires that $\partial_\mu \theta^a$ be nonsingular. These two conditions give through (2.32) two equations, which determine the function $N(\tau)$ and the trajectory $X(\tau)$. We show in an example how this works.

While we require that $e_\mu^a = \partial_\mu \theta^a$ be nonsingular, there still remains great freedom in fixing its form, i.e., of selecting θ^a . A natural choice is $e_\mu^a = \delta_\mu^a$, reflecting the fact that $R = 0$ and the space-time is flat. (Of course *any* form for θ^a gives a zweibein that describes flat space-time.) Hence we take

$$\theta^a(x) = x^a \quad (2.34)$$

and therefore

$$e_\mu^a = \delta_\mu^a, \quad (2.35a)$$

$$\omega_\mu = 0, \quad (2.35b)$$

$$a_\mu = \frac{1}{2} \epsilon_{\mu\nu} x^\nu. \quad (2.35c)$$

Note that a_μ is like an electromagnetic vector potential for a constant field $\epsilon_{\nu\mu}$, which, as is well known, produces a central extension in the algebra of translations. [Another choice, popular in the "black hole" literature, is $e_\mu^a(t, \sigma) = e^{\nu\lambda\sigma} \delta_\mu^a$. To achieve this, it is necessary to perform a local Lorentz gauge transformation as well as a local translation.]

Once $\theta^a(x)$ is chosen as in (2.34), the form of the orbit $X^\mu(\tau)$ becomes fixed by the requirement that the Poincaré coordinate $q^a(\tau)$ vanishes, i.e., in the "unitary," physical gauge. From (2.32) and (2.34) it follows that

$$\theta^a(X(\tau)) = \hat{p}^a \int^\tau d\tau' N(\tau') - \epsilon^a_b \hat{q}^b, \quad (2.36a)$$

$$X^a(\tau) = \hat{p}^a \int^\tau d\tau' N(\tau') + \hat{X}^a, \quad (2.36b)$$

where we have renamed the constant $-\epsilon^a_b \hat{q}^b$ as \hat{X}^a .

The form of the Lagrange multiplier multiplet is obtained by substituting into (2.33) the $\theta^a(x)$ of (2.34) and the $X^a(\tau)$ of (2.36b). Finally, our choice of parameterization in (2.28) and (2.36b) fixes $N(\tau)$ to be constant:

$$N(\tau) = \frac{1}{\hat{p}^0} \quad (2.37)$$

so that

$$X(t) = \pm vt + \hat{X}, \quad (2.38)$$

where $v = |\hat{p}^1/\hat{p}^0| \leq 1$ and we see that the particle is free.

Thus all aspects of the problem now attain an explicit analytic and geometric description. Notice that by virtue of (2.36a), where the condition is stated that the Poincaré coordinate vanishes *after* the gauge transformation, the Poincaré coordinate *before* the gauge transformation (2.27) has the same form as the particle path (apart from an ϵ twist).

The two-particle problem does not provide any new structure. Upon introducing an action such as (2.14) for each particle, we find that there is no interaction between the particles. We shall see that in the quantum theory the same physics holds.

III. QUANTIZATION

We quantize $I_g + I_p$ using symplectic methods appropriate to first-order Lagrangians [11] and we solve the constraints as in vector gauge theories. In the matter action, we choose the parametrization $X^0(\tau) = \tau$, so that there is a common time $t \equiv x^0$ for both the gravity and matter Lagrange densities, which may be taken as

$$\begin{aligned} \mathcal{L} = & \frac{1}{4\pi G} \{ \eta_a \dot{e}_1^a + \eta_2 \dot{\omega}_1 + \eta_3 \dot{a}_1 \} + e_0^a G_a + \omega_0 G_2 + a_0 G_3 \\ & + \left\{ p_a \dot{q}^a + p_a \epsilon^a_b (q^b \omega_1 - e_1^b) \dot{X} - \frac{1}{2} N (p^a p_a + m^2) \right\} \\ & \times \delta(\sigma - X), \end{aligned} \quad (3.1)$$

where the Gauss constraints G_A read

$$G_a = \frac{1}{4\pi G} (\eta'_a + \epsilon_a^b \eta_b \omega_1 + \eta_3 \epsilon_{ab} e_1^b) + \epsilon_a^b p_b \delta(\sigma - X), \quad (3.2a)$$

$$G_2 = -G^3 = \frac{1}{4\pi G} (\eta'_2 + \eta_a \epsilon^a_b e^b_1) - q^a \epsilon_a^b p_b \delta(\sigma - X), \quad (3.2b)$$

$$G_3 = -G^2 = \frac{1}{4\pi G} \eta'_3. \quad (3.2c)$$

We remind that the fields are functions of t and $x^1 \equiv \sigma$. The particle variables p_a , q^a , and $X \equiv X^1$ are functions

$$\mathcal{L} = \frac{1}{4\pi G} (\eta_a \dot{e}_1^a + \eta_2 \dot{\omega}_1 + \eta_3 \dot{a}_1) + (p_a \dot{q}^a + \Pi \dot{X}) \delta(\sigma - X) + e_0^a G_a + \omega_0 G_2 + a_0 G_3 - \left\{ \frac{1}{2} N (p^a p_a + m^2) + u [\Pi - p_a \epsilon^a_b (q^b \omega_1 - e^b_1)] \right\} \delta(\sigma - X). \quad (3.3)$$

The algebra of constraints reflects the algebraic underpinnings of the theory. The four Gauss law generators reproduce the Lie algebra (2.1). The nonvanishing commutators are, as expected,

$$[G_a(\sigma), G_b(\sigma')] = i \epsilon_{ab} G_3(\sigma) \delta(\sigma - \sigma'), \quad (3.4a)$$

$$[G_a(\sigma), G_2(\sigma')] = i \epsilon_a^b G_b(\sigma) \delta(\sigma - \sigma'). \quad (3.4b)$$

(In fact the above commutators, valid for any coupling constant $4\pi G$, hold separately for the gravity part and for the matter part of G_A .) Moreover the mass shell constraint (enforced by N) and the momentum constraint (enforced by u) commute with G_A and with each other. Thus all the constraints are first class and can be imposed as conditions on physical quantum states. This we now proceed to do, to begin with in the next section just for the gravity portion and then, in the following section, for the combined gravity-particle system.

IV. GRAVITATIONAL STATES

In this section, we delete the matter (particle) variables and discuss the quantum states of pure gravity [5,6]. From (3.3) it is seen that the Hamiltonian density consists solely of the Gauss constraints $G_A = (G_a, G_2, G_3)$ enforced by $A_0^A = (e_0^a, \omega_0, a_0)$. Since the algebra (3.4) shows the constraints to be first class, they may be imposed on states, and the quantum theory has no further structure. Before imposing the Gauss law constraints, let us first discuss in greater detail how gauge transformations act in the quantum theory.

Examining the explicit expressions for the G_A , we recognize that they generate by commutation the

only of t . The overdot and prime denote, respectively, differentiation with respect to t and σ .

From (3.1) we see that the field "coordinates" are (e_1^a, ω_1, a_1) , while their conjugate "momenta" are, respectively, $[-1/(4\pi G)](\eta_a, \eta_2, \eta_3)$. Also, p_a is conjugate to the Poincaré coordinate q^a . So that X possesses a conjugate momentum, we call Π the coefficient of \dot{X} in (3.1), and enforce that definition with another Lagrange multiplier u . Thus the Lagrange density that we quantize is

relevant gauge transformations on the dynamical variables, i.e., the infinitesimal forms of (2.9) and (2.12). However, we further note that whereas the full generators are needed to implement the gauge transformation on the "coordinates" $A_1^A = (e_1^a, \omega_1, a_1)$, the derivative parts of generators $\propto (\eta'_a, \eta'_2, \eta'_3)$ commute with the "momenta" $\eta_A \propto (\eta_a, \eta_2, \eta_3)$ and are not needed for effecting the gauge transformation on the "momenta." (This, of course, merely reflects the circumstance that the "coordinates" are connections, which experience inhomogeneous gauge transformations, while the "momenta" transform covariantly.)

A consequence of this difference emerges when we consider, before enforcing the Gauss law, quantum states in the Schrödinger representation as functionals either of the "coordinates" or the "momenta." Let us act on such functionals with the unitary operator \mathcal{U} that implements a finite gauge transformation U :

$$\mathcal{U} = \exp(i \int d\sigma \theta^a G_a) \exp(i \int d\sigma \alpha G_2) \exp(i \int d\sigma \beta G_3). \quad (4.1)$$

Acting on functionals of "coordinates" (i.e., connections A_1^A), \mathcal{U} gauge transforms the argument of the functional. However, when \mathcal{U} acts on functionals of "momenta" (i.e., Lagrange multipliers η_A), in addition to a gauge transformation on the argument of the functional, there arises a multiplicative phase. This can also be seen from the Fourier transform relation between functionals of "coordinates" $\Phi(A_1)$ and functionals of "momenta" $\Psi(\eta)$ [12]:

$$\Psi(\eta) = \int \mathcal{D}A_1 \exp \left[-\frac{i}{4\pi G} \int d\sigma \langle \eta, A_1 \rangle \right] \Phi(A_1), \quad (4.2)$$

$$\begin{aligned} \mathcal{U}^{-1} \Psi(\eta) &= \int \mathcal{D}A_1 \exp \left[-\frac{i}{4\pi G} \int d\sigma \langle \eta, A_1 \rangle \right] \mathcal{U}^{-1} \Phi(A_1) \\ &= \exp \left[\frac{i}{4\pi G} \int \langle \eta, dUU^{-1} \rangle \right] \int \mathcal{D}A_1 \exp \left[-\frac{i}{4\pi G} \int d\sigma \langle \eta, UA_1U^{-1} \rangle \right] \Phi(A_1) \\ &= \exp \left[\frac{i}{4\pi G} \int \langle \eta, dUU^{-1} \rangle \right] \Psi(\eta^U). \end{aligned} \quad (4.3)$$

Of course, the Gauss law demands that physical states be annihilated by the generators G_A and left invariant by \mathcal{U} . Thus, states in the "coordinate" representation are gauge invariant, while those in the "momentum" representation are gauge invariant up to a phase; i.e., they satisfy, according to (4.3),

$$\Psi(\eta^U) = \exp \left[-\frac{i}{4\pi G} \int \langle \eta, dUU^{-1} \rangle \right] \Psi(\eta). \quad (4.4)$$

It turns out to be more convenient to work in the "momentum" representation, so we seek functionals that obey (4.4), with η^U given in (2.12). Such functionals are readily constructed by satisfying the infinitesimal version of (4.4), i.e., by solving the constraint that Gauss generators (3.2) annihilate physical states:

$$\left[\eta'_a(\sigma) + i4\pi G \epsilon_a{}^b \eta_b(\sigma) \frac{\delta}{\delta \eta_2(\sigma)} + i4\pi G \eta_3(\sigma) \epsilon_{ab} \frac{\delta}{\delta \eta_b(\sigma)} \right] \Psi(\eta) = 0, \quad (4.5a)$$

$$\left[\eta'_2(\sigma) + i4\pi G \eta_a(\sigma) \epsilon^a{}_b \frac{\delta}{\delta \eta_b(\sigma)} \right] \Psi(\eta) = 0, \quad (4.5b)$$

$$\eta'_3(\sigma) \Psi(\eta) = 0. \quad (4.5c)$$

The solution to these equations is

$$\Psi(\eta) = \delta(\eta'_3) \delta([\eta^a \eta_a - 2\eta_2 \eta_3]') e^{i\Omega} \psi, \quad (4.6)$$

where ψ depends in an arbitrary fashion on the constant parts of the invariants $\eta^a \eta_a - 2\eta_2 \eta_3$ and η_3 , and the phase Ω is given by

$$\Omega = \frac{1}{8\pi G} \int \epsilon^{ab} \eta_a d\eta_b / \eta_3. \quad (4.7)$$

The only gauge noninvariant portion of (4.6) is its phase, and one easily confirms that under the gauge transformation (2.12), (4.4) is true. The phase may be reexpressed by noting that η_3 is an invariant, only whose constant part survives in (4.6); call it λ . Thus physical no-particle states are described by states of the form

$$\Psi \sim \exp \left[\frac{i}{8\pi G \lambda} \int \epsilon^{ab} \eta_a d\eta_b \right] \psi(M, \lambda), \quad (4.8)$$

where M is the constant part of the invariant $\eta_a \eta^a - 2\eta_2 \eta_3$. When reference is made to the geometrical formulation of the model, e.g., (1.3), it is established that λ is just the cosmological constant. In the *gauge* theory, this is not a parameter, but a possible value of a dynamical variable [3,4]. Also M plays the role of the "black hole" mass in the classical solution [1,2]; in the quantized gauge theory it too is a variable.

The phase (4.7) has the following group theoretical significance.

It is known that the Lie algebra for a group can be obtained from the canonical one-form $\langle K, dg g^{-1} \rangle$. Here K is a constant element of the Lie algebra, g a group element and \langle, \rangle defines an invariant inner product on the Lie algebra. [For semisimple groups this would be the

Cartan-Killing metric; otherwise, for example, in our extended Poincaré group, we use another metric, as in (2.2).] When the group generators Q are defined to be $Q = g^{-1} K g$, one finds that their Poisson brackets, as determined by the above one-form and by the symplectic two-form $d\langle K, dg g^{-1} \rangle = \langle K, dg g^{-1} dg g^{-1} \rangle$, reproduce the Lie algebra. The two-form is the Kirillov-Kostant symplectic two-form, and we similarly name the one-form. (This one-form in general is well defined only locally.)

We now show that Ω is precisely $[-1/(4\pi G)] \int \langle K, dg g^{-1} \rangle$, where K is any fixed element in the maximal Abelian subalgebra spanned by the generators J, I of the extended Poincaré group and η is identified with

$$\eta = g^{-1} K g. \quad (4.9)$$

We require that under a gauge transformation U , K is invariant while g transforms as $g \rightarrow g^U = gU$, so that $\eta \rightarrow \eta^U = U^{-1} \eta U$. It follows that

$$\begin{aligned} \langle K, dg g^{-1} \rangle &\rightarrow \langle K, dg g^{-1} + gdUU^{-1}g^{-1} \rangle \\ &= \langle K, dg g^{-1} \rangle + \langle g^{-1} K g, dUU^{-1} \rangle \\ &= \langle K, dg g^{-1} \rangle + \langle \eta, dUU^{-1} \rangle. \end{aligned} \quad (4.10)$$

Hence

$$\Psi(\eta) \propto \exp \left[-\frac{i}{4\pi G} \int \langle K, dg g^{-1} \rangle \right] \psi(M, \lambda) \quad (4.11)$$

transforms as required by (4.4) with $K = -\lambda J + (M/2\lambda)I$ and $g = \exp(\eta_a \epsilon^{ab} P_b / \lambda)$. Notice that g is defined in (4.9) only up to a left multiplication by a element h in the maximal Abelian subgroup. By replacing g with hg the phase is shifted by the boundary term $-(i/4\pi G) \int d\langle K, \ln h \rangle$, which may induce topological effects.

Explicit evaluation, when U is given as in (2.8), confirms the above, and η_a parametrizes the two-dimensional (co)adjoint orbit of the group; indeed the η_a are just the Darboux (canonical) coordinates on the reduced phase space [13].

V. WAVE FUNCTIONALS IN THE PRESENCE OF MATTER

In this section we extend the results of Sec. IV, by including matter degrees of freedom, to begin with a single point particle. We remain with the "momentum" representation for the gravity variables, but describe the particle by position variables, so the state is a functional of η_A and a function of q^a and X , while p_a and Π are realized by differentiation. The gauge transformation generators now include a matter contribution j_A , and their exponential acting on arbitrary functionals again gauge transforms the argument, and multiplies the wave functional with the same phase as in (4.3):

$$\mathcal{U}^{-1}\Psi(\eta, q, X) = \exp\left[\frac{i}{4\pi G} \int \langle \eta, dUU^{-1} \rangle\right] \Psi(\eta^U, q^U, X). \quad (5.1)$$

$$\Psi(\eta^U, q^U, X) = \exp\left[-\frac{i}{4\pi G} \int \langle \eta, dUU^{-1} \rangle\right] \Psi(\eta, q, X). \quad (5.2)$$

Thus application of the Gauss law, which requires the left side of (5.1) to be $\Psi(\eta, q, X)$, constrains the wave functional to satisfy

Once again solving this constraint is best accomplished from its infinitesimal version. We impose the requirement that the Gauss law generators annihilate the state. The resultant differential equations are as in (4.5), except the right sides now contain matter contributions:

$$\left[\eta'_a(\sigma) + i4\pi G \epsilon_a{}^b \eta_b(\sigma) \frac{\delta}{\delta \eta_2(\sigma)} + i4\pi G \eta_3(\sigma) \epsilon_{ab} \frac{\delta}{\delta \eta_b(\sigma)}\right] \Psi(\eta, q, X) = -4\pi G \delta(\sigma - X) \epsilon_a{}^b \frac{\partial}{i\partial q^b} \Psi(\eta, q, X), \quad (5.3a)$$

$$\left[\eta'_2(\sigma) + i4\pi G \eta_a(\sigma) \epsilon^a{}_b \frac{\delta}{\delta \eta_b(\sigma)}\right] \Psi(\eta, q, X) = 4\pi G \delta(\sigma - X) q^a \epsilon_a{}^b \frac{\partial}{i\partial q^b} \Psi(\eta, q, X), \quad (5.3b)$$

$$\eta'_3(\sigma) \Psi(\eta, q, X) = 0. \quad (5.3c)$$

Equations (5.3a) and (5.3c) are solved by

$$\Psi(\eta, q, X) = e^{i\Omega} \delta(\eta'_3) \hat{\Psi}(\eta_a \eta^a - 2\eta_2 \eta_3, \lambda, \rho, X), \quad (5.4)$$

where the phase is given as before by (4.7) and ρ is defined by

$$\rho^a = q^a + \eta^a(X)/\eta_3(X). \quad (5.5)$$

By virtue of (5.3c) and the functional δ function in (5.4), η_3 is the constant λ .

Imposing the remaining constraint (5.3b) leads to the following equation for $\hat{\Psi}$:

$$\left\{(\eta^a \eta_a - 2\eta_2 \eta_3)'(\sigma) + 8\pi G \lambda \delta(\sigma - X) \rho^a \epsilon_a{}^b \frac{\partial}{i\partial \rho^b}\right\} \hat{\Psi} = 0. \quad (5.6)$$

Note that ρ responds only to the Lorentz rotation part of a general gauge transformation. Hence (5.6) is a gauge invariant equation, and so also is (5.3c).

Solving (5.6) is accomplished by diagonalizing the operator $\rho^a \epsilon_a{}^b \partial / i\partial \rho^b$ so that it acquires the [continuous] eigenvalue ν :

$$\rho^a \epsilon_a{}^b \frac{\partial}{i\partial \rho^b} \Psi = \nu \hat{\Psi}. \quad (5.7)$$

This fixes the ‘‘angular’’ ρ dependence of Ψ and then (5.6) is solved by a functional δ function that evaluates $\eta^a \eta_a - 2\eta_2 \eta_3$, leaving still undetermined an arbitrary gauge invariant function of $\rho_a \rho^a$ and X :

$$\Psi(\eta, \rho, X) = e^{i\Omega} \delta(\eta'_3) \delta[(\eta^a \eta_a - 2\eta_2 \eta_3)' + 8\pi G \lambda \nu \delta(\sigma - X)] \left[\frac{\rho^0 + \rho^1}{\rho^0 - \rho^1}\right]^{i\nu/2} \psi_\nu(M, \lambda, \rho^a \rho_a, X) \quad (5.8)$$

where M is the constant part of the invariant $\eta^a \eta_a - 2\eta_2 \eta_3$.

It now remains to solve the momentum and mass shell constraints. We consider first the former—as will be seen it does not lead to any new structure, but merely eliminates the X dependence of ψ . That constraint reads

$$[\Pi + \omega_1(X) q^a \epsilon_a{}^b p_b + p_a \epsilon^a{}_b e_1^b(X)] \Psi = 0. \quad (5.9a)$$

Since Ψ satisfies the translational constraint (5.3a), $e_1^a(X) \Psi$ may be evaluated from that equation, whereupon (5.9a) becomes

$$\left[\Pi - \frac{\eta'_a(X)}{\eta_3(X)} p^a + \omega_1(X) \rho^a \epsilon_a{}^b p_b\right] \Psi = 0. \quad (5.9b)$$

Next moving the phase and the functional δ functions that are present in Ψ across the operator in (5.9b) and evaluating $\rho^a \epsilon_a{}^b \partial / i\partial \rho^b$ on its eigenvalue ν exposes the nontrivial content of the momentum constraint, as a differential equation for $\psi_\nu(M, \lambda, \rho^a \rho_a, X)$:

$$\left[\frac{\partial}{i\partial X} - \frac{\eta'_a(X)}{\lambda} \frac{\partial}{i\partial \rho^a}\right] \left[\frac{\rho^0 + \rho^1}{\rho^0 - \rho^1}\right]^{i\nu/2} \times \psi_\nu(M, \lambda, \rho^a \rho_a, X) = 0. \quad (5.9c)$$

Since ρ^a depends on X through its definition (5.5) [with $\eta_3(X) = \lambda$], we see that (5.9c) merely states that ψ_ν has no explicit X dependence. Thus the one-particle gravitational state is described by the functional

$$\begin{aligned} \Psi(\eta, q, X) = & \exp \left[\frac{i}{8\pi G \lambda} \int \epsilon^{ab} \eta_a d\eta_b \right] \delta(\eta'_3) \delta[(\eta^a \eta_a - 2\eta_2 \eta_3)' + 8\pi G \lambda \nu \delta(\sigma - X)] \\ & \times \left[\frac{\rho^0 + \rho^1}{\rho^0 - \rho^1} \right]^{i\nu/2} \psi_\nu(M, \lambda, \rho^a \rho_a) \end{aligned} \quad (5.10a)$$

with the gauge-invariant function ψ_ν to be determined by the mass shell constraint. Note that $[(\rho^0 + \rho^1)/(\rho^0 - \rho^1)]^{i\nu/2}$ may also be presented as

$$\exp \left[\frac{i}{2} \int d\sigma \nu \delta(\sigma - X) \ln \frac{\rho^0 + \rho^1}{\rho^0 - \rho^1} \right],$$

and by virtue of the second δ function, we can replace $\nu \delta(\sigma - X)$ by $-[1/(8\pi G \lambda)](\eta^a \eta_a - 2\eta_2 \eta_3)'$. Therefore after an integration by parts, the exponential reads

$$\exp \left[-\frac{i}{8\pi G \lambda} \int (\eta^c \eta_c - 2\eta_2 \eta_3) \epsilon_{ab} \hat{\rho}^a d\hat{\rho}^b \right],$$

where $\hat{\rho}^a$ is ρ^a/ρ . Hence the following expression is equivalent to (5.10a):

$$\begin{aligned} \Psi(\eta, q, X) = & \exp \left[\frac{i}{8\pi G \lambda} \int [\epsilon^{ab} \eta_a d\eta_b - (\eta^c \eta_c - 2\eta_2 \eta_3) \epsilon_{ab} \hat{\rho}^a d\hat{\rho}^b] \right] \\ & \times \delta(\eta'_3) \delta[(\eta^a \eta_a - 2\eta_2 \eta_3)' + 8\pi G \lambda \nu \delta(\sigma - X)] \psi_\nu(M, \lambda, \rho^a \rho_a). \end{aligned} \quad (5.10b)$$

The last constraint is enforced by N in Eq. (3.3):

$$\left[-\frac{\partial}{\partial \rho^a} \frac{\partial}{\partial \rho_a} + m^2 \right] \Psi = 0, \quad (5.11)$$

and with the diagonalization (5.7) it implies a second order differential equation for ψ_ν :

$$\left[\frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} - \left[\frac{m}{2} \right]^2 \frac{1}{z} + \left[\frac{\nu}{2} \right]^2 \frac{1}{z^2} \right] \psi_\nu(z) = 0. \quad (5.12)$$

The two linearly independent solutions are Bessel functions of the second type:

$$\psi_\nu(M, \lambda, \rho^a \rho_a) \propto \begin{cases} I_{i\nu}(m \sqrt{\rho^a \rho_a}), \\ K_{i\nu}(m \sqrt{\rho^a \rho_a}). \end{cases} \quad (5.13)$$

They differ in their asymptotic behavior: for large positive value of $\rho^a \rho_a$, the function $I_{i\nu}$ diverges exponentially while $K_{i\nu}$ decays exponentially. We saw in Sec. II that the classical solution does not specify the classical path until the physical gauge $q^a = 0$ is chosen. Alternatively with $q^a \neq 0$, but in the nonsingular gauge where $e_\mu^a \neq 0$, one may identify the classical trajectory with $-\epsilon_{ab} q^b$.

Since the quantal wave function depends on $\rho^a = q^a + \eta^a/\lambda$, we may interpret $\rho^a \rho_a$ as $X^2 - t^2$. The physical requirement that wave functions do not diverge at large distance would then disallow the $I_{i\nu}$ solution. This point, as well as the similarity of the quantal description to a free particle in 1+1 dimensions, are detailed in the Appendix.

Let us briefly comment on the case of several matter particles. We add in the action one interaction term (2.14) per particle. The different masses, trajectories and momenta are labeled by an index: $m_{(n)}, X_{(n)}, \Pi_{(n)}$. We also introduce a Poincaré coordinate $q_{(n)}^a$ (and the corresponding momentum $p_{(n)}^a$) per particle. [Alternatively, we can view q^a as a function of space-time, which enters in this system only through its value on the trajectories, $q^a(t, X_{(n)}) \equiv q_{(n)}^a(t)$. In the case of field theory, the complete function $q^a(t, \sigma)$ would appear [4.]. Notice that no specific nongravitational interaction between the particles has been considered.

The Gauss laws (5.3a) and (5.3b) have now on the right side a sum of δ functions peaked on the different trajectories and we obtain one momentum constraint (5.9a) and one mass-shell constraint (5.11) for each particle. The physical state for several particles thus “factorizes” and is

$$\begin{aligned} \Psi(\eta, q_{(n)}, X_{(n)}) = & \exp \left[\frac{i}{8\pi G \lambda} \int \epsilon^{ab} \eta_a d\eta_b \right] \delta(\eta'_3) \delta \left[(\eta^a \eta_a - 2\eta_2 \eta_3)' + 8\pi G \lambda \sum_n \nu_{(n)} \delta(\sigma - X_{(n)}) \right] \\ & \times \prod_n \left[\frac{\rho_{(n)}^0 + \rho_{(n)}^1}{\rho_{(n)}^0 - \rho_{(n)}^1} \right]^{i\nu_{(n)}/2} K_{i\nu_{(n)}}(m_{(n)}) \sqrt{\rho_{(n)}^a \rho_{a(n)}}, \\ \rho_{(n)}^a = & q_{(n)}^a + \eta^a(X_{(n)})/\lambda, \end{aligned} \quad (5.14)$$

which indicates that there is no interaction between the particles.

VI. DISCUSSION

Our quantization procedure does not give evidence of any gravitational force between the matter particles moving on a line. We believe that this conclusion cannot be avoided, as long as gauge invariance is maintained. The possibility of “not . . . unique quantization of dilaton gravity” [7] is eliminated by the gauge principle.

Let us however call attention to a subtle effect, that is not apparent in what has been done above, but may be relevant in other situations. The effect that we wish to discuss is most readily seen in the gauge $\eta_a=0$. When this gauge is elected, the wave function simplifies to

$$\delta(\eta'_3)\delta[\eta'_2-4\pi G\lambda v\delta(\sigma-X)] \\ \times \left[\frac{q^0+q^1}{q^0-q^1} \right]^{iv/2} K_{iv}(m\sqrt{q^a q_a}),$$

the translational gauge freedom generated by G_a is fixed, but one must also take into account the nontrivial nature of the $[G_a(\sigma), \eta_b(\sigma')]$ bracket, which is $i\epsilon_{ab}\eta_3(\sigma)\delta(\sigma-\sigma')$. Since $\ln \det[G_a, \eta_b]$ is effectively $2\delta(0)\int d\sigma \ln \eta_3(\sigma)$, the wave functional possesses a further factor $\exp(-\delta(0)\int d\sigma \ln \eta_3(\sigma))$. In our case this factor is invisible because η_3 is, according to (5.3c), the constant λ , and the factor is an irrelevant constant. Constancy of η_3 , in the presence of matter, is a consequence of the absence of a matter coupling to a_μ ; viz. j_3 vanishes. However, as we have remarked already, it is possible to introduce a nonminimal matter gravity interaction $\mathcal{B}a_\mu \dot{X}^\mu$, which changes $\eta_3(\sigma)$ to $\lambda+2\pi G\mathcal{B}\epsilon(\sigma-X)$. The finite part of $\int d\sigma \ln \eta_3(\sigma)$ may be evaluated by first differentiating with respect to X :

$$\frac{d}{dX} \int d\sigma \ln \eta_3(\sigma) = - \int \frac{d\sigma}{\eta_3(\sigma)} 4\pi G\mathcal{B}\delta(\sigma-X) \\ = - \frac{4\pi G\mathcal{B}}{\lambda},$$

and we conclude that the wave function acquires the singular factor $\exp([\delta(0)/\lambda]4\pi G\mathcal{B}X)$. [In fact the same factor emerges when the calculations of Sec. V are repeated without choosing the $\eta_a=0$ gauge but in the presence of the nonminimal $\mathcal{B}a_\mu \dot{X}^\mu$ interaction. Specifically the singular factor is encountered when solving the momentum constraint.] We do not know how to assess this singularity, which, to reiterate, does not affect the model considered in the body of this paper.

ACKNOWLEDGMENTS

We thank Dong-Su Bak and Washington Taylor IV for instructive conversations about the Kirillov-Kostant theory. This work was supported in part by funds provided by the U.S. Department of Energy (DOE) under Contract No. DE-AC02-76ER03069 (R.J.), by the U.S. National Science Foundation (NSF) under Contract

PHY-89-15286, and by the Swiss National Science Foundation (D.C.).

APPENDIX

In this Appendix we present a quantal description for the free motion of a relativistic particle in (1+1)-dimensional space-time. Our purpose is to exhibit in this familiar context formulas identical to those in the body of the paper derived from “string-inspired” gravity.

The Lagrangian is

$$L_{\text{particle}} = -\Pi_\mu \dot{X}^\mu - \frac{N}{2}(\Pi^\mu \Pi_\mu - m^2). \quad (\text{A1})$$

It contains the mass-shell constraint and is parametrization invariant. One may quantize in a parametrization invariant fashion, imposing the constraint on covariant wave functions with Π_μ replaced by $\partial/i\partial X^\mu$. In this way one is led to the equation

$$(\square + m^2)\psi(X) = 0. \quad (\text{A2})$$

Since the Lorentz generator in this theory, $\mathcal{M} = X^\mu \epsilon_\mu{}^\nu \Pi_\nu$, commutes with the mass-shell constraint, it may be additionally imposed:

$$X^\mu \epsilon_\mu{}^\nu \frac{\partial}{i\partial X^\nu} \psi(X) = -\nu \psi(X). \quad (\text{A3})$$

Clearly (A1) and (A2) are identical with (5.11) and (5.7); they possess the solution (5.13) with X^μ identified with $-\epsilon^\mu{}_\nu p^\nu$.

An alternative point of view, within which one may also justify the selection of the K -Bessel solution over the I -Bessel solution, is provided by solving the constraint first and choosing the parametrization $X^0(\tau)=\tau$. We then have

$$\Pi_0 = \sqrt{\Pi_1^2 + m^2}$$

and

$$L_{\text{particle}} \rightarrow \Pi \dot{X} - \sqrt{\Pi^2 + m^2}, \quad (\text{A4})$$

where $X^0=t$, $X^1=X$, $\Pi_1 \equiv \Pi$, with X and Π carrying a t dependence. We are not interested in energy eigenstates. Rather we seek to diagonalize the Lorentz generator, which in the parametrized formalism reads

$$\mathcal{M} = -t\Pi + X\sqrt{\Pi^2 + m^2}. \quad (\text{A5})$$

Solution of the Lorentz eigenvalue problem in X space is difficult owing to the nonlocality of the energy operator. Therefore we introduce the momentum-space wave functions $\varphi(t,p)$ and impose the symmetrized version of (A5) as an eigenvalue condition:

$$\left[-tp + \frac{i}{2} \frac{\partial}{\partial p} \sqrt{p^2 + m^2} + \sqrt{p^2 + m^2} \frac{i}{2} \frac{\partial}{\partial p} \right] \varphi_\nu = -\nu \varphi_\nu. \quad (\text{A6})$$

The solution of this first order differential equation is unique:

$$\varphi_\nu(t,p) = \exp(-it\sqrt{p^2+m^2}) \frac{(p+\sqrt{p^2+m^2})^{i\nu}}{(p^2+m^2)^{1/4}}, \quad (\text{A7})$$

where a normalization constant is fixed by

$$\int \frac{dp}{2\pi} \varphi_\nu^*(t,p) \varphi_\nu(t,p) = \delta(\nu-\nu'). \quad (\text{A8})$$

We wish to compare this with the solution within the parametrization-independent formalism. To that end, define the transform

$$\psi_\nu(t,X) = \int \frac{dp}{2\pi} \frac{e^{ipX}}{(p^2+m^2)^{1/4}} \varphi_\nu(t,p). \quad (\text{A9})$$

The reason for the additional factor of $(p^2+m^2)^{-1/4}$ in the measure is understood as follows. If $\psi_\nu(t,X)$ is to be identified with the parametrization-independent solution, it should satisfy the Klein-Gordon equation, and indeed the function ψ_ν in (A9) does so, since the time dependence of φ_ν is $\exp(-it\sqrt{p^2+m^2})$. Klein-Gordon solutions are normalized by

$$\delta(\nu-\nu') = \frac{i}{2} \int dX [\psi_\nu^*(t,X) \dot{\psi}_\nu(t,X) - \dot{\psi}_\nu^*(t,X) \psi_\nu(t,X)] \quad (\text{A10})$$

and this is seen to require the measure as in (A9) when φ_ν is normalized by (A8).

Carrying out the integration in (A9) gives

$$\psi_\nu(t,X) = e^{\mp i\nu\pi/2} \frac{m^{i\nu}}{\pi} \left[\frac{X+t}{X-t} \right]^{i\nu/2} K_{i\nu}(m\sqrt{X^2-t^2}) \quad (\text{A11})$$

with the upper [lower] sign if $(X-t) > 0$ [$(X-t) < 0$]. Thus we arrive unambiguously at the solution that is well behaved in spacelike directions, and in this way motivate the choice made in the text of discarding the I solution.

Finally we remark that the addition of the nonminimal $\mathcal{B}a_\mu \dot{X}^\mu$ interaction to the quantum field theory results in a wave functional that coincides, apart from the previously mentioned singular factor, with the wave function of a particle moving in an external electric field \mathcal{B} , in flat $(1+1)$ -dimensional space-time.

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