PHYSICAL REVIEW D VOLUME 50, NUMBER 6 15 SEPTEMBER 1994

General relativity as an effective field theory: The leading quantum corrections

John F. Donoghue
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01002
(Received 26 May 1994)

I describe the treatment of gravity as a quantum effective field theory. This allows a natural separation of the (known) low energy quantum effects from the (unknown) high energy contributions. Within this framework, gravity is a well-behaved quantum field theory at ordinary energies. In studying the class of quantum corrections at low energy, the dominant effects at large distance can be isolated, as these are due to the propagation of the massless particles (including gravitons) of the theory and are manifested in the nonlocal and/or nonanalytic contributions to vertex functions and propagators. These leading quantum corrections are parameter-free and represent necessary consequences of quantum gravity. The methodology is illustrated by a calculation of the leading quantum corrections to the gravitational interaction of two heavy masses.

PACS number(s): 04.60.Ds, 11.10.Lm

I. INTRODUCTION

We are used to the situation where our theories are only assumed to be provisional. They have been tested and found to be valid over a limited range of energies and distances. However, we do not know that they hold in more extreme situations. There are many examples of theories which have been superseded by new theories at higher energies, and we expect this process to continue. It is interesting to look at the incompatibility of general relativity and quantum mechanics in this light. It would not be surprising if there are new ingredients at high energy in order to have a satisfactory theory of quantum gravity. However, are there any conflicts between gravity and quantum mechanics at the energy scales that are presently accessible? If there are, it would be a major concern because it would mean our present theories are wrong in ways which cannot be blamed on new physics at high energy.

There is an apparent technical obstacle to addressing the compatibility of quantum mechanics and gravity at present energies, i.e., the nonrenormalizability of quantum gravity. Quantum fluctuations involve all energy scales, not just the energy of the external particles. Perhaps our lack of knowledge of the true high energy theory will prevent us from calculating quantum effects at low energy. In the class of renormalizable field theories, low-energy physics is shielded from this problem because the high-energy effects occur only in the shifting of a small number of parameters. When these parameters are measured experimentally, and results expressed in terms the measured values, all evidence of high-energy scales disappears or is highly suppressed [1]. However in some nonrenormalizable theories, the influence of high energy remains. For example, in the old Fermi theory of weak interactions, the ratio of the neutron decay rate to that of the muon has a contribution which diverges logarithmically at one loop. It is not the divergence itself which is the problem, as the ratio becomes finite in the standard model (with a residual effect of order $\alpha \ln M_Z^2$). More bothersome is the sensitivity to the high-energy theory—the low-energy ratio depends on whether the scale of the new physics is M_Z or 10^{14} GeV.

However, quantum predictions can be made in nonrenormalizable theories. The techniques are those of effective field theory, which has been assuming increasing importance as a calculational methodology. The calculations are organized in a systematic expansion in the energy. Effects of the high-energy theory again appear in the form of shifts in parameters which however are determined from experiment. To any given order in the energy expansion there are only a finite number of parameters, which can then be used in making predictions. Using the techniques of effective field theory, it is easy to separate the effects due to low-energy physics from those of the (unknown) high-energy theory. Indeed, even the phrasing of the question raised in the opening paragraph is a by-product of the way of thinking about effective field theory.

General relativity fits naturally into the framework of effective field theory. The gravitational interactions are proportional to the energy, and are easily organized into an energy expansion. The theory has been quantized on smooth enough background metrics [2–4]. We will explore quantum gravity as an effective field theory and find no obstacle to its successful implementation.

In the course of our study we will find a class of quantum predictions which are parameter-free (other than Newton's constant G) and which dominates over other quantum predictions in the low-energy limit. These "leading quantum corrections" are the first modifications due to quantum mechanics, in powers of the energy or inverse factors of the distance. Because they are independent of the eventual high-energy theory of gravity, depending only on the massless degrees of freedom and their low-energy couplings, these are true predictions of quantum general relativity.

The plan of the paper is as follows. In Sec. II, we briefly review general relativity and its quantization. Sec. III is devoted to the treatment of general relativity as an effective field theory, while the leading quantum corrections are described in more detail in Sec. IV. We give more details of a previously published example [5], that of the gravitational interaction around flat space, in Sec. V. Some speculative comments on the *extreme* low-energy limit, where the wavelength is on order the size and/or lifetime of the Universe, are given in Sec. VI with concluding comments in Sec. VII. An appendix gives some of the nonanalytic terms needed for the leading quantum corrections arising in loop integrals.

II. GENERAL RELATIVITY AND ITS QUANTIZATION

In this paper the metric convention is such that flat space is represented by $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$ [6,7]. The Einstein action is

$$S_{\text{grav}} = \int d^4x\sqrt{-g}\left[\frac{2}{\kappa^2}R\right] , \tag{1}$$

where $\kappa^2 = 32\pi G, g = \det g_{\mu\nu}, g_{\mu\nu}$ is the metric tensor, and $R = g^{\mu\nu}R_{\mu\nu}$:

$$\begin{aligned} R_{\mu\nu} &= \partial_\nu\Gamma^\lambda_{\mu\lambda} - \partial_\lambda\Gamma^\lambda_{\mu\nu} + \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\lambda\sigma} , \\ \Gamma^\lambda_{\alpha\beta} &= \frac{g^{\lambda\sigma}}{2}\left(\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}\right) . \end{aligned} \tag{2}$$

Heavy spinless matter fields interact with the gravitational field as described by the action

$$S_{\text{matter}} = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right] . \tag{3}$$

The quantum fluctuations in the gravitational field may be expanded about a smooth background metric $\bar{g}_{\mu\nu}$, with one possible choice being

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} , \\ g^{\mu\nu} &= \bar{g}^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^\mu_\lambda h^{\lambda\nu} + \cdots . \end{aligned} \tag{4}$$

The indices here and in subsequent formulas are raised and lowered with the background metric. In order to quantize the field $h_{\mu\nu}$, we need to fix the gauge. Following 't Hooft and Veltman [3] this entails a gauge-fixing term

$$\mathcal{L}_{GF} = \sqrt{-\bar{g}}\left\{\left(h_{\mu\nu}{}^{;\nu} - \frac{1}{2}h_{;\mu}\right)\left(h^{\mu\lambda}{}_{;\lambda} - h^{;\mu}\right)\right\} , \tag{5}$$

with $h \equiv h^\lambda_\lambda$, and with the semicolon denoting covariant differentiation on the background metric. The ghost Lagrangian is

$$\mathcal{L}_{\text{ghost}} = \sqrt{-\bar{g}}\eta^{*\mu}\left\{\eta_{\mu;\lambda}{}^{;\lambda} - \bar{R}_{\mu\nu}\eta^\nu\right\} \tag{6}$$

for the complex Faddeev-Popov ghost field η_μ.

The expansion of the Einstein action takes the form [3,8]

$$S_{\text{grav}} = \int d^4x\sqrt{-\bar{g}}\left[\frac{2\bar{R}}{\kappa^2} + \mathcal{L}^{(1)}_g + \mathcal{L}^{(2)}_g + \cdots\right] \tag{7}$$

where

$$\begin{aligned} \mathcal{L}^{(1)}_g &= \frac{h_{\mu\nu}}{\kappa}\left[\bar{g}^{\mu\nu}\bar{R} - 2\bar{R}^{\mu\nu}\right] , \\ \mathcal{L}^{(2)}_g &= \frac{1}{2}h_{\mu\nu;\alpha}h^{\mu\nu;\alpha} - \frac{1}{2}h_{;\alpha}h^{;\alpha} + h_{;\alpha}h^{\alpha\beta}{}_{;\beta} - h_{\mu\beta;\alpha}h^{\mu\alpha;\beta} + \bar{R}\left(\frac{1}{4}h^2 - \frac{1}{2}h_{\mu\nu}h^{\mu\nu}\right) + \left(2h^\lambda_\mu h_{\nu\lambda} - hh_{\mu\nu}\right)\bar{R}^{\mu\nu} . \end{aligned} \tag{8}$$

A similar expansion of the matter action yields

$$S_{\text{matter}} = \int d^4x\sqrt{-\bar{g}}\left\{\mathcal{L}^0_m + \mathcal{L}^{(1)}_m + \mathcal{L}^{(2)}_m + \cdots\right\} \tag{9}$$

with

$$\begin{aligned} \mathcal{L}^{(0)}_m &= \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) , \\ \mathcal{L}^{(1)}_m &= -\frac{\kappa}{2}h_{\mu\nu}T^{\mu\nu} , \\ T_{\mu\nu} &\equiv \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\bar{g}_{\mu\nu}\left(\partial_\lambda\phi\partial^\lambda\phi - m^2\phi^2\right) , \\ \mathcal{L}^{(2)}_m &= \kappa^2\left(\frac{1}{2}h^{\mu\nu}h^\nu_\lambda - \frac{1}{4}hh^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi \\ &\quad -\frac{\kappa^2}{8}\left(h^{\lambda\sigma}h_{\lambda\sigma} - \frac{1}{2}hh\right)\left[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right] . \end{aligned} \tag{10}$$

If the background metric satisfies Einstein's equation

$$\bar{R}^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}\bar{R} = -\frac{\kappa^2}{4}T^{\mu\nu} , \tag{11}$$

the linear terms in $h_{\mu\nu}, \mathcal{L}^{(1)}_g + \mathcal{L}^{(1)}_m$, will vanish.

In calculating quantum corrections at one loop, we need to consider the Lagrangian to quadratic order:

$$\begin{aligned} S_0 &= \int d^4x\sqrt{-g}\left\{\frac{2\bar{R}}{\kappa^2} + \mathcal{L}^{(0)}_m\right\} \\ S_{\text{quad}} &= \int d^4x\sqrt{-\bar{g}}\left\{\mathcal{L}^{(2)}_g + \mathcal{L}_{GF} + \mathcal{L}_{\text{ghost}} + \mathcal{L}^{(2)}_m\right\} . \end{aligned} \tag{12}$$

The integration over the gravitational degrees of freedom,

$$W[\phi] = e^{iZ[\phi]} = \int [dh_{\mu\nu}][d\eta_\mu]e^{i(S_0+S_{\text{quad}})} , \tag{13}$$

yields a functional which in general is nonlocal and also divergent. The identification of the quantum degrees of freedom and the definition of a quantum theory is no more difficult than the quantization of Yang-Mills theory, at least for small quantum fluctuations. The difficulties arise in applying this result. Because of the dimensionful coupling κ and the nonlinear nature of the theory, divergencies appear in places which cannot be absorbed into the basic parameters introduced this far. Since the coupling grows with energy, the theory is strongly coupled at very high energy, $E > M_{\rm Pl}$, and badly behaved in perturbation theory. We also do not have known techniques for dealing with large fluctuations in the metric, which may in principle be topology-changing in nature. However, the low-energy fluctuations are very weakly coupled, with a typical strength $\kappa^2 q^2 \sim 10^{-40}[10^{-70}]$ for $q^2 \sim 1/(1\ \text{fm})^2\ [q^2 \sim 1/(1\ \text{m})^2]$. Since small quantum fluctuations at ordinary energies behave normally in perturbation theory, it is natural to try to separate these quantum corrections from the problematic (and most likely incorrect) high-energy fluctuations. Effective field theory is the tool to accomplish this separation.

III. GRAVITY AS AN EFFECTIVE FIELD THEORY

Effective field theory techniques [9,10] have become common in particle physics. The method is not a change in the rules of quantum mechanics, but is rather a procedure which organizes the calculation and separates out the effects of high energy from the known quantum effects at low energy. General relativity is a field theory which naturally lends itself to such a treatment. Perhaps the known manifestation of an effective field theory which is closest in style to gravity is chiral perturbation theory [10], representing the low-energy limit of QCD. Both are nonlinear, nonrenormalizable theories with a dimensionful coupling constant. If the pion mass were taken to zero, as can be easily achieved theoretically, long distance effects similar to those from graviton loops would be found. In addition we have had the benefit of detailed calculations and experimental verification in the chiral case, so that the workings of effective field theory are transparent. In this section, I transcribe the known properties of effective field theory to the gravitational system.

The guiding principle underlying general relativity is that of gauge symmetry, i.e., the local invariance under coordinate transformations. This forces the introduction of geometry, and requires us to define the action of the theory using quantities invariant under the general coordinate transformations. However, this is not sufficient to completely define the theory, as many quantities are invariant. For example, each term in the action

$$S_{\rm grav} = \int d^4x\sqrt{-g}\left\{\Lambda + \frac{2}{\kappa^2}R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + \cdots\right\} \qquad (14)$$

(where $\Lambda, \kappa^2, c_1, c_2$ are constants and the ellipses denote higher powers of R, $R_{\mu\nu}$, and $R_{\mu\nu\alpha\beta}$), is separately invariant under general coordinate transformations. Other physics principles must enter in order to simplify the action. For example, the constant Λ is proportional to the cosmological constant ($\lambda = -8\pi G\Lambda$), which experiment tells us is very small [11]. We therefore set (the renormalized value of) $\Lambda = 0$ for the rest of this paper. Experiment tells us very little about the dimensionless constants c_1, c_2, bounding $c_1, c_2 \leq 10^{74}$ [12], and coefficients of yet higher powers of R have essentially no experimental constraints. Einstein's theory can be obtained by setting $c_1, c_2 = 0$ as well as forbidding all higher powers. However, it is very unlikely that in fact $c_1, c_2 = 0$. For example, quantum corrections involving the known elementary particles (whether or not gravity itself is quantized) will generate corrections to c_1, c_2, etc. Unless an infinite number of "accidents" occur at least some of the higher order coefficients will be nonzero.

In practice there is no known reason to require that c_1, c_2 vanish completely. We can instead view the gravitational action as being organized in an energy expansion, and then reasonable values of c_1, c_2 do not influence physics at low energies. In order to set up the energy expansion, we note that the connection $\Gamma^\lambda_{\alpha\beta}$ is first order in derivatives and the curvature is second order. When matrix elements are taken, derivatives turn into factors of the energy or momentum $i\partial_\mu \sim p_\mu$, so that the curvature is said to be of order p^2. Terms in the action involving two powers of the curvature are of order p^4. The graviton energy can be arbitrarily small and at sufficiently low energies terms of order p^4 are small compared to those of order p^2. The higher order Lagrangians will have little effect at low energies compared to the Einstein term R. This is why the experimental bounds on c_1, c_2 are so poor; reasonable values of c_1, c_2 give modifications which are unmeasurably feeble. In a pure gravity theory the expansion scale might be expected to be the Planck mass $M_{\rm Pl}^2 \simeq G_N^{-1} \approx (1.2 \times 10^{19}\ \text{GeV})^2$. For example, if we just consider the R and R^2 terms in the Lagrangian, Einstein's equation is modified to

$$(1 + 32\pi c_1 G_N R)\left(R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + 32\pi c_1 G_N\left[R_{;\lambda}{}^{\lambda} - R_{;\mu\nu} + \frac{1}{4}g_{\mu\nu}R^2\right] = -8\pi G T^{\mu\nu}. \qquad (15)$$

Unless $c_1 R$ or $c_1 R_{;\lambda}{}^{\lambda}/R \geq 10^{70}\ \text{m}^{-2}$, the influence of the $c_1 R^2$ terms is negligible.

In the literature [13] there are discussions of problems with $R + R^2$ theories. These include negative metric states, unitarity violation, an inflationary solution, and an instability of flat space. However Simon[14] has shown that these problems do not appear when the theory is treated as an effective field theory. Essentially, the problems arise from treating the $R + R^2$ description (without any higher terms) as a fundamental theory at high energy when the curvature is of order the Planck mass squared. Then the R^2 contribution is comparable to that of R.

Of course, then yet higher powers of R would also be comparable to the R^2 and R terms, so that in this region we would not be able to say anything about the full $R + R^2 + R^3 + R^4 + \cdots$ expansion. In the low-energy region the effect of R^2 is just a small correction to the behavior of the pure Einstein theory and no bad behavior is introduced.

The most general gravitational action will have an infinite number of parameters such as κ^2, c_1, c_2. At the lowest energy, only κ^2 is important. However, we can imagine in principle that we could do experiments with such high precision that we could also see the first corrections and measure c_1, c_2. If we knew the ultimate correct theory of gravity, we might be able to predict κ, c_1, c_2. With our incomplete knowledge at low energy, we must treat them as free parameters. Quantum effects from both high-energy and low-energy particles have the potential to produce shifts in κ, c_1, c_2 and it is the final (renormalized) value which experiments would determine.

It is crucial to differentiate the quantum effects of heavy particles from those of particles which are massless. The issue is one of scale. Virtual heavy particles cannot propagate long distances at low energies; the uncertainty principle gives them a range $\Delta r \sim 1/M_H$. On distance scales much larger than this, their effects will look local, as if they were described by a local Lagrangian. Even the slight nonlocality can be accounted for by Taylor expanding the interactions about a point. In a simple example, a particle propagator can be Taylor expanded:

$$\frac{1}{q^2 - M_H^2} = \frac{-1}{M_H^2} - \frac{q^2}{M_H^4} - \frac{q^4}{M_H^6} + \cdots . \qquad (16)$$

In the coordinate space propagator obtained by a Fourier transform, the constant $1/M_H^2$ generates a δ function, hence a local interaction and the factors of q^2 are replaced by derivatives in a local Lagrangian. The quantum effects of virtual heavy particles then appear as shifts in the coefficients of most general possible local Lagrangian.

On the other hand, the quantum effects of massless particles cannot all be accounted for in this way. Some portions of their quantum corrections—for example, the results of high energy propagation in loop integrals—do shift the parameters in the Lagrangian and are local in that respect. However, the low-energy manifestations of massless particles are not all local, as the particles can propagate for long distances. A simple example is again the propagator, now $1/q^2$, which cannot be Taylor expanded about $q^2 = 0$. The low-energy particles (massless ones or ones whose mass is comparable to or less than the external energy scale) cannot be integrated out of the theory but must be included explicitly in the quantum calculations.

Our direct experience in physics covers distances from 10^{-17} m to cosmic distance scales. Although gravity is not well tested over all of those scales, we would like to believe that both general relativity and quantum mechanics are valid in that range, with likely modifications coming in at $\sim 10^{-39}$ m $\sim 1/M_{\rm pl}$. For reasons discussed more fully in Sec. VI, I would like to imagine quantizing the theory in a very large, but not infinite, box. Roughly speaking, this is to avoid asking questions about wavelengths of order the size of the Universe, i.e., reaching back to the big bang singularity. However, the volume is taken large enough that we may ignore edge effects. We assume that any particles which enter this quantization volume (e.g., the remnants of the Big Bang) are either known or irrelevant. The curvature is assumed to be small and smooth throughout this space-time volume. This situation then represents the limits of our "known" confidence in both general relativity and quantum mechanics, and we would like to construct a gravitational effective field theory (GEFT) in this region.

The dynamical information about a theory can be obtained from a path integral. The results of the true theory of gravity will be contained in a generating functional

$$\begin{aligned} W[J] &= e^{Z[J]} \\ &= \int [d\phi]\,[d(\text{gravity})]\, e^{iS_{\rm true}(\phi,(\text{gravity}),J)} , \end{aligned} \qquad (17)$$

where (gravity) represents the fields of a full gravity theory, ϕ represents matter fields, and J can be a set of source fields added to the Lagrangian (i.e., $\Delta\mathcal{L} = -J\phi$) in order to allow us to probe the generating functional. The gravitational effective field theory is defined to have the same result:

$$W[J] = e^{Z[J]} = \int [d\phi]\,[d\,h_{\mu\nu}]\, e^{iS_{\rm eff}(\phi,\bar{g},h,J)} . \qquad (18)$$

Here $S_{\rm eff}$ is constructed as the most general possible Lagrangian containing g, ϕ, and J consistent with general covariance. It contains an infinite number of free parameters, such as κ, c_1, c_2 described above. The effects of the high-energy part of the true theory are accounted for in these constants. However, as discussed above, the low-energy degrees of freedom must be accounted for explicitly, hence their inclusion in the path integral. Since we are only interested in the small fluctuation and low-energy configurations of $h_{\mu\nu}$, we do not need to address issues of the functional measure for large values of $h_{\mu\nu}$. Any measure and regularization scheme which does not violate general covariance may be used. Because the coupling of the low-energy fluctuations is so very weak, the path integral has a well-behaved perturbative expansion. We have implicitly assumed that gravitons are the only low-energy particles which are remnants of the full gravity theory. If any other massless particles result, they would also need to be included.

The most general effective Lagrangian will contain both gravitational and matter terms and will written in a derivative expansion

$$\begin{aligned} S_{\rm eff} &= \int d^4x\sqrt{-g}\,\mathcal{L}_{\rm eff} , \\ \mathcal{L}_{\rm eff} &= \mathcal{L}_{\rm grav} + \mathcal{L}_{\rm matter} , \\ \mathcal{L}_{\rm grav} &= \mathcal{L}_{g0} + \mathcal{L}_{g2} + \mathcal{L}_{g4} + \cdots , \\ \mathcal{L}_{\rm matter} &= \mathcal{L}_{m0} + \mathcal{L}_{m2} + \cdots . \end{aligned} \qquad (19)$$

The general gravitational component has already been written down:

$$\mathcal{L}_{g0} = \Lambda \ ,$$
$$\mathcal{L}_{g2} = \frac{2}{\kappa^2} R \ ,$$
$$\mathcal{L}_{g4} = c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} \ . \tag{20}$$

The first two terms in the general matter Lagrangian for a massive field are

$$\mathcal{L}_{m0} = \frac{1}{2}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right] \ ,$$
$$\mathcal{L}_{m2} = d_1 R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + R\left(d_2\partial_\mu\phi\partial^\mu\phi + d_3 m^2\phi^2\right) . \tag{21}$$

Note that derivatives acting on a massive matter field ϕ are not small quantities—the ordering in the derivative expansion only counts derivatives which act on the light fields, which in this case is only the gravitons. In contrast, if the matter field were also massless, the ordering in energy would be different:

$$\bar{\mathcal{L}}_{m0} = 0 \ ,$$
$$\bar{\mathcal{L}}_{m2} = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \bar{d}_3 R\phi^2 \ ,$$
$$\bar{\mathcal{L}}_{m4} = \bar{d}_1 R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \bar{d}_2 R\partial_\mu\phi\partial^\mu\phi + \left(\bar{d}_4 R^2 + \bar{d}_5 R_{\mu\nu}R^{\mu\nu}\right)\phi^2 \ , \tag{22}$$

where the overbar is meant to indicate the parameters in the massless theory. For $\bar{d}_3 = \frac{-1}{12}$, we have the "improved" action of Callan, Coleman, and Jackiw [15]; however, any value of $\bar{d}_3$ is consistent with general covariance and the energy expansion. Note that since we are only working to $\mathcal{O}(p^4)$, we may use the lowest order equations of motion to simplify $\mathcal{L}_{m2}, \bar{\mathcal{L}}_{m4}$.

Let us now discuss the expected size of the parameters in the effective action. Those parameters which are accessible to realistic measurements (Λ, κ, m) have been labeled by their conventional names. From the standpoint of the energy expansion it is of course a great shock that the renormalized value of the cosmological constant is so small ($\lambda = -8\pi G\Lambda$). The cosmological bound is $\mid \Lambda \mid = 10^{-46}$ GeV^4 [11], where as the standard expectation would be a value 40–60 orders of magnitude larger. Effective field theory has nothing special to say about this puzzle. However, it does indicate that at ordinary scales, Λ is unimportant and that the energy expansion for gravity starts at two derivatives with $\mathcal{L}_{g2}$. The constants c_1, c_2 are dimensionless. They determine the scale of the energy expansion of pure gravity which, in general, is

$$1 + \kappa^2 q^2 c_i \approx 1 + \frac{q^2}{\Lambda^2_{\text{grav}}}. \tag{23}$$

We of course have no direct experience with this scale, but the expectation that Λ_{grav} is of order of the Planck mass would correspond to $c_1, c_2 \approx 1$. The phenomenological bound [12], $c_1, c_2 \leq 10^{74}$, is of course nowhere close to being able to probe this possibility.

For the constants in the matter Lagrangian, d_i, the expectations are a bit more complicated, as we must distinguish between point particles which have only gravitational interactions and those which have other interactions and/or a substructure. The constants d_i have dimension $1/(\text{mass})^2$, and we will see by explicit calculation in Sec. V that they contribute to the form factors in the energy-momentum vertex for the ϕ particle, being equivalent to the charge radius in the well-known electromagnetic form factor. Loop diagrams involving gravitons shift d_i by terms of order $\kappa^2 \approx 1/M^2_{\text{Pl}}$. In the absence of interactions other than gravity, it is consistent to have $d_i = O(1/M^2_{\text{Pl}})$. However, for particles that have other interactions, the energy and momentum will be spread out due to quantum fluctuations and a gravitational charge radius will result. The expectation in this case is

$$d_1, d_2, d_3 = O\left(\langle r^2\rangle_{\text{grav}}\right) . \tag{24}$$

For composite particles, such as the proton, this will be of size of the physical radius, $\langle r^2\rangle^{\text{proton}}_{\text{grav}} \approx 1$ fm^2. For interacting point particles, such as the electron, this would be of order the scale of quantum correction $d_1, d_2, d_3 \approx \alpha/m_e^2$.

The gravitational effective field theory is a full quantum theory and loop diagrams are required in order to satisfy general principles, such as unitarity. The finite infrared part of loop diagrams will be discussed more fully in the next section. Also obtained in the usual calculation of many loop diagrams are ultraviolet divergences. These arise in a region where the low-energy effective theory may not be reliable, and hence the divergences may not be of deep significance. However as a technical matter they must be dealt with without influencing low-energy predictions. The method is known from explicit calculations of the divergences in gravity [3,4,16,17], and from general effective field theory practice. The divergences are consistent with the underlying general covariance, and must occur in forms similar to terms in the most general possible effective Lagrangian. They can then be absorbed into renormalized values for the unknown coefficients which appear in this general Lagrangian. Moreover, it can be shown that for loops involving low-order terms in the energy expansion, that the renormalization involves the coefficients which appear at higher order.

In a background field method, one can compactly summarize the one-loop quantum corrections. The classical background field $\bar{g}(x)$ is determined by the matter fields and sources by the equations of motion:

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = 16\pi G \left.\frac{\delta S_{\text{eff,matter}}}{\delta \bar{g}^{\mu\nu}}\right|_{h_{\alpha\beta}=0} . \tag{25}$$

The effective action is thus rendered into quadratic form in $h_{\mu\nu}$:

$$S_{\text{eff}} = \int d^4_x \sqrt{\bar{g}} \left\{ \bar{\mathcal{L}}(\bar{g}) - \frac{1}{2} h_{\alpha\beta} D^{\alpha\beta\gamma\delta} h_{\gamma\delta} + \cdots \right\}, \tag{26}$$

where

$$D^{\alpha\beta\gamma\delta} = I^{\alpha\beta,\mu\nu} d_\lambda d^\lambda I_{\mu\nu,}{}^{\gamma\delta} - \frac{1}{2}\bar{g}^{\alpha\beta} d_\lambda d^\lambda \bar{g}^{\gamma\delta} + \bar{g}^{\alpha\beta} d^\gamma d^\delta + d^\alpha d^\beta \bar{g}^{\gamma\delta} - 2 I^{\alpha\beta,\mu\nu} d_\sigma d_\lambda I_\mu{}^{\sigma,\gamma\delta}$$
$$+\bar{R}\left(I^{\alpha\beta,\gamma\delta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta}\right) + \left(\bar{g}^{\alpha\beta}\bar{R}^{\gamma\delta} + \bar{R}^{\alpha\beta}\bar{g}^{\gamma\delta}\right) - 4 I^{\alpha\beta,\lambda\mu}\bar{R}_{\mu\nu} I_\lambda{}^{\nu,\gamma\delta} \quad (27)$$

with d_μ being a covariant derivative and

$$I^{\alpha\beta,\gamma\delta} = \frac{1}{2}\left(\bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta} + \bar{g}^{\alpha\delta}\bar{g}^{\beta\gamma}\right). \quad (28)$$

Formally integrating over $h_{\mu\nu}$ one finds

$$Z[\phi, J] = \mathrm{Tr}\ \ln D. \quad (29)$$

While some of the finite portions of Z are difficult to extract (see the next section), since $\ln D$ is in general a nonlocal functional, the divergences are local and are readily calculated. One-loop divergences are known for gravity coupled to matter fields and the two-loop result has been found for pure gravity. At one loop, the divergences due to gravitons have the form [3]

$$\Delta\mathcal{L}_0^{(1)} = \frac{1}{8\pi^2}\frac{1}{\epsilon}\left\{\frac{1}{120}R^2 + \frac{7}{20}R_{\mu\nu}R^{\mu\nu}\right\}, \quad (30)$$

where $\epsilon = 4 - d$ within dimensional regularization. This produces the following minimal subtraction renormalization of the gravitational parameters:

$$c_1^{(r)} = c_1 + \frac{1}{960\pi^2\epsilon},$$
$$c_2^{(r)} = c_2 + \frac{7}{160\pi^2\epsilon}. \quad (31)$$

At two loops, the divergence of pure gravity is [16]

$$\Delta\mathcal{L}^{(2)} = \frac{209\,\kappa}{2880(16\pi^2)^2}\frac{1}{\epsilon}\sqrt{-g}R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}. \quad (32)$$

The key feature is that higher powers of R are involved at higher loops. This is a consequence of the structure of the energy expansion in a massless theory. A simple example can illustrate the essentials of this fact. Consider a four graviton vertex, Fig. 1(a). Since each graviton brings a factor of κ [see Eq. (4) and recall that $\kappa \sim 1/M_{\mathrm{Pl}}$], the Einstein action gives this a behavior

$$M_{\mathrm{Ein}} \sim \kappa^2 p^2, \quad (33)$$

where p is representative of the external momentum, whereas the Lagrangian at order E^4 have the behavior:

$$M_{\mathrm{HO}} \sim c_1\kappa^4 p^4. \quad (34)$$

If we use two of the Einstein vertices in a loop diagram, Fig. 1(b), the momenta could be either external or internal: for example,

$$M_{\mathrm{loop}} \sim \int d^4 l\,\kappa^2 l^2 \frac{1}{l^2}\frac{1}{(l-p)^2}\kappa^2 l^2$$
$$\sim \kappa^4 I(p) \sim \kappa^4 p^4. \quad (35)$$

If we imagine that the divergent integrals are regularized by dimensional regularization (which preserves the general covariance and which only introduces new scale-dependent factors in logarithms, not in powers), the Feynman integral must end up being expressed in terms of the external momenta. Because no masses appear for gravitons, the momentum power of the final diagrams can be obtained easily by counting powers of κ. The result is a divergence at order p^4 and can be absorbed into c_1, c_2. Loop diagrams involve more gravitons than tree diagrams, hence more factors of κ, which by dimensional analysis means that they are the same structure as higher order terms in the energy expansion.

If we were to attempt a full phenomenological implementation of gravitational effective field theory at one-loop order, the procedure would be as follows.

(1) Define the quantum degrees of freedom using the lowest order $[O(p^2)]$ effective Lagrangian, as done in Sec. II.

(2) Calculate the one-loop corrections.

(3) Combine the effects of the order p^2 and p^4 Lagrangians (given earlier in this section) at the tree level with the one-loop corrections. The divergences (and some accompanying finite parts) of the loop diagrams may be absorbed into renormalized coefficients of the Lagrangian (m, c_i, d_i), using some renormalization scheme which does not violate general covariance.

(4) Measure the unknown coefficients by comparison with some experimental measurement.

(5) Having determined the parameters of the theory, one can make predictions for other experimental observables, valid to $O(p^4)$ in the energy expansion.

In practice the difficulty arises at step (4): there is no observable that I am aware of which is sensitive to reasonable values of any of the $O(p^4)$ coefficients. However, the low-energy content of the gravitational effective field theory is not just contained in these parameters. There is a distinct class of quantum corrections, uncovered in the above procedure, which are independent of the unknown coefficients. Moreover this class, the "leading quantum

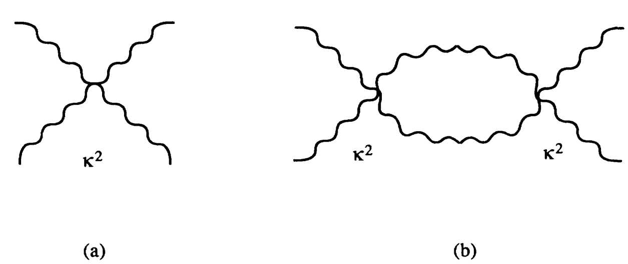

FIG. 1. (a) A tree level vertex of order $\kappa^2 p^2$; (b) a one-loop vertex of order $\kappa^4 p^4$.

corrections," are generally dominant at large distances over the other one-loop gravitational corrections. These are discussed more fully in the next section.

IV. LEADING QUANTUM CORRECTIONS

Although the ultraviolet behavior of quantum gravity has been heavily studied to learn about the behavior of general relativity as a fundamental theory, from the standpoint of effective field theory it is rather the infrared behavior which is more interesting. In the preceding section, the renormalization of the parameters in the effective action was described. Although a technical necessity, this has no predictive content. However the low-energy propagation of massless particles leads to long distance quantum corrections which are distinct from the effects of the local effective Lagrangian.

A crucial distinction in this regard is whether the effective action may be expanded in a Taylor expansion in the momentum (or equivalently in powers of derivatives). If the result is analytic, it may be represented by a series of local Lagrangians with increasing powers of derivatives. However nonanalytic effects cannot be equivalent to local contributions, and hence are unmistakable signatures of the low-energy particles. Moreover, the nonanalytic effects can be dominant in magnitude over analytic corrections. The expansion of the gravitational action is in powers of q^2 so that the first two terms of a matrix element will be

$$M_{\rm local} = A\,q^2\left(1 + \alpha\kappa^2 q^2 + \cdots\right) \ . \tag{36}$$

As we will see in the next section, a graviton loop will have a logarithmic nonanalytic modification around flat space:

$$M_{\rm full} = A\,q^2[1 + \alpha\kappa^2 q^2 + \beta\kappa^2 q^2\, ln\,(-q^2) + \cdots]. \tag{37}$$

When massive matter fields are included in loops with gravitons we may also have nonanalytic terms of the form $m/\sqrt{-q^2}$ instead of the logarithm [18]. Both of these effects have the property that they pick up imaginary components for timelike values of q^2 (i.e., for $q^2 > 0$ in this metric), as they are then part of the loop diagrams which are required for the unitarity of the S matrix. The imaginary pieces arise from the rescattering of on-shell intermediate states, and can never be contained in a local Lagrangian. In addition, since q^2 can become very small, the nonanalytic pieces will satisfy $\mid \ln(-q^2) \mid\gg 1$ and $\mid m/\sqrt{-q^2} \mid\gg 1$ at low energy, thereby dominating over the analytic effect. We can see that the nonanalytic terms have a distinct status as the leading quantum corrections due to long distance effects of massless particles.

The leading nonanalytic effects have the extra advantage that they involve only the massless degrees of freedom and the low-energy couplings of the theory, both of which are known independent of the ultimate high-energy theory. The massless particles are the gravitons, photons, and maybe neutrinos. Only the lowest energy couplings are needed, since higher order effects at the vertices introduce more powers of q^2. The low-energy couplings are contained in the Einstein action and only depend on the gravitational constant G_N. So in distinction to the analytic contributions, which depend on the unknown parameters $c_1, c_2, \ldots$, the leading quantum corrections are parameter free.

Although our prime interest above has been the quantum corrections within the gravitational part of the theory, we note that similar comments can be made if interactions other than just gravity are present. For example a theory with massless particles, such as photons in QED, can also generate nonanalytic behavior in loop amplitudes when the photons are coupled to gravity. Let us call these class II nonanalytic corrections as compared to the class I nonanalytic effects found due to the quantum behavior of gravitons. In addition there is a district type of quantum predictions (class III) which may also be predictions of the low-energy theory once we allow interactions in addition to gravity. This occurs for analytic terms in the energy expansion which are accompanied by a parameter with dimension $1/(\text{mass})^n$ $n \geq 1$. The parameters d_i in Eq. (21) are examples. The low-energy theory can generate contributions to the parameter with inverse powers of a light mass, while the Appelquist-Carazzone theorem [1] tells us that the effects of a high-energy theory would generally produce inverse powers of a heavy mass. Therefore, the low-energy contribution can be dominant, and the uncertainty caused by unknown high-energy theory is minimal. In the case of the d_i parameters, if the particles were strongly interacting QCD would generate a gravitational charge radius corresponding to $d_i \approx 1/(1\text{ GeV})^2$, which would be unlikely to be changed by the underlying quantum theory of gravity. Other examples in the case of QED plus gravity have been worked out by Behrends and Gastmans [18]. Classes II and III corrections (if present) are most often larger than the gravitational leading quantum corrections (Class I) because their form need not be expansions in the small quantity G_N. [An exception is the gravitational potential at large distance, where analytic corrections have no effect on the $1/r^3$ term.]

V. EXAMPLE: THE GRAVITATIONAL POTENTIAL

In this section, I describe in detail an example which demonstrates the extraction of the leading quantum corrections. The gravitational interaction of two heavy objects close to rest is described in lowest order by the Newtonian potential energy

$$V(r) = -\frac{Gm_1m_2}{r} \ . \tag{38}$$

This is modified in general relativity by higher order effects in v^2/c^2 and by nonlinear terms in the field equations of order $\frac{Gm}{rc^2}$ (which are of the same order as v^2/c^2). While a simple potential is not an ideal relativistic concept, the general corrections would be of the form [7]

$$V(r) = -\frac{Gm_1m_2}{r}\left[1 + a\frac{G(m_1+m_2)}{rc^2} + \cdots\right] . \quad (39)$$

The constant a would depend on the precise definition of the potential and would be calculable in the post-Newtonian expansion. At some level there will be quantum corrections also. By dimensional analysis, we can figure out the form that these should take. Since they arise from loop diagrams, they will involve an extra power of $\kappa^2 \sim G$, and if they are quantum corrections they will be at least linear in $\hbar$. If the effects are due to long-range propagation of massless particles, the other dimensionful parameter is the distance r. The combination

$$\frac{G\hbar}{r^2c^3} \quad (40)$$

is dimensionless and provides an expansion parameter for the long-distance quantum effects. We then expect a modification to the potential of the form

$$V(r) = -\frac{Gm_1m_2}{r}\left[1 + a\frac{G(m_1+m_2)}{rc^2} + b\frac{G\hbar}{r^2c^3} + \cdots\right] \quad (41)$$

and our goal is to calculate b for an appropriate definition of the potential [5].

The Newtonian potential can be found as the nonrelativistic limit of graviton exchange; see Fig. 2. In the harmonic gauge, the graviton propagator is

$$iD_{\mu\nu\alpha\beta}(q) = \frac{i}{q^2 - i\epsilon}P_{\mu\nu,\alpha\beta} \quad (42)$$

with

$$P_{\mu\nu,\alpha\beta} = \frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta}\right] \quad (43)$$

with $\eta_{\mu\nu}$ being the flat space matrix $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. The matter stress tensor has the on-shell matrix element

$$V_{0\mu\nu}(q) \equiv \langle p' \mid T_{\mu\nu} \mid p\rangle = p_\mu p'_\nu + p'_\mu p_\nu + \frac{1}{2}q^2\eta_{\mu\nu} \quad (44)$$

in our normalization convention

$$\langle p' \mid p\rangle = 2E(2\pi)^3\delta^3(p - p') . \quad (45)$$

[Here the subscript 0 indicates that this form holds before the inclusion of radiative corrections.] Graviton exchange

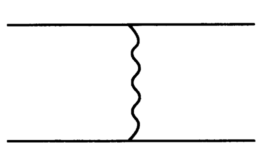

FIG. 2. One graviton exchange for the Newtonian potential. The matter fields are represented by solid lines and wavy lines represent gravitons.

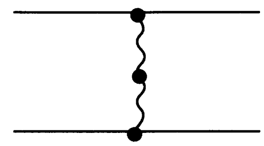

FIG. 3. The set of corrections included in the one particle reducible potential.

then yields

$$M_{12} = \frac{\kappa^2}{4}V^{(1)}_{0\mu\nu}(q)D^{\mu\nu,\alpha\beta}(q)V^{(2)}_{0\alpha\beta}(-q). \quad (46)$$

The static limit corresponds to $q_\mu = (0,\mathbf{q})_\mu$ and

$$\frac{1}{2m_1}V^{(1)}_{\mu\nu}(q) = m_1\delta_{\mu 0}\delta_{\nu 0} , \quad (47)$$

where the $\frac{1}{2m_1}$ accounts for the covariant normalization factor. The Newtonian potential is then found from the Fourier transform of

$$\frac{1}{2m_1 2m_2}M_{12} \sim -\frac{\kappa^2}{8}\frac{m_1m_2}{\mathbf{q}^2} , \quad (48)$$

which in coordinate space yields

$$V(r) = -\int\frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{r}}\frac{\kappa^2}{8}\frac{m_1m_2}{\mathbf{q}^2} = -G\frac{m_1m_2}{r}. \quad (49)$$

Of course, despite the description of quantizing, gauge fixing etc., this is purely a classical result.

In order to define a quantum potential one can consider the set of one particle reducible graphs of Fig. 3, where the heavy dots signify the full set of radiative corrections to the vertex function and the graviton propagator. These corrections are given explicitly in Figs. 4 and 5. It is this set which we will examine. Fortunately we will be able to extract the information that we need for the vacuum polarization from the work of others. This leaves the vertex correction to be worked out here.

The vertices required for the calculation follow from the Lagrangians given previously. For the vertices pictured in Fig. 6, we find

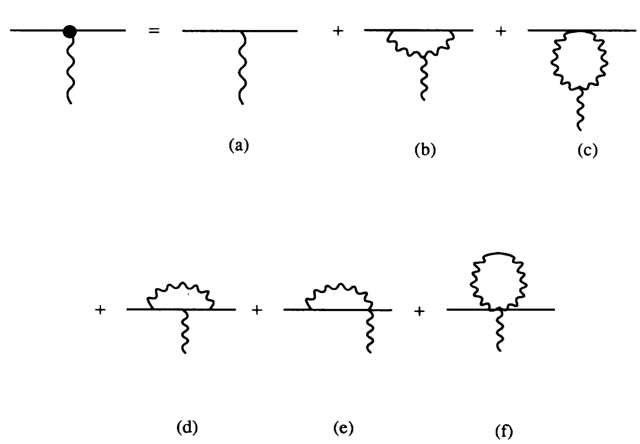

FIG. 4. The diagrams involved in the vertex correction.

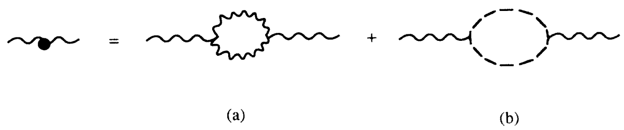

FIG. 5. The diagrams in the graviton vacuum polarization. The dotted lines indicate the ghost fields.

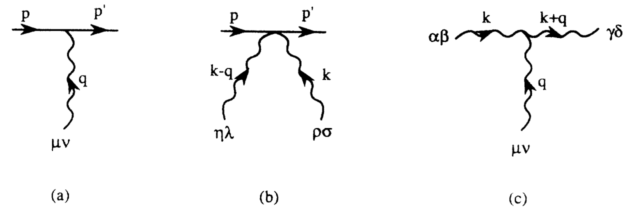

FIG. 6. Vertices required for the Feynman diagrams.

$$\text{Fig. 6(a)}: \qquad \tau_{\mu\nu}(p,p') = \frac{-i\kappa}{2}\left(p_\mu p'_\nu + p'_\mu p_\mu - g_{\mu\nu}[p\cdot p' - m^2]\right) \tag{50}$$

and

$$\text{Fig. 6(b)}: \qquad V_{\eta\lambda,\rho\sigma} = \frac{i\kappa^2}{2}\left\{ I_{\eta\lambda,\alpha\delta} I^{\delta}{}_{\beta,\rho\sigma}\left(p^\alpha p'^\beta + p^\alpha p'^\beta\right) - \frac{1}{2}\left(\eta_{\eta\lambda} I_{\rho\sigma,\alpha\beta} + \eta_{\rho\sigma} I_{\eta\lambda,\alpha\beta}\right) p'^\alpha p^\beta \right.$$
$$\left. -\frac{1}{2}\left(I_{\eta\lambda,\rho\sigma} - \frac{1}{2}\eta_{\eta\lambda}\eta_{\rho\sigma}\right)\left[p\cdot p' - m^2\right]\right\} , \tag{51}$$

where

$$I_{\alpha\beta,\gamma\delta} \equiv \frac{1}{2}\left(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma}\right) . \tag{52}$$

The graviton vertex is found most easily by using Eq. (8) plus Eq. (5) with the background metric being expanded as $\bar{g}(x) = \eta_{\mu\nu} + \kappa H^{\rm ext}_{\mu\nu}(x)$, where we pick out the vertex with one external field and two quantum fields. After some work, this can be put into the form

$$\tau^{\mu\nu}_{\alpha\beta,\gamma\delta} = \frac{i\kappa}{2}\left(P_{\alpha\beta,\gamma\delta}\left[k^\mu k^\nu + (k-q)^\mu (k-q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu} q^2\right] \right.$$
$$+2q_\lambda q_\sigma\left[I^{\lambda\sigma,}{}_{\alpha\beta} I^{\mu\nu,}{}_{\gamma\delta} + I^{\lambda\sigma,}{}_{\gamma\delta} I^{\mu\nu,}{}_{\alpha\beta} - I^{\lambda\mu,}{}_{\alpha\beta} I^{\sigma\nu,}{}_{\gamma\delta} - I^{\sigma\nu,}{}_{\alpha\beta} I^{\lambda\mu,}{}_{\gamma\delta}\right]$$
$$+\left[q_\lambda q^\mu\left(\eta_{\alpha\beta} I^{\lambda\nu,}{}_{\gamma\delta} + \eta_{\gamma\delta} I^{\lambda\nu,}{}_{\alpha\beta}\right) + q_\lambda q^\nu\left(\eta_{\alpha\beta} I^{\lambda\mu,}{}_{\gamma\delta} + \eta_{\gamma\delta} I^{\lambda\mu,}{}_{\alpha\beta}\right)\right.$$
$$\left. -q^2\left(\eta_{\alpha\beta} I^{\mu\nu,}{}_{\gamma\delta} + \eta_{\gamma\delta} I^{\mu\nu,}{}_{\alpha\beta}\right) - \eta^{\mu\nu} q^\lambda q^\sigma\left(\eta_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta} I_{\alpha\beta,\lambda\sigma}\right)\right]$$
$$+\left[2q^\lambda\left(I^{\sigma\nu,}{}_{\alpha\beta} I_{\gamma\delta,\lambda\sigma}(k-q)^\mu + I^{\sigma\mu,}{}_{\alpha\beta} I_{\gamma\delta,\lambda\sigma}(k-q)^\nu - I^{\sigma\nu,}{}_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} k^\mu - I^{\sigma\mu,}{}_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} k^\nu\right)\right.$$
$$\left. +q^2\left(I^{\sigma\mu,}{}_{\alpha\beta} I_{\gamma\delta,\sigma}{}^{\nu} + I_{\alpha\beta,\sigma}{}^{\nu} I^{\sigma\mu,}{}_{\gamma\delta}\right) + \eta^{\mu\nu} q^\lambda q_\sigma\left(I_{\alpha\beta,\lambda\rho} I^{\rho\sigma,}{}_{\gamma\delta} + I_{\gamma\delta,\lambda\rho} I^{\rho\sigma,}{}_{\alpha\beta}\right)\right]$$
$$+\left\{\left(k^2 + (k-q)^2\right)\left(I^{\sigma\mu,}{}_{\alpha\beta} I_{\gamma\delta,\sigma}{}^{\nu} + I^{\sigma\nu,}{}_{\alpha\beta} I_{\gamma\delta,\sigma}{}^{\mu} - \frac{1}{2}\eta^{\mu\nu} P_{\alpha\beta,\gamma\delta}\right)\right.$$
$$\left.\left. -\left(k^2 \eta_{\gamma\delta} I^{\mu\nu,}{}_{\alpha\beta} + (k-q)^2 \eta_{\alpha\beta} I^{\mu\nu,}{}_{\gamma\delta}\right)\right\}\right) \tag{53}$$

with $I_{\alpha\beta,\gamma\delta}$ defined in Eq. (52) and $P_{\alpha\beta,\gamma\delta}$ defined in Eq. (42).

The diagrams involved for the vertex correction are given in Fig. 4. We will argue below that Figs. 4(d)–4(f) will not contribute to the leading quantum corrections, so that we need to calculate only diagrams 4(b) and 4(c). These have the form

$$M^{\mu\nu}(3b) = \int \frac{d^4k}{(2\pi)^4} i\tau_{\eta\lambda}(p, p'-k)\frac{i}{(k-p')^2 - m^2 + i\epsilon} i\tau_{\rho\sigma}(p'-k, p') iD^{\eta\lambda,\alpha\beta}(k-q) i\tau^{\mu\nu}_{\alpha\beta,\gamma\delta} iD^{\gamma\delta,\rho\sigma}(k),$$

$$M^{\mu\nu}(3c) = \int \frac{d^4k}{(2\pi)^4} iV_{\eta\lambda,\rho\sigma} iD^{\eta\lambda,\alpha\beta}(k-q) i\tau^{\mu\nu}_{\alpha\beta,\gamma\delta} iD^{\gamma\delta,\rho\sigma}(k) . \tag{54}$$

Before proceeding, it is worth examining the structure of the answer. The general vertex may be described by two form factors:

$$\begin{aligned} V_{\mu\nu} &\equiv \langle p' \mid T_{\mu\nu} \mid p\rangle \\ &= F_1(q^2)\left[p_\mu p'_\nu + p'_\mu p_\nu + q^2\frac{\eta_{\mu\nu}}{2}\right] \\ &\quad + F_2(q^2)\left[q_\mu q_\nu - g_{\mu\nu}q^2\right] \end{aligned} \tag{55}$$

with normalization condition $F_1(0) = 1$. The expansion in energy corresponds to an expansion of the form factors in powers of q^2. The one-loop diagrams of Fig. 6 have an extra power of κ^2 compared to the tree level vertex, and κ^2 has dimensions $(\text{mass})^{-2}$, so that $\kappa^2 m^2$ and $\kappa^2 q^2$ form dimensionless combinations. However loop diagrams will also produce nonanalytic terms with the form $\ln(-q^2)$ and $\sqrt{\frac{m^2}{-q^2}}$, which also are dimensionless. Also contributing to the form factors are the terms in the higher order Lagrangian as these give extra factors of q^2. By working out these contributions and taking the general form of the loop diagrams from dimensional considerations we obtain the form factors

$$\begin{aligned} F_1(q^2) &= 1 + d_1 q^2 + \kappa^2 q^2\left(\ell_1 + \ell_2 \ln\frac{(-q^2)}{\mu^2}\right. \\ &\quad \left. + \ell_3\sqrt{\frac{m^2}{-q^2}}\right) + \cdots , \\ F_2(q^2) &= -4(d_2 + d_3)m^2 + \kappa^2 m^2\left(\ell_4 + \ell_5 \ln\frac{(-q^2)}{\mu^2}\right. \\ &\quad \left. + \ell_6\sqrt{\frac{m^2}{-q^2}}\right) + \cdots , \end{aligned} \tag{56}$$

where ℓ_i $(i = 1, 2, \ldots, 6)$ are numbers which come from the computation of the loop diagrams. There can be no corrections in $F_1(q^2)$ of the form $\kappa^2 m^2$ because of the normalization condition $F_1(0) = 1$. The constant μ^2 can be chosen arbitrarily, with a corresponding shift in the constants ℓ_1 and ℓ_4. The ellipses denote higher powers of q^2. The constants ℓ_1 and ℓ_4 will in general be divergent, while ℓ_2, ℓ_3, ℓ_5, and ℓ_6 must be finite. For q^2 timelike, $\ln(-q^2)$ and $\sqrt{\frac{m^2}{-q^2}}$ pick up imaginary parts which correspond to the physical (on-shell) intermediate states as described by unitarity. Recall that d_i represents the unknown effects of the true high-energy theory, while ℓ_1 and ℓ_4 come largely from the high-energy end of the loop integrals. For these high energies we have no way of knowing if the loop integrals are well represented by the low-energy vertices and low-energy degrees of freedom—almost certainly they are not. Therefore it is logical, as well as technically feasible, to combine ℓ_1 and ℓ_4 with the constants d_i, producing renormalized values

$$\begin{aligned} d_1^{(r)}(\mu^2) &= d_1 + \kappa^2\ell_1 , \\ d_2^{(r)}(\mu^2) + d_3^{(r)}(\mu^2) &= d_1 + d_3 - \kappa^2\frac{\ell_4}{4} . \end{aligned} \tag{57}$$

It is these renormalized values which would be (in principle) measured by experiment, and the μ^2 labeling $d_i^{(r)}(\mu^2)$ indicates that the measured value would depend on the choice of μ^2 in the logarithms, although all physics would be independent of μ^2.

In forming a gravitational interaction of two particles, one combines the vertices with the propagator. Temporarily leaving aside the vacuum polarization, one has

$$\begin{aligned} \frac{\kappa^2}{4}V^{(1)}_{\mu\nu}(q)D^{\mu\nu,\alpha\beta}(q)V^{(2)}_{\alpha\beta}(-q) &= \frac{\kappa^2}{2q^2}\left[F_1^{(1)}(q^2)F_1^{(2)}(q^2)\left\{p_1\cdot p_2 p'_1\cdot p'_2 + p_1\cdot p'_2 p_2\cdot p'_1 - m_1^2 m_2^2\right\}\right. \\ &\quad \left. + \frac{q^2}{2}\left\{F_1^{(1)}(q^2)F_2^{(2)}(q^2)m_1^2 + F_1^{(2)}(q^2)F_2^{(1)}(q^2)m_2^2\right\}\right] \\ &\approx \frac{\kappa^2 m_1^2 m_2^2}{2}\left\{\frac{1}{q^2} + 2\left(d_1 - 2d_2 - 2d_3\right)\right. \\ &\quad \left. + \kappa^2\left[(2\ell_1 - \ell_4) + (2\ell_2 - \ell_5)\ln\left(\frac{-q^2}{\mu^2}\right) + (2\ell_3 - \ell_6)\sqrt{\frac{m^2}{-q^2}}\right]\right\} , \end{aligned} \tag{58}$$

where the second line is the approximate result in the static limit. Linear analytic terms in q^2 in the form factors yield *constants* in the interactions, which in turn correspond to a point (δ function) interaction

$$\int \frac{d^3q}{(2\pi)^3}e^{-i\mathbf{q}\cdot\mathbf{r}} = \delta^3(x) . \tag{59}$$

The nonanalytic terms however lead to *power law* behavior since

$$\begin{aligned} \int \frac{d^3q}{(2\pi)^3}e^{-i\mathbf{q}\cdot\mathbf{r}}\frac{1}{q} &= \frac{1}{2\pi^2 r^2} , \\ \int \frac{d^3q}{(2\pi)^3}e^{-i\mathbf{q}\cdot\mathbf{r}}\ln\mathbf{q}^2 &= \frac{-1}{2\pi^2 r^3} . \end{aligned} \tag{60}$$

Therefore, the long distance corrections to the gravitational interactions come uniquely from the nonanalytic terms in the loop diagrams.

A similar result holds for the vacuum polarization diagram. If we temporarily suppress the Lorentz indices and relative constants of order unity, the generic form of the vacuum polarization follows from dimensional counting,

$$\pi(q^2) = \kappa^2 q^4\left[c_1 + c_2 + \ell_7 + \ell_8\ln(-q^2)\right] , \tag{61}$$

such that the propagator is modified as

$$\frac{1}{q^2} + \frac{1}{q^2}\pi(q^2)\frac{1}{q^2} + \cdots = \left\{\frac{1}{q^2} + \kappa^2[c_1 + c_2 + \ell_7 + \ell_8\ln(-q^2)]\right\}. \tag{62}$$

In these formulas c_1 and c_2 are the unknown parameters from the higher order Lagrangian of Eq. (20), and ℓ_7, ℓ_8 are constants calculable in the vacuum polarization diagram. Again ℓ_7 is divergent, but the combination $(c_1 + c_2 + \ell_7)$ forms a renormalized parameter which could in principle be measured. As above, constants in this propagator lead to a $\delta^3(x)$ interaction, while the logarithm corresponds to a long range effect.

By focusing only on the nonanalytic terms, we simplify the calculation somewhat. These are independent of the regularization scheme. The nonanalytic pieces of the relevant Feynman integrals are given in the Appendix. We note that there are several useful properties that we can exploit. For example, any factor of k^2 or $(k-q)^2$ in the numerator automatically removes any nonanalytic behavior. For example,

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k-q^2)} \frac{k^2}{(k-p')^2 - m^2} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k-q)^2} \frac{1}{(k-p')^2 - m^2}$$
$$= \int \frac{d^4k'}{(2\pi)^4} \frac{1}{k'^2} \frac{1}{(k'-p)^2 - m^2} = I(p^2) \ . \tag{63}$$

Where in the second line we have shifted to $k' = k - q$. The result is a function of m^2 only and has no q^2 dependence. Thus the nonanalytic terms vanish for any integrands which would vanish with on-shell gravitons. This is a reflection of the Cutkosky rules and is not surprising as the nonanalytic terms accompany imaginary parts in the amplitudes, and these could be seen using on-shell states and unitarity. As a result, the vertex function simplifies slightly as all the components in the curly brackets of Eq. (53) do not contribute. Also, factors of $k \cdot q$ may be written as $2k \cdot q = k^2 - (k-q)^2 + q^2 \to q^2$ in the loop integrals. With these results, it is not hard to show that

$$q_\mu \tau^{\mu\nu}_{\alpha\beta,\gamma\delta} = 0 \tag{64}$$

in any diagram, where in fact the vanishing of the results occurs individually for each of the terms in the square brackets of Eq. (53). These conditions provide a set of checks on the calculation that were found to be useful.

The calculation of the nonanalytic terms is straightforward although a bit tedious due to the lengthy form of the triple graviton coupling. For the diagram of Fig. 4(b), I find

$$\Delta F_1(q^2) = \frac{\kappa^2 q^2}{32\pi^2} \left\{ \left[\frac{1}{4} - 2 + 1 + 0 \right] \ln(-q^2) + \left[\frac{1}{16} - 1 + 1 + 0 \right] \frac{\pi^2 m}{\sqrt{-q^2}} \right\}$$
$$= \frac{\kappa^2 q^2}{32\pi^2} \left\{ -\frac{3}{4} \ln(-q^2) + \frac{1}{16} \frac{\pi^2 m}{\sqrt{-q^2}} \right\} ,$$
$$\Delta F_2 = \frac{\kappa^2 m^2}{32\pi^2} \left\{ [1 - 3 + 8 - 3] \ln(-q^2) + \left[\frac{7}{8} - 1 + 2 - 1 \right] \frac{\pi^2 m}{\sqrt{-q^2}} \right\}$$
$$= \frac{\kappa^2 m^2}{32\pi^2} \left\{ 3 \ln(-q^2) + \frac{7}{8} \frac{\pi^2 m^2}{\sqrt{-q^2}} \right\} , \tag{65}$$

where the sequence of numbers in the first version of F_i refers to the four sets of terms in square brackets in Eq. (53), respectively. For Fig. 4(c), I obtain

$$\Delta F_1 = \frac{\kappa^2 q^2}{32\pi^2} [0 + 2 + 0 - 2] \ln(-q^2)$$
$$= 0 \tag{66}$$

$$\Delta F_2 = \frac{\kappa^2 m^2}{32\pi^2} \left[-\frac{25}{3} + 0 + 2 + 2 \right] \ln(-q^2)$$
$$= \frac{\kappa^2 m^2}{32\pi^2} \left[-\frac{13}{3} \ln(-q^2) \right] . \tag{67}$$

The diagrams of Figs. 4(d)–(f) do not have any nonanalytic terms of the form considered here because the matter fields are massive or the loop integrals are independent of q^2. Figure 4(d) does have an infrared divergence similar to that of the vertex correction in QED. This can be handled in the same fashion as the QED case [20], i.e., by including soft radiative effects. If we were actually to attempt to apply the quantum potential phenomenologically, it would be important to include such effects. However, for our purposes we do not need to consider them further. The resulting nonanalytic contributions to F_1, F_2 are then

$$F_1(q^2) = 1 + \frac{\kappa^2}{32\pi^2} q^2 \left[-\frac{3}{4} \ln(-q^2) + \frac{1}{16} \frac{\pi^2 m}{\sqrt{-q^2}} \right] ,$$
$$F_2(q^2) = \frac{\kappa^2 m^2}{32\pi^2} \left[-\frac{4}{3} \ln(-q^2) + \frac{7}{8} \frac{\pi^2 m}{\sqrt{-q^2}} \right] . \tag{68}$$

The nonanalytic terms in the vacuum polarization can be obtained by the following procedure. The divergent parts of the vacuum polarization have been calculated, by 't Hooft and Veltman [3] employing the same gauge-fixing scheme as used in the present paper, using dimensional regularization. When only massless particles appear in the diagram, the $\ln(-q^2)$ terms can be read off of the coefficients of $1/(d-4)$ using a relatively well-known trick. The vacuum polarization graph has dimension of $(\mathrm{mass})^2$ and will be calculated from Feynman integrals of the form

$$I(q^2) = \kappa^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}f(k,q) = \kappa^2 q^4\left(\frac{\mu}{q}\right)^{4-d}\left[\frac{a}{d-4}+b\right] \tag{69}$$

for some integrand $f(k,q)$. The arbitrary scale factor μ, with dimension of $(\mathrm{mass})^1$, has been inserted in order to maintain the proper dimension for the overall integral. The second form then follows uniquely from dimensional analysis (since q is the only other dimensionful quantity), where a and b are constants which may depend on d but do not contain further poles as $d\to 4$. Since

$$\frac{1}{d-4}\left(\frac{\mu}{q}\right)^{4-d} = \frac{1}{d-4}e^{\frac{d-4}{2}\ln\left(\frac{q^2}{\mu^2}\right)} \tag{70a}$$

$$= \frac{1}{d-4}+\frac{1}{2}\ln\left(\frac{q^2}{\mu^2}\right)+O(d-4)\ , \tag{70b}$$

the logarithm will always share the coefficient of $\frac{1}{d-4}$ in the combination given in Eq. 70(b). 't Hooft and Veltman find that the divergent part of the graviton plus ghost vacuum polarization diagrams is equivalent to the Lagrangian in Eq. (30). This is therefore equivalent to the gravitational logs in the diagram being given by

$$\Pi_{\alpha\beta,\gamma\delta} = -\frac{\kappa^2}{32\pi^2}\ln(-q^2)\left\{\frac{21}{120}q^4 I_{\alpha\beta\gamma\delta}+\frac{23}{120}q^4\eta_{\alpha\beta}\eta_{\gamma\delta}-\frac{23}{120}(\eta_{\alpha\beta}q_\gamma q_\delta+\eta_{\gamma\delta}q_\alpha q_\beta) +\frac{21}{240}(q_\alpha q_\delta\eta_{\beta\gamma}+q_\alpha q_\gamma\eta_{\beta\delta}+q_\beta q_\gamma\eta_{\alpha\delta}+q_\beta q_\delta\eta_{\alpha\gamma})+\frac{11}{30}q^\alpha q^\beta q^\gamma q^\delta\right\}+(\text{nonlogs})\ . \tag{71}$$

When we construct the potential, a related form will be needed

$$P_{\mu\nu,\alpha\beta}\Pi^{\alpha\beta,\gamma\delta}P_{\gamma\delta,\rho\sigma} = \frac{\kappa^2 q^4}{32\pi^2}\left[\frac{21}{120}(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\nu\rho}\eta_{\mu\sigma})+\frac{1}{120}\eta_{\mu\nu}\eta_{\rho\sigma}\right][-\ln(-q^2)]+\cdots\ , \tag{72}$$

where all terms involving q_μ, q_ρ, etc. can be dropped since q_μ contracted with the vertex function gives a vanishing result.

If we combine these diagrams as shown in Fig. 3 in order to form a one-particle-reducible gravitational interaction, we find

$$-\frac{\kappa^2}{4}\frac{1}{2m_1}V^{(1)}_{\mu\nu}(q)\left[iD^{\mu\nu,\alpha\beta}(q)+iD^{\mu\nu,\rho\sigma}i\Pi_{\rho\sigma,\eta\lambda}iD^{\eta\lambda,\alpha\beta}\right]V_{\alpha\beta}(q)\frac{1}{2m_2} \approx 4\pi G m_1 m_2\left[\frac{i}{\mathbf{q}^2}-\frac{i\kappa^2}{32\pi^2}\left[-\frac{127}{60}\ln(\mathbf{q}^2)+\frac{\pi^2(m_1+m_2)}{2\sqrt{\mathbf{q}^2}}\right]+\text{const}\right], \tag{73}$$

where in the second line we have taken the nonrelativistic limit. This can be converted into the coordinate space form by Fourier transforming, with the result

$$V(r) = -\frac{Gm_1m_2}{r}\left[1-\frac{G(m_1+m_2)}{rc^2}-\frac{127}{30\pi^2}\frac{G\hbar}{r^2c^3}\right]. \tag{74}$$

This conforms to the general structure of the gravitational interactions given in Eq. (41).

One interesting consequence of the quantum correction is that it appears that there is no such thing as a purely classical source for gravity. In electrodynamics, the corrections to the vertex function are such that as one takes the particles' mass to infinity, all quantum corrections will vanish. Therefore by taking the $m\to\infty$ limit to the full theory, one obtains a purely classical source for electromagnetism. In the case of gravity this does not work, as the quantum corrections remain even as $m\to\infty$ and in fact share the same dependence on m as does the classical theory. This is because the gravitational coupling itself grows with m so that the expected decrease of quantum effects as $m\to\infty$ is compensated for by the increased coupling constant.

VI. THE EXTREME LOW-ENERGY LIMIT

The gravitational effective field theory appears to behave normally over distance and/or energy scales where we have experience with gravity and quantum theory. We have imagined quantizing in a very large box in order to exclude wavelengths of the scale of the Universe, and the effective field theory methodology separates out the high-energy effects. We expect modifications to the theory on the high-energy side. Of significant interest is whether there is a fundamental incompatibility between gravity and quantum mechanics at the extreme infrared side if we remove the low-energy cut off. In this section, I briefly discuss the reasons for suspecting that there might be a problem, although I do not resolve the issues.

Gravitational effects can build up over long distances and/or times. Most disturbing in this regard are the singularity theorems of Hawking and Penrose [21], which state loosely that a matter distribution evolved in spacetime by the Einstein action almost always has a true singularity in either the past or future. [The exact assumptions are more precisely stated in the original works, but apply to essentially all situations that we care about, although simple cases such as Minkowski space or a single stable star are exceptions.] Thus while it may be possible to locally specify a smooth set of coordinates with a small curvature, if we try to extend this condition to the whole spacetime manifold using the order E^2 Einstein equations, there will be at least one location in the distant past or future where the curvature becomes singular. The Big Bang in our standard cosmology is an example. From the standpoint of effective field theory, it is not the singularity itself which is the concern. Once the curvature becomes large, the R^2, R^3, etc. terms in the action become important and the evolution is different than is assumed in the derivation of the theorem. There is no longer any indication that a true singularity must develop beyond this scale. However it *is* bothersome that the curvature must get large. For gravitational effective field theory, the content of the singularity theorems is that it is difficult to specify a space-time where the curvature is everywhere small. The gravitational effective field theory would work over scales which have small curvatures, but would not be able to work on scales which encompass the putative singularity. However, it may be possible to modify the effective field theory treatment to include the singular region as an extended gravitational source, much as solitonic Skyrmions can be treated as a heavy chiral source in chiral perturbation theory.

Another global potential problem in the extreme infrared is the existence of horizons. Black holes can exist which have their horizon in low curvature regions. This is not a problem in a local patch near the horizon, but may lead to difficulties if we extend the region to all of space-time. Quantum information that would have otherwise flowed to infinity disappears into the black hole. Quantum coherence is lost on the largest time scales. It is not clear that this is a contradiction with the ultimate quantum theory, but at the least we end up with a situation which is beyond our experience in physics and for which we are unsure how to apply quantum ideas.

VII. SUMMARY

In addition to being a useful calculational tool, effective field theory provides a good way to think about the different energy scales of a theory. In the case of quantum gravity it allows one to separate the effects of the unknown high-energy theory from the known degrees of freedom at low energy. Gravity and quantum mechanics seem to be compatible over a large range of energies corresponding to our range of experience in physics. In this range quantum predictions can be extracted from generally covariant theories in the same way as done in other effective field theories.

A particular class of one-loop effects have been isolated and shown to give the leading quantum correction in an expansion in the energy or inverse distance. These nonanalytic terms come uniquely from the long distance propagation of the massless particles. The example of the leading corrections to the vertex and vacuum polarization diagrams in flat space has been discussed in detail, and these have been combined to form an effective potential.

The class of leading quantum corrections is not optional. The set of intermediate states is a known consequence of the low-energy theory. It will also be found in a full quantum gravity theory as long as our Universe is reproduced in the low-energy limit.

ACKNOWLEDGMENTS

I would like to thank Jennie Traschen and David Kastor for numerous discussions on this topic and G. Esposito-Farese, S. Deser, H. Dykstra, E. Golowich, B. Holstein, G. Leibbrandt, and J. Simon for useful comments.

APPENDIX

Below are the nonanalytic terms which arise in many Feynman integrals needed in Sec. V. These are given the "mostly negative" metric $q^2 = q_0^2 - \mathbf{q}^2$. I use the abbreviations $L = \ln(-q^2), S = \pi^2 m/\sqrt{-q^2}$:

$$J = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k-q)^2} = \frac{i}{32\pi^2}[-2L] + \cdots ,$$

$$J_\mu = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{k^2(k-q^2)} = \frac{i}{32\pi^2} q_\mu [-L] + \cdots ,$$

$$J_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{k^2(k-q^2)} = \frac{i}{32\pi^2}\left[q_\mu q_\nu \left\{-\frac{2}{3}L\right\} - g_{\mu\nu}\, q^2 \left\{-\frac{1}{6}L\right\}\right] + \cdots . \tag{A1}$$

In the following, the external momentum p' is on shell as is $p = p' - q$:

$$I = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(k-q)^2}\frac{1}{[(k-p')^2-m^2]} = \frac{i}{32\pi^2 m^2}[-L-S] ,$$

$$I_\mu = \int \frac{d^4k}{(2\pi)^4}\frac{k_\mu}{k^2(k-q)^2[(k-p')^2-m^2]} = \frac{i}{32\pi^2 m^2}\left[p'_\mu\left\{\left(1+\frac{q^2}{2m^2}\right)L + \frac{1}{4}\frac{q^2}{m^2}S\right\} + q_\mu\left\{-L-\frac{1}{2}S\right\}\right] ,$$

$$I_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{k^2(k-q)^2[(k-p')^2+m^2]} = \frac{i}{32\pi^2 m^2}\left[p'_\mu p'_\nu\left\{-\frac{q^2}{2m^2}L \right.\right.$$
$$\left.\left. -\frac{q^2}{8m^2}S\right\} + (p'_\mu q_\nu + p'_\nu q_\mu)\left\{\frac{1}{2}\left(1+\frac{q^2}{m^2}\right)L + \frac{3}{16}\frac{q^2}{m^2}S\right\} + q_\mu q_\nu\left\{-L-\frac{3}{8}S\right\} + q^2 g_{\mu\nu}\left\{\frac{1}{4}L+\frac{1}{8}S\right\}\right] + \cdots$$

$$I_{\mu\nu\alpha} = \int \frac{d^4k}{(2\pi)^4}\frac{k_\mu k_\nu k_\alpha}{k^2(k-q)^2[(k-p')^2+m^2]}$$
$$= \frac{i}{32\pi^2 m^2}\left\{\left[p'_\mu p'_\nu p'_\alpha\left\{\frac{1}{6}\frac{q^2}{m^2}L\right]\right\} + (p'_\mu p'_\nu q_\alpha + p'_\mu q_\nu p'_\alpha + q_\mu p'_\nu p'_\alpha)\left\{-\frac{1}{3}\frac{q^2}{m^2}L - \frac{q^2}{16m^2}S\right\}\right.$$
$$+(q_\mu q_\nu p'_\alpha + p'_\mu q_\nu q'_\alpha + q_\mu p'_\nu q_\alpha)\left[\frac{1}{3}L\right] + q_\mu q_\nu q_\alpha\left[-L-\frac{5}{16}S\right]$$
$$\left. +\left(g_{\mu\nu}p'_\alpha + g_{\mu\alpha}p'_\nu + g_{\nu\alpha}p'_\mu\right)\left[-\frac{q^2}{12}L\right] + \left(g_{\mu\nu}q_\alpha + g_{\mu\alpha}q_\nu + g_{\nu\alpha}q_\mu\right)\left[\frac{q^2 L}{6}+\frac{q^2 S}{16}\right]\right\} + \cdots . \quad \text{(A2)}$$

In the latter set of integrals there are further nonanalytic terms with higher powers of q^2/m^2, which are not needed for the calculation of leading effects in the text. Note that the nonanalytic parts of the integrals satisfy various "mass shell" constraints, such as

$$q^\mu I_\mu = \frac{q^2}{2}I + \cdots ,$$
$$q^\mu I_{\mu\nu} = \frac{q^2}{2}I_\nu + \cdots ,$$
$$q^\mu I_{\mu\nu\alpha} = \frac{q^2}{2}I_{\nu\alpha} + \cdots ,$$
$$q^\mu J_\mu = \frac{q^2}{2}J + \cdots ,$$
$$q^\mu J_{\mu\nu} = \frac{q^2}{2}J_\nu + \cdots ,$$
$$g^{\mu\nu}I_{\mu\nu} = 0 + \cdots ,$$
$$g^{\mu\nu}I_{\mu\nu\alpha} = 0 + \cdots ,$$
$$g^{\mu\nu}J_{\mu\nu} = 0 + \cdots , \quad \text{(A3)}$$

where the ellipses denote analytic terms. Here $p'\cdot q = q^2/2$ has been used, following from $p^2 = (p'-q)^2$ on shell. Some of these relations arise using $2k\cdot q = k^2 - (k-q)^2 + q^2$, and the fact that integrals with factors of k^2 or $(k-q)^2$ in the numerator can be shifted to remove all dependence on q.

[1] T. Appelquist and J. Carazzone, Phys. Rev. D **11**, 2856 (1975).

[2] B.S. DeWitt, Phys. Rev. **160**, 1113 (1967); **162**, 1195 (1967); **162**, 1239 (1967). R.P. Feynman, Acta. Phys. Pol. **24**, 697 (1963); Caltech lectures 1962–63 (unpublished).

[3] G. 't Hooft and M. Veltman, Ann. Inst. H. Poincaré **A20**, 69 (1974).

[4] D.M. Capper, G. Leibbrandt, and M. Ramon Medrano, Phys. Rev. D **8**, 4320 (1973); M.R. Brown, Nucl. Phys. **B56**, 194 (1973); D.M. Capper, M.J. Duff, and L. Halpern, Phys. Rev. D **10**, 461 (1944); S. Deser and P. van Nieuwenhuizen, Phys. Rev. Lett. **32**, 245 (1974); Phys. Rev. D **10**, 401 (1974); **10**, 411 (1974); S. Deser, H.-S. Tsao, and P. van Nieuwenhuizen, *ibid.* **10**, 3337 (1974).

[5] J.F. Donoghue, Phys. Rev. Lett. **72**, 2996 (1994).

[6] The $(1,-1,-1,-1)$ metric is used in P.A.M. Dirac, *General Theory of Relativity* (Wiley, New York, 1975).

[7] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[8] M. Veltman, in *Methods in Field Theory*, Proceedings of the Les Houches Summer School, 1975, Les Houches, France, 1975, edited by R. Balian and J. Zinn-Justin, Les Houches Summer School Proceedings Vol. XXVIII (North-Holland, Amsterdam, 1976).

[9] S. Weinberg, Physica (Amsterdam) **96A**, 327 (1979); J.F. Donoghue, in *Effective Field Theories of the Standard Model*, edited by U.-G. Meissner (World Scientific, Singapore, 1992), p. 3; A. Cohen, in *Proceedings of the 1993 TASI Summer School*, edited by S. Raby (World Scientific, Singapore, 1994); H. Leutwyler and S. Weinberg, in *Proceedings of the XXVIth International Conference on High Energy Physics*, Dallas, Texas, 1992, edited by J. Sanford, AIP Conf. Proc. No. 272 (AIP, NY, 1993), pp. 185, 346.

[10] J.F. Donoghue, E. Golowich, and B.R. Holstein, *Dynamics of the Standard Model* (Cambridge University Press, Cambridge, England, 1992); J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985); H. Georgi, *Weak Interactions and Modern Particle Theory* (Benjamin/Cummings, Menlo Park, 1984).

[11] Particle Data Group, K. Hikasa *et al.*, Phys. Rev. D **45**,

S1 (1992), p. I.1.
[12] K.S. Stelle, Gen. Relativ. Gravit. **9**, 353 (1978).
[13] A. Pais and G.E. Uhlenbeck, Phys. Rev. **79**, 145 (1950); K.S. Stelle, Phys. Rev. D **16**, 953 (1977); A.A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[14] J. Simon, Phys. Rev. D **41**, 3720 (1990); **43**, 3308 (1991).
[15] C.G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) **59**, 42 (1970).
[16] M. Goroff and A. Sagnotti, Nucl. Phys. **B266**, 799 (1986).
[17] N. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1992).
[18] In chiral perturbation theory, the $m/\sqrt{-q^2}$ effects are cut off by the pion mass and become the $1/\sqrt{m_\pi^2}$ contributions described by L.F. Li and H. Pagels, Phys. Rev. Lett. **26**, 1204 (1971); H. Pagels, Phys. Rep. C **16**, 219 (1975).
[19] F. Behrends and R. Gastmans, Phys. Lett. **55B**, 311 (1975); Nucl. Phys. **B88**, 99 (1975).
[20] This is discussed in most field theory texts; see, e.g., J.D. Bjorken and S. Drell, *Relativistic Quantum Field Theory* (McGraw-Hill, New York, 1964).
[21] S.W. Hawking and R. Penrose, Proc. R. Soc. London Ser. A **314**, 529 (1970); S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).