Generalized magnetic universe solutions

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We apply the techniques that have been developed over the last few decades for generating non-trivially new solutions of the Einstein-Maxwell equations from seed solutions for simple spacetimes. The simple seed spacetime that we choose is the "magnetic universe" to which we apply the Ehlers transformation. Three interesting nonsingular metrics are generated. Two of these may be described as "rotating magnetic universes" and the third as an "evolving magnetic universe." Each is causally complete, in that all timelike and lightlike geodesics do not end in a finite time or affine parameter. We also give the electromagnetic field in each case. For the two rotating stationary cases we give the projection with respect to a stationary observer of the electromagnetic field into electric and magnetic components.

I. INTRODUCTION

Begining in the late 1950s and continuing into the 1980s an interesting technique was developed for generating new vacuum, and Einstein-Maxwell, solutions from a given "seed" solution having at least one Killing vector. This was of special interest when the Killing vector field corresponds to an azimuthal symmetry about an axis. Even more fruitful was the case when the seed metric had stationary axial symmetry about an axis with two corresponding Killing vector fields $\partial/\partial\phi$ and $\partial/\partial t$. The technique, as we shall apply it, starts with the differential form K corresponding to a given Killing vector field $\mathbf{K}$. This Killing form in combination with forms associated with the self-dual part of the Maxwell field, when present in the seed spacetime, allows one to define succesively two scalar potentials: one the Killing-Maxwell potential Φ and the other the Killing-Einstein potential $\mathcal{E}$ defined by Ernst. Certain transformations of the scalar potentials of the seed metric lead to nontrivially new potentials, a new Killing form, and a new tetrad basis defining a new metric.

The transformations from old to new solutions were found piecemeal by various investigators. One of the earliest, given by Ehlers [1], is the one we shall apply in the present paper. Others were found by Harrison [2,9] and Geroch [3]. Finally all were together codified in a group structure by Kinnersley [4]. They have also been investigated extensively first by Ernst and then by Hauser and Ernst [5], whose procedure we follow here.

It is to be noted that even when a seed solution is well behaved, even altogether free from singularities, the resulting solutions often are not well behaved, and therefore not of great physical interest. In this paper we apply the Ehlers transformation using the magnetic universe [6] as a seed solution. The magnetic universe is globally well behaved with no singularities. In the magnetic universe there are three Killing vector fields which, applied individually, allow the generation of three distinct new spacetimes. Each of these spacetimes is timelike and lightlike geodesically complete (nonsingular). Two of them are stationary-axisymmetric representing rotating magnetic universes. The third is cylindrically symmetric but nonstationary, evolving in time.

In Sec. II we describe the seed metric, the magnetic universe. We then describe the derivation of the potentials, and the Ehlers transformation. In Sec. III we apply the Ehlers transformation to obtain the three new [9] metrics. In Sec. IV we examine global structure and demonstrate that each spacetime is nonsingular. We also discuss the properties of the electromagnetic fields of each solution. Section V contains our conclusions.

II. SEED METRIC AND TRANSFORMATION

Among configurations of matter or field that are in static equilibrium under their own gravitational attraction one of the simplest is a parallel bundle of magnetic flux. Associated with the equilibrium magnetic field distribution is a well-determined geometry with cylindrical symmetry [6]. The field distribution together with its associated geometry has come to be called "the magnetic universe." The geometry of the cylindrical magnetic universe is given by

$$ds^2 = v^2\left(-dt^2 + dz^2 + d\rho^2\right) + \frac{\rho^2}{v^2}d\phi^2 = e^z \bullet e^z + e^\phi \bullet e^\phi + e^\rho \bullet e^\rho - e^t \bullet e^t \,, \tag{2.1}$$

where z, ϕ, ρ and t are the usual cylindrical spacetime coordinates. Here v is given by $v \equiv 1 + \rho^2$, and an orthonormal basis of one-forms is

$$e^z \equiv v\,dz, \quad e^\phi \equiv \frac{\rho}{v}\,d\phi, \quad e^\rho \equiv v\,d\rho, \quad e^t \equiv v\,dt . \quad (2.2)$$

The volume form is $\epsilon = v^2\rho\,dt \wedge dz \wedge d\rho \wedge d\phi$. We use a notation where vectors are written in boldface type and forms in italic type. The Maxwell field, which varies only with radius ρ, is given by

$$F = \frac{2\rho}{v^2}\,d\rho \wedge d\phi$$
$$= -d\left(\frac{1}{v}\right) \wedge d\phi = \frac{2}{v^2}\,e^\rho \wedge e^\phi \equiv \frac{2}{v^2}\,e^{\rho\phi}, \quad (2.3)$$

where we define the symbol $e^{ab} \equiv e^a \wedge e^b$. The coordinates and metric are dimensionless because we are measuring all lengths in units of the "range radius" $\bar{a} = 2/B_0$ (gravitational length units) and where B_0 is the magnitude of the magnetic field on the axis (measured in Gauss). In cgs units,

$$\bar{a} = \frac{2c^2}{B_0 G^{1/2}} = \frac{1.96}{B_0} \times 10^{24}\text{cm} .$$

The three Killing vector fields are $\partial/\partial\mathbf{z}$, $\partial/\partial\boldsymbol{\phi}$, and $\partial/\partial\mathbf{t}$ with corresponding one-forms $v^2 dz$, $(\rho/v)^2 d\phi$, and $-v^2 dt$. The three Killing fields give rise to three conserved quantities for the geodesic equation. These quantities have been used to solve the geodesic equation in closed form [7].

We now discuss the solution generating technique. Starting with a solution of the Einstein-Maxwell equations with a Killing field, one produces two scalar potentials Φ and $\mathcal{E}$. One then uses these potentials with a solution generating technique (e.g., the Ehlers transformation) to find an orthonormal tetrad of one-forms for the new metric.

First we introduce some notation, the so-called "step product" $M \ulcorner N$ of two forms which leads to a contracted form. This is only different from zero when the degree of the form M on the left is not greater than the degree of N. The step product of two one-forms is the dot product. Given a one-form K and a two-form Q which is the wedge product of two one-forms, $Q \equiv u \wedge v$, we define the step product

$$K \ulcorner Q \equiv K \ulcorner (u \wedge v) = u\,(K \cdot v) - v\,(K \cdot u), \quad (2.4)$$

where the dot represents the usual inner product of two fields, the same for vectors and the corresponding one-forms.

Now let W be a self-dual two-form (where the dual of a two-form F is $-i^*F$). Then one can show that

$$2 \times \text{self-dual of } [K \wedge (K \ulcorner W)] = - (K \cdot K)W. \quad (2.5)$$

This relation will be particularly useful in enabling us to reconstruct the electromagnetic field from a Killing-vector-dependent scalar Maxwell potential Φ which we shall now define.

From the Maxwell two-form we define the self-dual Maxwell two-form W^M by

$$W^M \equiv F - i^*F . \quad (2.6)$$

For example, in the special case of the magnetic universe,

$$W^M = \frac{2}{v^2}\left(e^{\rho\phi} - ie^{tz}\right) = \frac{2\rho}{v^2}\,d\rho \wedge d\phi \; - \; 2idt \wedge dz . \quad (2.7)$$

Now we define what we may call the "Killing-Maxwell one-form" $K \ulcorner W^M$ and the complex scalar Maxwell potential Φ by

$$d\Phi = K \ulcorner W^M . \quad (2.8)$$

The integrability conditions for this equation are satisfied as a consequence of the source-free Maxwell equations, provided the Lie derivative of F vanishes along the associated non-null Killing field; i.e., the electromagnetic field is constant along the Killing trajectories. The integrability is shown as follows: we have, for any two-form W, the expression of the Lie derivative in terms of the exterior derivative and the step product [8]:

$$\mathcal{L}_{\mathbf{K}}W = K \ulcorner dW - d(K \ulcorner W) .$$

Here, in our case, W^M is linear and homogeneous in F and $^*F$ and $-$ with $dW^M = dF = d^*F = 0$ as well as $\mathcal{L}_{\mathbf{K}}F = \mathcal{L}_{\mathbf{K}}{}^*F = \mathcal{L}_{\mathbf{K}}W^M = 0$; it follows that $d\left(K \ulcorner W^M\right) = 0$.

Next we define the self-dual Einstein-Maxwell-Harrison-Ernst two-form W^E by

$$W^E \equiv - \left(dK + 2\bar{\Phi}F\right) + i\left(^*dK + 2\bar{\Phi}{}^*F\right) , \quad (2.9)$$

where $\bar{\Phi}$ is the complex conjugate of Φ. Then we define the Killing-Einstein one-form $K \ulcorner W^E$ and the scalar Ernst potential $\mathcal{E}$ by

$$d\mathcal{E} = K \ulcorner W^E . \quad (2.10)$$

The integrability conditions for this equation are satisfied as a consequence of the Einstein-Maxwell equations using the vanishing Killing Lie derivative conditions analogous to before. The constant in $\mathcal{E}$ is fixed by requiring

$$\text{Re}\mathcal{E} = f - |\Phi|^2 \quad (2.11)$$

where f is minus the norm of the Killing field; i.e., $f \equiv -\mathbf{K} \cdot \mathbf{K}$.

For a hypersurface orthogonal Killing vector $\mathbf{K}$ a useful relation is found by substituting Eq. (2.9) into Eq. (2.10) giving

$$d\mathcal{E} = df - 2\bar{\Phi}d\Phi .$$

This implies

$$d\text{Im}\mathcal{E} = i\left(\bar{\Phi}d\Phi - \Phi d\bar{\Phi}\right) . \quad (2.12)$$

Thus, as is true in each of our three cases, if Φ is pure real or pure imaginary, $\text{Im}\mathcal{E}$ is a constant which may be taken as zero. In more general situations $\text{Im}\mathcal{E}$ will be determined by Eq. (2.12).

We will also need only the one-forms A^E and M^{EE}

defined by

$$dA^E = W^E \tag{2.13}$$

and

$$dM^{EE} = 2\mathrm{Re}\left(\bar{\mathcal{E}}W^E\right) . \tag{2.14}$$

These form potentials are chosen to satisfy the gauge conditions $K \ \llcorner \ A^E = \mathcal{E}$ and $K \ \llcorner \ M^{EE} = |\mathcal{E}|^2$. The consistency of these two conditions can be seen by applying the K contraction to Eq. (2.14) and noting that, with the Lie derivatives of M^{EE} and A^E vanishing, the K contraction commutes with the exterior derivative and we have

$$d\left(K \ \llcorner \ M^{EE}\right) = 2\mathrm{Re}\left[\bar{\mathcal{E}}d\left(K \ \llcorner \ A^E\right)\right] = 2\mathrm{Re}\left(\bar{\mathcal{E}}d\mathcal{E}\right) = d\left(|\mathcal{E}|^2\right) .$$

We are now ready to discuss the Ehlers transformation. Let β be a real number. Define the scalar Λ by

$$\Lambda \equiv 1 + i\beta\mathcal{E} .$$

The Ehlers transformation, expressed in the group formalism developed by Kinnersley [4] is

$$\begin{pmatrix} 1 \\ \mathcal{E}' \\ \Phi' \end{pmatrix} = \frac{1}{\Lambda} \begin{pmatrix} 1 & i\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \mathcal{E} \\ \Phi \end{pmatrix} = \begin{pmatrix} 1 \\ \mathcal{E}/\Lambda \\ \Phi/\Lambda \end{pmatrix}$$

which implies, for the norm of the transformed Killing field in the new spacetime,

$$f' = \mathrm{Re}\mathcal{E}' + |\Phi'|^2 = \frac{\mathrm{Re}\mathcal{E} + |\Phi|^2}{|\Lambda|^2} = \frac{f}{|\Lambda|^2} . \tag{2.15}$$

The more general formulas given in [5] simplify in the case of the Ehlers transformation. It can be shown that the

transform K' of the Killing form K satisfies the equation

$$f'^{-1}K' = f^{-1}K|\Lambda|^2 + \beta^2 M^{EE} - 2\beta\,\mathrm{Im}A^E . \tag{2.16}$$

In all three cases that occur with the Killing vectors of the magnetic universe, which we will develop in the following, $\mathcal{E}$ is purely real and $M^{EE} = \mathcal{E}^2 dx^a$, where x^a is the coordinate for the relevant Killing field. The vector field is $\partial/\partial x^a$ and the form field is

$$K^a = -fdx^a = -\frac{f}{\sqrt{|f|}}\, e^a \tag{2.17}$$

($f < 0$ for x^a spacelike; $f > 0$ for x^a timelike). Equation (2.16) then simplifies to

$$f'^{-1}K^{a'} = -dx^a |\Lambda|^2 + \beta^2\mathcal{E}^2 dx^a - 2\beta\,\mathrm{Im}A^E$$

which, upon using $|\Lambda|^2 = 1 + \beta^2\mathcal{E}^2$, can be written

$$f'^{-1}K^{a'} = -\left(dx^a + 2\beta\,\mathrm{Im}A^E\right) = -\left(\frac{1}{\sqrt{|f|}}\, e^a + 2\beta\,\mathrm{Im}A^E\right) . \tag{2.18}$$

Now a convenient tetrad in the new spacetime can be constructed from the tetrad used to describe the original spacetime. The three tetrad forms orthogonal to K transform as

$$e^a \to e^{a'} = |\Lambda|e^a \quad (e^a \perp K) \tag{2.19}$$

The remaining tetrad form which we shall designate e^k ($e^k \parallel K$) transforms as

$$e^K \to e^{K'} = \frac{\sqrt{|f|}}{|\Lambda|}\left(dx^K + 2\beta\,\mathrm{Im}A^E\right) = -\frac{\sqrt{|f|}}{|\Lambda|}\, f'^{-1}K' . \tag{2.20}$$

The Maxwell scalar potential transforms as

$$\Phi' = \frac{\Phi}{1 + i\beta\mathcal{E}} = \frac{\Phi(1 - i\beta\mathcal{E})}{|\Lambda|^2} \tag{2.21}$$

and Eq. (2.5) yields

$$W \to W' = f'^{-1}\left\{K' \wedge [K' \ \llcorner \ W'] - i^*[K' \wedge (K' \ \llcorner \ W')]\right\}$$
$$= f'^{-1}\left(K' \wedge d\Phi' - i^*[K' \wedge d\Phi']\right) \tag{2.22}$$

or, separating out the real part on both sides of the equation, we have

$$F \rightarrow F' = f'^{-1}K' \wedge d\,(\mathrm{Re}\Phi') + *\left[f'^{-1}K' \wedge d\,(\mathrm{Im}\Phi')\right]$$

$$= -\frac{|\Lambda|}{\sqrt{|f|}}\left(e^{k'} \wedge d\,(\mathrm{Re}\Phi') + *\left[e^{k'} \wedge d\,(\mathrm{Im}\Phi')\right]\right) . \qquad (2.23)$$

In the two types of cases of interest to us here: (1) the seed Φ pure real and equal to $\mathcal{E}$ (the case of the ϕ metric); (2) the seed Φ pure imaginary, $\mathcal{E}$ real (the case of the t and z metrics), the general equation reduces as follows:

$$\Phi \text{ real} = \mathcal{E}: \quad F' = -\frac{|\Lambda|}{|\Lambda|^4\sqrt{|f|}}\left[\left(1 - \beta^2\mathcal{E}^2\right)e^{k'} \wedge d\Phi - 2\beta\Phi^*\left(e^{k'} \wedge d\Phi\right)\right] ,$$

$$\Phi \text{ imaginary, } \mathcal{E} \text{ real}: \quad F' = -\frac{|\Lambda|}{|\Lambda|^4\sqrt{|f|}}\left(\beta\left[\mathcal{E}\left(1 + \beta^2\mathcal{E}^2\right)e^{k'} \wedge d|\Phi| + |\Phi|\left(1 - \beta^2\mathcal{E}^2\right)e^{k'} \wedge d\mathcal{E}\right]\right.$$

$$\left. + *\left[\left(1 + \beta^2\mathcal{E}^2\right)e^{k'} \wedge d|\Phi| - 2\beta^2|\Phi|\mathcal{E}e^{k'} \wedge d\mathcal{E}\right]\right) .$$

Choosing an observer with four-velocity **u** the Maxwell tensor can be decomposed into the electric and magnetic fields measured by this observer. The electric and magnetic fields are given by

$$E = u \ulcorner F ,$$

$$B = -u \ulcorner {}^*F ,$$

where u is the one-form corresponding to the vector **u**. In our case we choose u to be minus the timelike tetrad form.

III. THREE GENERALIZED MAGNETIC UNIVERSES

We now apply the Ehlers transformation to the magnetic universe to produce new [9] spacetimes. We get a different spacetime for each of the Killing vectors of the magnetic universe.

A. The ϕ metric

The two-form W^M for the magnetic universe is given by

$$W^M = \frac{2\rho}{v^2}\,d\rho \wedge d\phi - 2idt \wedge dz . \qquad (3.1)$$

Using the Killing vector $\mathbf{K} = \partial/\partial\phi$ it follows that

$$K \ulcorner W^M = \frac{2\rho}{v^2}\,d\rho$$

and therefore that $\Phi = -v^{-1}$. We then find that W^E is given by

$$W^E = \frac{2\rho}{v^2}d\rho \wedge d\phi - 2idz \wedge dt . \qquad (3.2)$$

Using Eq. (2.10) we then find

$$\mathcal{E} = -v^{-1} .$$

We then find, using Eqs. (2.9) and (2.13), that

$$A^E = -v^{-1}d\phi - 2izdt .$$

Then using Eqs. (2.10) and (2.14) we find

$$M^{EE} = v^{-2}\,d\phi .$$

The new tetrad is given as

$$e^{1'} = \left(v^2 + \beta^2\right)^{1/2}\,dz ,$$

$$e^{2'} = \rho\left(v^2 + \beta^2\right)^{-1/2}\,[d\phi + 4\beta zdt] ,$$

$$e^{3'} = \left(v^2 + \beta^2\right)^{1/2}\,d\rho ,$$

$$e^{4'} = \left(v^2 + \beta^2\right)^{1/2}\,dt .$$

Thus the metric is given by

$$ds^2 = \left(v^2 + \beta^2\right)\left[-dt^2 + dz^2 + d\rho^2\right] + \rho^2\left(v^2 + \beta^2\right)^{-1}[d\phi + 4\beta zdt]^2 . \qquad (3.3)$$

The new Maxwell tensor is given in terms of the new tetrad by

$$F' = 2\frac{v^2 - \beta^2}{\left(v^2 + \beta^2\right)^2}\,e^{3'} \wedge e^{2'} + \frac{4\beta v}{\left(v^2 + \beta^2\right)^2}\,e^{4'} \wedge e^{1'} . \qquad (3.4)$$

The corresponding electric and magnetic fields are

$$E = -\frac{4\beta v}{\left(v^2 + \beta^2\right)^2}\,e^{1'} ,$$

$$B = 2\frac{v^2 - \beta^2}{\left(v^2 + \beta^2\right)^2}\,e^{1'} . \qquad (3.5)$$

The electric and magnetic fields point in the (plus or minus) z direction. Note that in the $\beta \rightarrow 0$ limit the

metric and Maxwell tensor reduce to those of the magnetic universe. Thus the parameter β can be regarded as the magnitude of the rotation.

B. The t metric

Using the Killing vector $\mathbf{K} = \partial/\partial \mathbf{t}$ it follows that

$$K \ulcorner W^M = 2idz, \quad \Phi = 2iz . \tag{3.6}$$

It then follows from Eq. (2.9) that

$$W^E = d\left(v^2 - 4z^2\right) \wedge dt + 4id\left(\rho^2 v^{-1} z\right) \wedge d\phi .$$

It then follows from Eq. (2.10) that

$$\mathcal{E} = v^2 - 4z^2 . \tag{3.7}$$

Now using Eqs. (2.9) and (2.13) we find

$$A^E = \mathcal{E} dt + 4i\rho^2 v^{-1} zd\phi . \tag{3.8}$$

Similarly using Eqs. (3.7) and (2.14) we find again that

$$M^{EE} = \mathcal{E}^2 dt .$$

We are now ready to compute the new tetrad. Since $\Lambda = 1 + i\beta\mathcal{E}$ and here $|\Lambda| = \left[1 + \beta^2\left(v^2 - 4z^2\right)^2\right]^{1/2}$ we find that three of the tetrad vectors are given by

$$e^{1'} = |\Lambda| \, vdz ,$$

$$e^{2'} = |\Lambda| \, \rho v^{-1} d\phi ,$$

$$e^{3'} = |\Lambda| \, vd\rho .$$

Finally using Eqs. (2.20) and (3.8) we find that the last tetrad vector is given by

$$e^{4'} = |\Lambda|^{-1} \left(vdt + 8\beta\rho^2 zd\phi\right) .$$

Thus the metric is given by

$$ds^2 = \left(1 + \beta^2\mathcal{E}^2\right) \left[v^2 dz^2 + v^2 d\rho^2 + \rho^2 v^{-2} d\phi^2\right] - \left(1 + \beta^2\mathcal{E}^2\right)^{-1} \left[vdt + 8\beta z\rho^2 d\phi\right]^2 . \tag{3.9}$$

The new Maxwell tensor is given in terms of the new tetrad by

$$F = 2|\Lambda|^{-4}v^{-2}\left[\left(1 + \beta^2\mathcal{E}\left[\mathcal{E} + 16z^2\right]\right) e^{3'} \wedge e^{2'} + 8\beta^2\mathcal{E}vz\rho e^{1'} \wedge e^{2'}\right]$$
$$+ 2|\Lambda|^{-4}v^{-2}\beta\left[\left(\mathcal{E}\left[1 + \beta^2\mathcal{E}^2\right] + 8z^2\left[\beta^2\mathcal{E}^2 - 1\right]\right) e^{1'} \wedge e^{4'} + 4vz\rho\left(1 - \beta^2\mathcal{E}^2\right) e^{3'} \wedge e^{4'}\right] . \tag{3.10}$$

The corresponding electric and magnetic fields are

$$E = 2|\Lambda|^{-4}v^{-2}\beta\left[\left(\mathcal{E}\left[1 + \beta^2\mathcal{E}^2\right] + 8z^2\left[\beta^2\mathcal{E}^2 - 1\right]\right) e^{1'} + 4vz\rho\left(1 - \beta^2\mathcal{E}^2\right) e^{3'}\right] ,$$

$$B = 2|\Lambda|^{-4}v^{-2}\left[\left(1 + \beta^2\mathcal{E}\left[\mathcal{E} + 16z^2\right]\right) e^{1'} - 8\beta^2\mathcal{E}vz\rho e^{3'}\right] . \tag{3.11}$$

Note that the electric and magnetic fields each have components in both the z and ρ directions. Here again the metric and Maxwell fields reduce to those of the magnetic universe in the limit as $\beta \to 0$.

C. The z metric

The steps used to find the new metric using the Killing vector $\partial/\partial \mathbf{z}$ are completely analogous to those used in the case of the $\partial/\partial \mathbf{t}$ Killing vector. In fact the z metric can be found from the t metric using the complex coordinate transformation $t \to iz$, $z \to it$. We will not repeat the derivation; but simply write down the metric:

$$ds^2 = \left(1 + \beta^2\mathcal{E}^2\right) \left[-v^2 dt^2 + v^2 d\rho^2 + \rho^2 v^{-2} d\phi^2\right]$$
$$+ \left(1 + \beta^2\mathcal{E}^2\right)^{-1} \left[vdz + 8\beta t\rho^2 d\phi\right]^2 , \tag{3.12}$$

where the scalar $\mathcal{E}$ is here given by $\mathcal{E} = -\left(v^2 + 4t^2\right)$.

IV. NONSINGULARITY: GEODESICS AND COMPLETENESS

In this section we demonstrate that these new metrics are nonsingular. It is clear from the coordinate components that the metrics are smooth for $\rho > 0$. We now show that the metrics are smooth on the axis as well. Note that in general cylindrical coordinates are badly behaved on the axis; so we must introduce Cartesian coordinates and show that the corresponding coordinate components of the metric are smooth. First consider the case of the ϕ metric. Introduce Cartesian coordinates x and y by

$$x \equiv \rho \cos \frac{\phi}{1 + \beta^2} , \tag{4.1}$$

$$y \equiv \rho \sin \frac{\phi}{1 + \beta^2} .$$

Then $\rho^2 = x^2 + y^2$ so any smooth function of ρ^2 is also a smooth function of x and y. Using Eq. (3.3) some straightforward but tedious algebra shows that the metric in these coordinates is

$$ds^2 = \left(v^2 + \beta^2\right)\left[-dt^2 + dz^2\right] + \left(v^2 + \beta^2\right)^{-1}8\beta\left[2\beta z\rho^2 dt^2 - \left(1 + \beta^2\right)dt\left(xdy - ydx\right)\right]$$
$$+ \left(v^2 + \beta^2\right)^{-1}\left[\left(1 + \beta^2\right)^2\left(dx^2 + dy^2\right) + (v + 1)\left(v^2 + 2\beta^2 + 1\right)\left(xdx + ydy\right)^2\right] . \tag{4.2}$$

The metric components are all smooth functions of the Cartesian coordinates (x, y, z, t). Thus the topology of this spacetime is R^4 and the metric is smooth everywhere.

Now consider the t metric. Here we introduce Cartesian coordinates x and y by $x \equiv \rho\cos\phi$, $y \equiv \rho\sin\phi$. Again $\rho^2 = x^2 + y^2$ so a smooth function of ρ^2 is a smooth function of x and y. The metric is given [using Eq. (3.9) and some straightforward but tedious algebra] by

$$ds^2 = \left(1 + \beta^2\mathcal{E}^2\right)v^2 dz^2 - v^2\left(1 + \beta^2\mathcal{E}^2\right)^{-1}\left[dt + 8\beta zv^{-1}\left(xdy - ydx\right)\right]^2$$
$$+ \left(1 + \beta^2\mathcal{E}^2\right)v^{-2}\left[dx^2 + dy^2 + (v + 1)\left(v^2 + 1\right)\left(xdx + ydy\right)^2\right] . \tag{4.3}$$

Again the metric components are smooth functions of the coordinates (x, y, z, t). The z metric can be obtained from the t metric by the transformation $t \to iz$, $z \to it$. Thus we have also demonstrated that the z metric is smooth.

We now study geodesics in the three spacetimes and demonstrate that the spacetimes are timelike and null geodesically complete (i.e., nonsingular). The geodesics will be found using the Hamilton-Jacobi equations. We first find a function S satisfying

$$g^{\mu\nu}S_\mu S_\nu + \kappa = 0 , \tag{4.4}$$

where $S_\mu \equiv \partial S/\partial x^\mu$ and κ is 0 for null geodesics and 1

for timelike geodesics. The trajectory of the particle (or light ray) is then given by solving the equation

$$\dot{x}^\mu = g^{\mu\nu}S_\nu .$$

Here x^μ gives the coordinates of the particle and an overdot denotes derivative with respect to affine parameter.

A. The ϕ metric

Using the form of the metric we find that the Hamilton Jacobi equation becomes

$$S_\rho^2 + S_z^2 + \frac{\left(v^2 + \beta^2\right)^2}{\rho^2}S_\phi^2 - \left(4\beta zS_\phi - S_t\right)^2 + \kappa\left(v^2 + \beta^2\right) = 0 . \tag{4.5}$$

This equation can be separated as follows:

$$S = \int \mathcal{Z}(z)dz + \int \mathcal{R}(\rho)d\rho + L\phi - Et . \tag{4.6}$$

Here L and E are constants. Substituting in the Hamilton-Jacobi equation (4.5) we find that the longitudinal and radial momentum functions $\mathcal{Z}$ and $\mathcal{R}$ are given by

$$\mathcal{Z} = \left[\left(4\beta zL + E\right)^2 - U^2\right]^{1/2} , \tag{4.7}$$

$$\mathcal{R} = \left[U^2 - \frac{\left(v^2 + \beta^2\right)^2}{\rho^2}L^2 - \kappa\left(v^2 + \beta^2\right)\right]^{1/2} .$$

Here U is a separation constant. The constants $-S_t = E$ and $S_\phi = L$ are conserved quantities related to the Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$, respectively. Essentially E is the energy of the particle and L is its angular momentum. In the limiting case, $\beta = 0$, when the solution returns to being that of the seed magnetic universe, the

separation constant $U = \left[E^2 - \mathcal{Z}^2\right]^{1/2}$ is the "transverse energy" [7]. The presence of a third conserved quantity U in this more general case is related to the fact that this spacetime admits a Killing tensor. The Killing tensor $K_{\mu\nu}$ is given by

$$K_{\mu\nu} = \left(v^2 + \beta^2\right)^2\left[\partial_\mu z\,\partial_\nu z - \partial_\mu t\,\partial_\nu t\right] . \tag{4.8}$$

The geodesic equation then becomes

$$\dot{\rho} = \left[U^2\left(v^2 + \beta^2\right)^{-2} - \rho^{-2}L^2 - \kappa\left(v^2 + \beta^2\right)^{-1}\right]^{1/2} ,$$
$$\dot{z} = \left(v^2 + \beta^2\right)^{-1}\left[\left(E + 4\beta zL\right)^2 - U^2\right]^{1/2} ,$$
$$\tag{4.9}$$
$$\dot{\phi} = \rho^{-2}\left(v^2 + \beta^2\right)L - 4\beta z\left(v^2 + \beta^2\right)^{-1}\left(E + 4\beta zL\right) ,$$
$$\dot{t} = \left(v^2 + \beta^2\right)^{-1}\left(E + 4\beta zL\right) .$$

First consider the case $L \neq 0$. Then ρ oscillates between a minimum and a maximum value. The coordinate z is then given by

$$\int \frac{d\lambda}{v^2 + \beta^2} = \int \frac{dz}{\sqrt{(E + 4\beta zL)^2 - U^2}} \, , \qquad (4.10)$$

where λ is the affine parameter of the geodesic. Thus at any finite value of λ the coordinate z remains finite. It then follows from Eqs. (4.9) that ϕ and t remain finite for all finite values of λ. Thus the geodesics are complete.

Now consider the case where $L = 0$. Then ρ is bounded unless $\kappa = 0$. (When $\kappa = L = 0$ the coordinate ρ still remains finite for finite λ.) The coordinate z is then given by

$$z = \int d\lambda \, (v^2 + \beta^2)^{-1} \sqrt{E^2 - U^2} \, . \qquad (4.11)$$

Thus z remains finite at finite λ. Using Eqs. (4.9) one can then show that ϕ and t also remain finite for finite λ. Thus the geodesics are complete. We have thus shown that the ϕ metric is timelike and null geodesically complete.

B. The t metric

Using the form of the metric we find that the Hamilton Jacobi equation becomes

$$S_\rho^2 + S_z^2 + \left(v^2 \rho^{-1} S_\phi - 8\beta z \rho v S_t\right)^2 - \left(1 + \beta^2 \mathcal{E}^2\right)^2 S_t^2 + \kappa \left(1 + \beta^2 \mathcal{E}^2\right) v^2 = 0 \, . \qquad (4.12)$$

This equation can be separated as follows:

$$S = \mathcal{H}(z, \rho) + L\phi - Et \, . \qquad (4.13)$$

Note that $\mathcal{H}$ cannot be further separated. This metric has no Killing tensor; so we have only the two constants of the motion E (the energy) and L (the angular momentum).

The geodesic equation becomes

$$\begin{aligned}
\dot{\rho} &= v^{-2} \left(1 + \beta^2 \mathcal{E}^2\right)^{-1} \mathcal{H}_\rho \, , \\
\dot{z} &= v^{-2} \left(1 + \beta^2 \mathcal{E}^2\right)^{-1} \mathcal{H}_z \, , \\
\dot{\phi} &= \left(1 + \beta^2 \mathcal{E}^2\right)^{-1} v\rho^{-1} \left(v\rho^{-1} L + 8\beta z \rho E\right) \, , \\
\dot{t} &= \left(1 + \beta^2 \mathcal{E}^2\right) v^{-2} E - \left(1 + \beta^2 \mathcal{E}^2\right)^{-1} 8\beta z \rho \left(v\rho^{-1} L + 8\beta z \rho E\right) \, .
\end{aligned} \qquad (4.14)$$

Using Eqs. (4.14) in the Hamilton-Jacobi equation (4.12) we find

$$\dot{\rho}^2 + \dot{z}^2 - v^{-4} E^2 + v^{-2} \left(1 + \beta^2 \mathcal{E}^2\right)^{-2} \left(v\rho^{-1} L + 8\beta z \rho E\right)^2 + \kappa \left(1 + \beta^2 \mathcal{E}^2\right)^{-1} v^{-2} = 0 \, .$$

It then follows that

$$\dot{\rho}^2 + \dot{z}^2 \le v^{-4} E^2$$

and since $v \ge 1$ we find that $|\dot{\rho}| \le |E|$ and $|\dot{z}| \le |E|$. Therefore ρ and z remain finite for all finite affine parameter λ. It then follows from Eqs. (4.14) that ϕ and t remain finite for all finite λ. Thus this spacetime is null and timelike geodesically complete.

This method cannot be used to show completeness of the z metric. However, the fact that the z and t metrics are related by a (complex) coordinate transformation leads us to believe that the z metric is also null and timelike geodesically complete.

TABLE I. Results: new solutions of the Einstein-Maxwell equations generated from the magnetic universe.

New metric	z metric	ϕ metric	t metric
Seed Killing form K	$K^z = v^2 dz$	$K^\phi = \left(\rho^2/v^2\right) d\phi$	$K^t = -v^2 dt$
Norm f	$f = -v^2$	$f = -\rho^2/v^2$	$f = v^2$
Maxwell potential Φ	$\Phi_z = -2it$	$\Phi_\phi = -1/v$	$\Phi_t = 2iz$
Einstein potential $\mathcal{E}$	$\mathcal{E}_z = -\left(v^2 + 4t^2\right)$	$\mathcal{E}_\phi = -1/v$	$\mathcal{E}_t = v^2 - 4z^2$
One-form A^E	$A_z^E = \mathcal{E}_z dz - 4i\rho^2 v^{-1} t d\phi$	$A_\phi^E = -v^{-1} d\phi - 2iz dt$	$A_t^E = \mathcal{E}_t dt + 4i\rho^2 v^{-1} z d\phi$
One-form M^{EE}	$M_z^{EE} = \mathcal{E}_z^2 dz$	$M_\phi^{EE} = v^{-2} d\phi$	$M_t^{EE} = \mathcal{E}_t^2 dt$
$\|\Lambda\|^2$	$1 + \beta^2 \mathcal{E}_z^2$	$1 + \beta^2 v^{-2}$	$1 + \beta^2 \mathcal{E}_t^2$
New tetrad form $e^{k'}$	$\|\Lambda\|^{-1} \left(v dz + 8\beta\rho^2 t d\phi\right)$	$\rho v^{-1} \|\Lambda\|^{-1} \left(d\phi + 4\beta z dt\right)$	$\|\Lambda\|^{-1} \left(v dt + 8\beta\rho^2 z d\phi\right)$
Tetrad forms $e^{a'} \perp e^{k'}$	$\|\Lambda\| e^a$	$\|\Lambda\| e^a$	$\|\Lambda\| e^a$

V. SUMMARY

The results are summarized in Table I. New Einstein-Maxwell spactimes are generated from the magnetic universe as a seed metric. The latter has three Killing vector fields $\mathbf{K_z}$, $\mathbf{K_\phi}$, $\mathbf{K_t}$. The new metrics are found by applying the Ehlers transformation with the Maxwell-Ernst and Einstein-Ernst potentials to find the new Killing form and the new tetrad basis forms.

ACKNOWLEDGMENTS

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[8] W. Slebodzinski, *Exterior Forms and Their Applications*, (PWN, Warszawa, 1970), p. 398, theorem 187.5. Another application of the theorem may be made as follows: In the literature the one-form of which W^M is the exterior derivative is labeled A^M and we have a *gauge condition*

$$\mathcal{L}_{\mathbf{K}} A^M = d\left(K \ulcorner A^M\right) - K \ulcorner W^M$$

$$= d\left(K \ulcorner A^M\right) - d\Phi = 0 .$$

Thus $K \ulcorner A^M = \Phi$ using the condition $\mathcal{L}_{\mathbf{K}} A^M = 0$, and fixing the constant of integration at zero.

[9] Note by M.A.M.: Apologies to my old friend Kent Harrison. Upon my return from travels in New Zealand and Australia it occurred to me to examine the paper (Ref. [2]) by B. K. Harrison. I found that, modulo some redundant parameters and a duality rotation mixing the magnetic and electric fields, the solution for the ϕ metric is indicated there, and designated "a twisted Melvin universe." It is included as a special case in a general family of solutions obtained from flat spacetime as a seed metric. No analysis of nonsingularity or completeness is given in Harrison's paper.