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Impact of frame dragging on the Kepler frequency of relativistic stars

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It has long been known that in general relativity the centrifugal force on an element in a rotating star involves the frequency of the star *relative* to the frequency at which the local inertial frame is dragged by the rotation. Intuitively, one would expect that this would increase the critical frequency at which rotation disrupts the star. Our analysis shows the opposite to be true and gives theoretical underpinning to a commonly used empirical formula for the Kepler frequency of a rotating star.

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I. INTRODUCTION

The dynamical effects of rotation in gravitational fields in some cases seem strange, and none is more strange than the reversal of the centrifugal force in the vicinity of a Schwarzschild black hole [1,2]. In this paper we discuss another but unrelated phenomenon associated with the rotation of a star, which, though less spectacular, runs counter to classical expectation and accounts in part for a numerical observation concerning the Kepler frequency. Two groups independently made the useful observation that the *fully relativistic* computation of the Kepler frequency of a *rotating* neutron star at the mass limit of a sequence can be approximated to an accuracy of better than 10% by a factor, less than unity, times the *classical* expression for the Kepler frequency of a satellite in circular orbit around the corresponding *spherical nonrotating* star [3,4]. The observation has been utilized in papers too numerous to cite and provides an enormous simplification of the problem because the solution of the numerically intensive and complicated general relativistic equations for a rotating star can be replaced by the solution of the much simpler Oppenheimer-Volkoff (OV) equations. The two groups of authors who provided this valuable observation did so on the basis of numerical solutions, and no hint was provided as to how this result could emerge from the general relativistic (GR) expression of the Kepler frequency of a rotating star, which is actually a self-consistency condition on the solution and is a very different expression from the classical one. We have given a partial explanation elsewhere [5], and that work provided the hint that frame dragging plays an important role and one that is *counter* to our classical intuition.

A satellite in stable circular orbit at the equator of a *nonrotating* star has a frequency in *general relativity* that is precisely equal to the classical one [6]:

$$\Omega^2 = M/R^3 \ . \tag{1}$$

In classical mechanics this expresses the balance of gravitational and centrifugal forces. Here M and R are the gravitational mass and radius of the star, and Ω is the uniform angular velocity of the satellite. In classical mechanics the same expression holds for the Kepler period of a satellite at the equator R of a *rotating* axially symmetric star, but in general relativity the situation is drastically altered, as is well known. Among the important effects is the phenomenon of dragging of local inertial frames by the rotating star [7–10]. Mach's critical attention to the concept of inertial forces no doubt played an important role in ultimately focusing attention on the effects of rotating matter. Thirring appears to have been the first to realize that in Einstein's theory a rotating mass shell drags the local inertial frames [7]. The effect was studied in greater generality by Brill and Cohen [10]. Shortly thereafter, Hartle incorporated the effect into his calculation of the equilibrium configurations of rotating stars [11]. He notes that the centrifugal force acting on a fluid element of the star is governed by the rate of rotation of the star, assumed to be uniform, *relative* to the *local* inertial frames, which are dragged by the star's rotation, in the same direction. The frequency with which the local inertial frames are dragged is largest at the center of the star, never exceeds the frequency of the star itself, and goes to zero at great distance from the star. It is this problem that we reexamine in this paper. The above statement by Hartle is correct, but the words by themselves imply that inasmuch as the centrifugal effects are governed by the *difference* of two frequencies of the same sign, the effects should be smaller; that is to say, the Kepler frequency is correspondingly increased and is larger than the value given by (1). This turns out to be incorrect. The reason that the quoted words of a quarter century ago do not convey the correct implication is discussed in the next section.

Of course, there are other factors that affect the Kepler frequency of a relativistic star, but they are not at issue and have been analyzed elsewhere [5]. Our analytic discussion progresses in three stages, with an improvement in the metric at each.

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II. ANALYTIC TREATMENT

While the classical result (1) holds for a particle in orbit around a *nonrotating* star also in general relativity, it is easy to understand why it cannot hold, for several reasons, for a *rotating* star in general relativity. The radially dependent dragging of local inertial frames must perforce affect the actual distribution of matter in the *rotating* star, and hence the metric of spacetime is altered by the rotation, that is, by the particular distribution of matter, determined by the condition of equilibrium or balance of forces. In classical mechanics space and time are assumed to be absolute. In general relativity the metric functions are dynamically determined by the distribution of mass, which itself responds to the metric. It should not be surprising therefore that the expression for the Kepler frequency does not resemble the classical one. Instead it is (cf. Appendix A)

$$\Omega_K = \omega + \frac{\omega'}{2\psi'} + e^{\nu-\psi}\left[\frac{\nu'}{\psi'} + \left(\frac{\omega'}{2\psi'}e^{\psi-\nu}\right)^2\right]^{1/2} . \quad (2)$$

The primes denote derivatives with respect to Schwarzschild radial coordinate r, and all functions on the right are evaluated at the star's equator. More than this, they depend also on Ω_K, so that the above is *not* an equation for Ω_K, but a transcendental relationship which the solution of the equations of stellar structure must satisfy if the star is rotating at its Kepler frequency. The frame dragging frequency $\omega(r)$ satisfies a particular boundary condition at the equator of the star that has been written before and is derived in Appendix D.

A. Restriction to Schwarzschild metric

To obtain an analytic solution to the problem, we shall, in a first step, take the metric which corresponds to that of a static spherically symmetric star, i.e., the Schwarzschild metric. This will provide a first orientation. Corrections to this metric will be considered in the next sections. Thus at the equator we take

$$e^{2\nu} = 1 - \frac{2M}{R} , \quad (3)$$

$$e^{2\psi} = R^2 , \quad (4)$$

where for our approximate solution to Eq. (2) we take M to be the mass of the rotating star and R its equatorial radius. (The second of these equations looks strange, but we follow an old precedent so as not to introduce confusion [12–14]. See Appendix A for the general form of the metric.) Combined with the condition that outside the star $\omega(r)$ must obey (cf. Appendix D)

$$\omega(r) = \frac{2I}{r^3}\Omega, \quad r > R \quad (5)$$

(where I is the moment of inertia), we are able to write an approximate solution to the transcendental equation for Ω_K: namely,

$$\Omega_K^2 = \left[1 + \frac{\omega(R)}{\Omega_K} - 2\left(\frac{\omega(R)}{\Omega_K}\right)^2\right]^{-1} \frac{M}{R^3}$$
$$= \left[1 + \frac{2I}{R^3} - 2\left(\frac{2I}{R^3}\right)^2\right]^{-1} \frac{M}{R^3} . \quad (6)$$

This approximate result has a very interesting structure, for it shows the classical result modified by a prefactor. The prefactor leads to a *reduction* in the relativistic Kepler frequency when $\omega(R)/\Omega_K < \frac{1}{2}$ or equivalently $4I/R^3 < 1$. There is no apparent reason why this limit *must* be obeyed, even if in practice it is (cf. Refs. [5,14,15]). Therefore we proceed to an improved metric.

B. Monopole-corrected metric

Here we carry the analytic investigation one step further by taking monopole corrections to the Schwarzschild metric into account [11,16] (see Appendix B). In this case Eq. (3) reads

$$e^{2\nu} = 1 - \frac{2M}{R} + \frac{2J^2}{R^4} , \quad (7)$$

while Eq. (4) remains unchanged. Here $J \equiv I\Omega$ is the angular momentum. From Eq. (2) one finds for the Kepler frequency

$$\Omega_K^2 = \left[1 + \frac{\omega(R)}{\Omega_K} - \left(\frac{\omega(R)}{\Omega_K}\right)^2\right]^{-1} \frac{M}{R^3}$$
$$= \left[1 + \frac{2I}{R^3} - \left(\frac{2I}{R^3}\right)^2\right]^{-1} \frac{M}{R^3} . \quad (8)$$

The prefactor in Eq. (8) *always* leads to a *reduction* of the Kepler frequency below its classical value because $\omega(R)/\Omega_K < 1$. The dragging frequency cannot exceed the star frequency [11]. This universal limit is a result of the improved metric.

It may be of some interest that Eq. (5) places a limit involving the moment of inertia and radius of a star:

$$\frac{2I}{R^3} < 1 . \quad (9)$$

C. Quadrupole-corrected metric

At the level of quadrupole corrections, there are certain terms that we can investigate only numerically. We describe this in the Appendix C. For a broad sample of 17 equations of state (see Ref. [5]), the terms not susceptible to analytic analysis are shown to alter the Kepler frequency generally by less than 3%. So we ignore them. Then the metric through to quadrupole corrections due to rotation are

$$e^{2\nu} = 1 - \frac{2M}{R} + \frac{J^2}{R^4}\left(3 + \frac{2M_s}{R} - \frac{R}{M_s}\right) ,$$
$$e^{2\psi} = R^2\left[1 + \frac{J^2}{R^4}\left(2 + \frac{R}{M_s}\right)\right] , \tag{10}$$

where M_s is the mass of the star at the mass limit of the *nonrotating* and therefore spherical sequence (solution to the OV equations). After considerable algebra, an equation similar to those derived above is obtained:

$$\Omega_K^2 = \left[1 + (1+\epsilon)\left(\frac{\omega(R)}{\Omega_K}\right) - (2+\eta)\left(\frac{\omega(R)}{\Omega_K}\right)^2\right]^{-1} \frac{M}{R^3} . \tag{11}$$

The expressions for ϵ and η are derived in the Appendix C. For a wide selection of models [4,5,14], we have computed these parameters which we record in Tables I and II, together with the ratio of frame dragging to Kepler frequency of the limiting mass star as computed in GR. The phenomenon of frame dragging causes a *reduction* in the Kepler frequency if $\omega/\Omega_K < (1+\epsilon)/(2+\eta)$ [obtained from Eq. (11)], which we see is indeed satisfied by a comfortable margin in all cases.

The results of the above three subsections reduce to Eq. (1) for a particle in a stable orbit around a static relativistic star, since in that case $\omega(r) \equiv 0$.

D. Empirical formula

We have shown above how the effect of frame dragging on the Kepler frequency can be expressed as a factor, slightly model dependent, times the classical expression for the balance between gravity and centrifuge at the equator R of a rotating star of mass M at the termination of the stable sequence. The empirical expression involves the radius and mass of the corresponding spherical *nonrotating* star [3,4]:

$$\Omega_K = \alpha\left(\frac{M_s}{R_s^3}\right)^{1/2} , \quad \alpha \approx 0.625 . \tag{12}$$

Elsewhere we have shown how the M/R^3 term in (11) is reduced to this final form by accounting for the radius and mass augmentation due to rotation [5].

TABLE I. Model-dependent parameters ϵ, η and the ratio $\omega(R)/\Omega$ all computed in GR. The last column is the limiting value of the ratio that leads to a reduction in Kepler frequency due to frame dragging.

Label[a]	ϵ	η	ω/Ω_K	$(1+\epsilon)/(2+\eta)$
1	0.031	0.356	0.17	0.44
2	0.040	0.110	0.20	0.49
3	0.025	0.358	0.16	0.43
4	0.022	0.448	0.15	0.42
5	0.035	0.191	0.19	0.47
6	0.021	0.468	0.15	0.41
7	0.049	0.058	0.22	0.51
8	0.026	0.388	0.16	0.43
9	0.029	0.318	0.17	0.44
10	0.078	−0.209	0.28	0.60
11	0.084	−0.217	0.29	0.61
12	0.050	−0.024	0.22	0.53
13	0.078	−0.102	0.27	0.57
14	0.087	−0.234	0.29	0.62
15	0.119	−0.415	0.35	0.71
16	0.128	−0.418	0.36	0.71
17	0.073	−0.201	0.27	0.60

[a]These labels refer to the equations of state of Ref. [5].

TABLE II. Model-dependent parameters ϵ, η and the ratio $\omega(R)/\Omega$ all computed in GR. The notation for the equations of state (EOS's) is from Ref. [4].

EOS	ϵ	η	ω/Ω_K	$(1+\epsilon)/(2+\eta)$
L	0.060	0.074	0.23	0.51
PAL1	0.015	0.660	0.10	0.38
D	0.028	0.390	0.14	0.43
C	0.041	0.232	0.17	0.47
PAL3	0.021	0.445	0.13	0.42
FP	0.061	−0.013	0.21	0.53
F	0.023	0.348	0.13	0.44
A	0.052	0.082	0.20	0.51
π	0.063	−0.040	0.22	0.54
B	0.049	0.100	0.19	0.50
G	0.046	−0.003	0.20	0.52

III. SUMMARY

In this work we showed that the dragging of local inertial frames caused by the rotation of any massive star *reduces* its Kepler (mass shedding) frequency relative to the Kepler period of a satellite in a circular orbit around a nonrotating star, contrary to the intuitive expectation that naturally follows from the fact that the centrifugal force on fluid elements of the star is determined by the frequency of the star *relative* to the local inertial frames which are dragged in the direction of the star's rotation.

This counterintuitive behavior can be understood mathematically as following from the fact that Eq. (2) is not a formula for Ω_K, but a transcendental equation, in which all quantities on the right depend also on Ω_K and on $\omega(r)$. Thus to say that the centrifugal effect on a fluid element of the star at r depends on $\Omega_K - \omega(r)$, while true, does *not* inform us that there is a *reduction* in the centrifugal effect with corresponding increase in the Kepler frequency. We mention that this counterintuitive behavior of the role of frame dragging, though a peculiar effect of rotation, has nothing to do with the still more bizarre "change in sign of the centrifugal force" in the vicinity of black holes, which, as the discoverers of this latter effect emphasize, has nothing to do with frame dragging since it holds for a satellite in orbit around a Schwarzschild black hole [1,2].

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APPENDIX A: KEPLER FREQUENCY IN GENERAL RELATIVITY

We are interested in models of compact stars that are uniformly rotating, axisymmetric fluid configurations. Therefore the spacetime is stationary and axisymmetric, which corresponds to respectively time translation and rotational symmetry. The line element can be written as [17,14,18]

$$ds^2 = -e^{2\nu(r,\theta;\Omega)}dt^2 + e^{2\psi(r,\theta;\Omega)}[d\phi - \omega(r,\theta;\Omega)dt]^2 + e^{2\mu(r,\theta;\Omega)}d\theta^2 + e^{2\lambda(r,\theta;\Omega)}dr^2 \ . \tag{A1}$$

As a consequence of the underlying symmetries, the metric functions ν, ψ, μ, and λ are independent of t and ϕ. The function $\omega(r,\theta;\Omega)$ denotes the angular velocity of the local inertial frames (dragging of the local inertial frames). As indicated, it depends on the radial coordinate r and the azimuthal coordinate θ, and is proportional to the star's rotational velocity Ω.

The frequency Ω is assumed to be constant throughout the star's fluid. The frequency $\bar{\omega}(r,\theta;\Omega) \equiv \Omega - \omega(r,\theta;\Omega)$, which is the star's rotational frequency relative to the frequency of the local inertial frames, is the one on which the centrifugal force acting on the mass elements of the rotating star's fluid depends [11]. It is this frequency relative to which the fluid inside the star moves.

From Eq. (A1) one finds, for a material particle rotating at the star's surface (constant r and θ coordinates),

$$1 = e^{2\nu}\left(\frac{dt}{d\tau}\right)^2 - e^{2\psi}\left(\frac{d\phi}{d\tau} - \omega\frac{dt}{d\tau}\right)^2 \ . \tag{A2}$$

For the purpose of brevity, the arguments of the functions here and in the following are omitted. From $u^\phi = \Omega u^t$, where $u^\phi \equiv d\phi/d\tau$ and $u^t \equiv dt/d\tau$, one obtains $d\phi/d\tau = \Omega dt/d\tau$. Thus the time component of the particle's four-velocity is given by

$$\frac{dt}{d\tau} = \frac{e^{-\nu}}{\sqrt{1-V^2}} \ , \tag{A3}$$

where

$$V \equiv e^{\psi-\nu}\bar{\omega} \tag{A4}$$

denotes the particle's orbital velocity ($u^r = u^\theta = 0$). Equation (A4) serves to express the star's rotational frequency in terms of V and the frame dragging frequency,

$$\Omega = e^{\nu-\psi}V + \omega \ , \tag{A5}$$

which, in other words, is the expression for the rotational frequency of a massive particle rotating in a stable orbit of constant radial distance, i.e., $r = R_{\rm eq}$ and $\theta = \pi/2$, from the star's origin. For its evaluation, knowledge of V is necessary. The relevant mathematical expression for V will be derived now. Since the particle path is a circular orbit, we can determine V simply as the extremal of $ds^2(t,r,\phi)$, i.e., $ds/dr = 0$. From Eq. (A1) one obtains

$$\psi' e^{2\nu}V^2 - \omega' e^{\psi+\nu}V - \nu' e^{2\nu} = 0 \ , \tag{A6}$$

where, according to Eq. (A4),

$$d\phi - \omega\, dt = (\Omega - \omega)dt = Ve^{\nu-\psi}dt \ .$$

Equation (A6) constitutes a quadratic equation in the equatorial velocity V. Its solutions are

$$V_{+,-} = \frac{\omega'}{2\psi'}e^{\psi-\nu} \pm \left[\frac{\nu'}{\psi'} + \left(\frac{\omega'}{2\psi'}e^{\psi-\nu}\right)^2\right]^{1/2} \ . \tag{A7}$$

The solution V_+ corresponds to corotation, which is the desired one in connection with the stability of the star to mass shedding. The other solution corresponds to a counterrotating satellite at its Kepler frequency.

In summary, Eqs. (A5) and (A7) are to be solved simultaneously in combination with the stellar structure equations by means of a self-consistent iteration procedure in order to find the general relativistic Kepler frequency of a rotating star model of given central density [14,15].

APPENDIX B: MONOPOLE CORRECTION TO THE METRIC

For our purpose we recall only the metric functions ν and ψ occurring in Eq. (A1). These are given by [11,16]

$$e^{2\nu(r,\theta;\Omega)} = e^{2\Phi(r)}\{1 + 2[h_0(r;\Omega) + h_2(r;\Omega)P_2(\cos\theta)]\} \ , \tag{B1}$$

$$e^{2\psi(r,\theta;\Omega)} = r^2\sin^2\theta\{1 + 2[v_2(r;\Omega) - h_2(r;\Omega)]P_2(\cos\theta)\} \ , \tag{B2}$$

where

$$e^{2\Phi(r)} = \left(1 - \frac{2M}{r}\right), \quad r \geq R \ . \tag{B3}$$

The functions h_l, m_l ($l = 0,2$), and v_2 of Eqs. (B1) and (B2) stand for the monopole and quadrupole perturbation functions, and the quantity P_2 is the second order Legendre polynomial, $P_2(x) = (3x^2 - 1)/2$. In the nonrotating limit, the perturbation functions vanish identically, and the metric functions reduce to those of a Schwarzschild star.

The monopole function h_0 is given by

$$e^{2\Phi(r)}h_0(r) = -\frac{\Delta M}{r} + \frac{J^2}{r^4}, \quad r \geq R\ . \tag{B4}$$

Neglecting the quadrupole perturbation functions in Eqs. (B1) and (B2), one obtains, for the metric functions at the star's equator,

$$e^{2\nu(R)} = 1 - \frac{2M}{R} + \frac{2J^2}{R^4}\ , \tag{B5}$$

$$e^{2\psi(R)} = R^2\ . \tag{B6}$$

The quantity ΔM in Eq. (B4) denotes the mass increase of a rotating star caused by rotation; J ($\equiv I\Omega$) refers to the star's angular momentum.

APPENDIX C: QUADRUPOLE CORRECTION TO THE METRIC

The quadrupole functions h_2 and v_2 of Eqs. (B1) and (B2) are given by

$$h_2(r) = \frac{J^2}{r^4}\left(1 + \frac{r}{M_s}\right) + AQ_2^2\left(\frac{r}{M_s} - 1\right), \quad r \geq R\ , \tag{C1}$$

$$v_2(r) = -\frac{J^2}{r^4} + A\frac{2M_s/r}{\sqrt{1 - 2M_s/r}}Q_2^1\left(\frac{r}{M_s} - 1\right)\ , \quad r \geq R\ . \tag{C2}$$

The quantities Q_2^1 and Q_2^2 denote associated Legendre polynomials of the second kind, and A is a constant [11].

As mentioned in Sec. II C, at the level of quadrupole corrections there are terms in the metric that can be investigated only numerically. These are the expressions proportional to the associated Legendre polynomials. From a numerical study we find that these modify the value of the general relativistic Kepler frequency, Eq. (2), by less than 2–3 %, depending on the equation of state. Ignoring them, the auxiliary functions ϵ and η occurring in Eq. (11) are given by

$$\epsilon = \frac{5}{2}\left(1 + \frac{2}{5}\frac{R}{M_s}\right)R^2\omega^2(R)\ , \tag{C3}$$

$$\eta = \epsilon + \tfrac{1}{2}\left(1 + \frac{1}{4}\frac{R}{M_s}\right)R^2\omega^2(R)\left(\frac{\Omega_K^2}{\omega^2(R)} - 1\right) - \frac{3}{2}\left(1 + \frac{5}{6}\frac{M_s}{R} - \frac{1}{4}\frac{R}{M_s}\right)\ . \tag{C4}$$

By means of the empirical formula of Eq. (12) one obtains, for Eqs. (C3) and (C4) ($\alpha = 0.625 \approx \sqrt{2/5}$),

$$\epsilon = \frac{2}{5}\left(1 + \frac{\Delta R}{R_s}\right)^3\left(1 + \frac{10}{4}\frac{M_s}{R}\right)\left(\frac{\omega(R)}{\Omega_K}\right)^2\ , \tag{C5}$$

$$\eta = \frac{1}{20}\left(1 + \frac{\Delta R}{R_s}\right)^3\left[1 + 4\frac{M_s}{R} + 7\left(1 + \frac{16}{7}\frac{M_s}{R}\right)\left(\frac{\omega(R)}{\Omega_K}\right)^2\right] - \frac{3}{2}\left(1 + \frac{5}{6}\frac{M_s}{R} - \frac{1}{4}\frac{R}{M_s}\right)\ , \tag{C6}$$

with the definition $\Delta R \equiv R - R_s$, where R_s denotes the radius of the nonrotating maximum-mass star.

APPENDIX D: FRAME DRAGGING FREQUENCY AT THE EQUATOR OF A ROTATING STAR

We derive the expression for the frequency of the local inertial frames, ω, at the equator of a rotating star, which rotates with frequency Ω. The result is accurate to order $O(J/r^4)$ [17], where J denotes the star's angular momentum (cf. Appendix B). We begin by deriving an expression for the moment of inertia of a stationary rotating, axisymmetric, relativistic star in equilibrium. Under these restrictions, the expression for the moment of inertia is given by [19]

$$I(\mathcal{A}, \Omega) \equiv \frac{1}{\Omega}\int_{\mathcal{A}} dr\, d\theta\, d\phi\, \mathcal{T}_3^0\sqrt{-g}\ . \tag{D1}$$

In the above equations, $\mathcal{A}$ denotes an axially symmetric region in the interior of a body where all matter is rotating with the same angular velocity Ω. The quantity g refers to the determinant of the metric tensor. For the metric of Eq. (A1), one finds [cf. Ref. [20] for details, except for sign convention errors, so that the $(\epsilon + P)$ terms of Eqs. (12), (13), (17), and (18) should have their sign changed. Also, the upper and lower indices on $\mathcal{T}$ should read as in this paper. Final result for I in both papers is correct.]

$$\sqrt{-g} = e^{\lambda+\mu+\nu+\psi}\ , \tag{D2}$$

$$\mathcal{T}_3^0 = (\epsilon + P)\bar{\omega}(r,\Omega)e^{2\psi(r,\Omega)} \times [e^{2\nu(r,\Omega)} - \bar{\omega}(r,\Omega)^2 e^{2\psi(r,\Omega)}]^{-1}\ . \tag{D3}$$

The expression for the moment of inertia of Eq. (D1) then leads to

$$I = 4\pi\int_0^{\pi/2} d\theta \int_0^{R(\theta)} dr \frac{e^{\lambda(r,\Omega)+\mu(r,\Omega)+\nu(r,\Omega)+\psi(r,\Omega)}[\epsilon + P(\epsilon)]}{e^{2\nu(r,\Omega)-2\psi(r,\Omega)} - \bar{\omega}(r,\Omega)^2}\frac{\bar{\omega}(r,\Omega)}{\Omega}\ , \tag{D4}$$

which reads, in the case of a rotationally nondeformed star,

$$J \equiv I\Omega = \frac{8\pi}{3}\int_0^R dr\, r^4 \frac{\epsilon + P(\epsilon)}{\sqrt{1-2m(r)/r}}\bar{\omega}(r,\Omega)e^{-\Phi(r)} . \quad \text{(D5)}$$

The quantity J denotes the star's angular momentum. From the field equation $\mathcal{R}_3^0 = 8\pi\mathcal{T}_3^0$, one obtains a differential equation for $\bar{\omega}$ [11]:

$$\frac{d}{dr}\left(r^4 j(r)\frac{d\bar{\omega}(r)}{dr}\right) + 4r^3\frac{dj(r)}{dr}\bar{\omega}(r) = 0 , \quad r \le R , \quad \text{(D6)}$$

where

$$j(r) = e^{-\Phi(r)}\left(1 - \frac{2m(r)}{r}\right)^{1/2} . \quad \text{(D7)}$$

From Eq. (D7) it follows that

$$\frac{dj}{dr} = -4\pi r(\epsilon + P)e^{-\Phi}/\sqrt{1-2m/r} , \quad \text{(D8)}$$

which is used to find

$$r^4 j\frac{d\bar{\omega}}{dr} = 6J, \quad r = R , \quad \text{(D9)}$$

from Eq. (D6). Throughout, R is used to denote the equatorial radius in the case of a rotating star and simply the radius of a nonrotating spherical star, when no confusion would arise; otherwise, in the latter case, R_s is used. Here use of Eq. (D5) has been made. For $r \ge R$ one has $j \equiv 1$, and one obtains, from Eq. (D6),

$$\bar{\omega} = -\frac{A}{r^3} + B . \quad \text{(D10)}$$

Since $\bar{\omega} \to \Omega$ for $r \to \infty$ (frame dragging vanishes at infinity), one gets $B = \Omega$. To determine the constant A in Eq. (D10), we compute $d\bar{\omega}/dr$ from Eq. (D10) and make use of Eq. (D9), evaluated at $r = R$, leading to $A = 2J$. Thus the angular velocity of the dragged inertial frames at the star's equator is given by

$$\omega = \frac{2I}{R^3}\Omega . \quad \text{(D11)}$$

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