Dimensionally continued wormhole solutions

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In this paper we consider wormhole solutions for the action of "special Lovelock gravity" recently discussed by Banados, Teitelboim, and Zanelli. This action is, in odd dimensions, the Chern-Simons form for the anti–de Sitter group and, in even dimensions, the Euler density constructed with the Lorentz part of the anti–de Sitter curvature tensor. We present a systematic study of classical wormhole solutions in the special Lovelock theory with various matter content, including a perfect fluid energy-momentum tensor, axionic field, and conformal scalar field.

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I. INTRODUCTION

The study of field equations in more than four dimensions has received a great deal of attention in recent years. The five-dimensional monopole solution has been explicitly constructed by Sorkin [1] and Gross and Perry [2] by embedding the Taub-NUT (Newman-Unti-Tamburino) gravitational instanton [4] into five-dimensional pure Kaluza-Klein theory. In recent years, we have extended this construction to higher-dimensional Kaluza-Klein theories [4]. We have exhibited new classes of instanton solutions to the empty space Einstein equation [5]. On the other hand, the multidimensional gravity Lagrangian could contain an arbitrary number of terms, consisting of the invariants which can be constructed from powers of the Riemann curvature tensor. Especially, the Lagrangian can be formed by a linear combination of dimensionally extended Euler densities [6], which can be considered as the topological generalization of the Hilbert-Einstein Lagrangian. This generalization is often called "Lovelock gravity" [7]. Moreover, the classical dynamical equations associated with the Lovelock Lagrangian are of the second-order field equations. Zwiebach [8] and Zumino [9] have shown that, if the low-energy limit of the supergravity obtained from string theory is to respect unitarity, the corrective terms have to be set in groups giving rise to these Euler densities in such a way that they would lead to ghost-free nontrivial interactions. Static spherically symmetric solutions for the Lovelock theory have been studied by several authors [10]. However, in those references no choice of Lovelock coefficients is made, making it difficult to extract physical information from the solution. Recently, the black hole solutions have been studied in Lovelock theory with a special choice of coefficients [11]. The action is, in odd dimensions, the Chern-Simons form of the anti–de Sitter group and, in even dimensions, the Euler density constructed with the Lorentz part of the anti–de Sitter curvature tensor [11].

There is much attention at present being focused on the problem of wormholes, which are D-dimensional Euclidean metrics that consist of two large asymptotically Euclidean regions joined by a narrow throat. It is believed that the wormhole might play an important role in the theory of quantum gravity. The possible consequences on low energy physics compared with the Planck scale are the possible loss of quantum coherence [12] or an additional indeterminacy in the constants of nature as Coleman argued [13] and the mechanism of setting down the natural constants, especially the cosmological constant [14]. Since the work of Giddings and Strominger [15], many wormhole solutions have been constructed such as axion fields [15], scalar fields with or without spontaneous breaking of global U(1) symmetry [16], Yang-Mills fields [17], and the coupled theory [18]. Wormholes in the Skyrme model [19], in string theory [20], in higher-derivative gravity theory [21], and in higher-dimensional spacetime [22] have also been investigated.

In this paper, our goal is the study of the wormhole solutions in Lovelock theory with the specialized Lovelock coefficients. This could open a way to solve the problems of Lovelock theories: for example, (i) the existence of several families of classical solutions allows the system to jump from one family to another in a random way [23], (ii) the multivaluedness in the inversion of the relation between the metric time derivatives and their associated momenta [23,24], and (iii) the existence of initial data sets for which the Cauchy problem is ill posed, leading to an unpredictable evolution [25]. All of this suggests that a special choice of the arbitrary coefficients should be made. The choice proposed here produces a unique solution, which are D-dimensional Euclidean metrics that consist of two large asymptotically Euclidean regions joined by a narrow throat. It is shown that the Lovelock corrections preserve the essential feature of Einstein gravity models of giving rise to an essentially unique wormhole solution. The outline of this paper is as follows. In Sec. II we introduce the equations of motion for an homogeneous and isotropic metric in the special Lovelock theory. In Sec. III we consider the generalized Tolman wormhole solutions. Section IV deals with the axionic wormhole solutions. The wormhole solutions with conformal scalar fields are studied in Sec. V. The results are summarized in Sec. VI. The Appendix contains the calculation of the existence of a throat.

II. EQUATIONS OF MOTION

The Lovelock gravitational action for a D-dimensional manifold M can be expressed

$$S=\int_M \mathcal{L}_D=\int_M \sum_{m=0}^{[(D-1)/2]} C_m \epsilon_{a_1 \cdots a_D} R^{a_1 a_2} \Lambda \cdots \Lambda R^{a_{2m-1}a_{2m}} \Lambda e^{a_{2m+1}} \Lambda \cdots e^{a_D} . \tag{2.1}$$

Here e^a is the D tetrad associated with the manifold M and $R^a_b=d\omega^a_b+\omega^a_c \Lambda \omega^c_b$ is the corresponding curvature two-form, and ω^a_b is the spin connection, $a_i=\{0,1,\ldots,D-1\}$, and $\epsilon_{a_1 \cdots a_D}$ is the Levi-Civita tensor. The coefficients C_m are arbitrary real constants; $[x]$ is the integral part of the x. As the first term in (2.1) must correspond to the Hilbert-Einstein Lagrangian, we take

$$C_1=\frac{1}{16\pi G_D} \tag{2.2}$$

with G_D the D-dimensional gravitational constant. G_D must not be confused with the effective four-dimensional gravitational constant for universes in which $D-4$ dimensions are compactified. In any case, assuming all Lovelock terms ($m>1$) to be corrections to the Hilbert-Einstein action in low energy limit, one must require C_1 to be positive; otherwise, the gravity theory would not be attractive in the limit.

In this paper, we take the metric of D-dimensional isotropic spacetime to be

$$ds^2=d\tau^2+a^2(\tau)d\Omega^2_{D-1} , \tag{2.3}$$

where τ is the Euclidean time, $a(\tau)$ is the scale factor, and $d\Omega^2_{D-1}$ denotes a line element of the unit $(D-1)$-sphere. D-dimensional anisotropic spacetimes, which could eventually lead to compactification of the extra $D-4$ dimensions, will be considered elsewhere.

Lovelock theory with the special choice of coefficients was recently discussed by Banados, Teitelboim, and Zanelli (BTZ) [11]. In odd-dimensional spacetime ($D=2n-1$), it is possible to construct a Lovelock Lagrangian invariant under the anti–de Sitter group by making a certain choice of the coefficients C_m. The construction method is very similar to the Chern-Simons action in three-dimensional spacetime. The idea of embedding D-dimensional gravity into SO$(D,1)$ was also discussed by MacCarthy and Pagels in the case $D=4$ [26]. The particular form can also be found in Ref. [27] and was earlier used in Ref. [28]. Following BTZ, one has

$$\mathcal{L}_{2n-1}=\sum_{m=0}^{n-1} \frac{\kappa}{D-2m} \begin{bmatrix} n-1 \\ m \end{bmatrix} l^{-D+2m} \epsilon_{a_1 \cdots a_D} R^{a_1 a_2} \Lambda \cdots \Lambda R^{a_{2m-1}a_{2m}} \Lambda e^{a_{2m+1}} \Lambda \cdots \Lambda e^{a_D} , \tag{2.4}$$

where l is a length.

There is no analogue of the Chern-Simons action in the even-dimensional spacetime ($D=2n$). It is necessary to break the full anti–de Sitter symmetry SO$(2n,1)$ in order to produce a nontrivial action [29]. The Lagrangian can be written as

$$\mathcal{L}_{2n}=\sum_{m=0}^{n-1} \kappa \begin{bmatrix} n \\ m \end{bmatrix} l^{-D+2m} \epsilon_{a_1 \cdots a_D} R^{a_1 a_2} \Lambda \cdots \Lambda R^{a_{2m-1}a_{2m}} \Lambda e^{a_{2m+1}} \Lambda \cdots \Lambda e^{a_{2n}} , \tag{2.5}$$

which may be regarded as the gravitational analogue of the Born-Infeld Lagrangian [11]. Equations (2.4) and (2.5) are both the special case of the Lovelock Lagrangian in Eq. (2.1). Taking the homogeneous and isotropic metric (2.3), the action (2.1) becomes

$$S=A_{D-1}(D-1)!\sum_m C_m \int dt\, a^{D-1-2m}[(D-2m)(1-\dot a^2)^m-2ma\ddot a(1-\dot a^2)^{m-1}] , \tag{2.6}$$

where $A_{D-1}=2\pi^{D/2}/\Gamma(D/2)$ is the area of the unit $(D-1)$-sphere. From action (2.6) we can get the equation of motion as

$$\sum_{m=0}^{[(D-1)/2]} C_m(D-2m)G_{(m)ab}=T_{ab} \tag{2.7}$$

where

$$G^0_{(m)0}=-(D-1)!\left[\frac{1-\dot a^2}{a^2}\right]^m , \tag{2.8}$$

$$G^i_{(m)j}=(D-2)!\left[\frac{1-\dot a^2}{a^2}\right]^{m-1} \times\left[2m\frac{\ddot a}{a}-(D-2m-1)\frac{1-\dot a^2}{a^2}\right]\delta^i_j , \tag{2.9}$$

and $i,j=1,\ldots,D-1$; and the specialized Lovelock coefficients are

$$C_m = \begin{cases} \dfrac{\kappa}{D-2m}\begin{bmatrix} n-1 \\ m \end{bmatrix} l^{-D+2m}, & D=2n-1, \\ \kappa \begin{bmatrix} n \\ m \end{bmatrix} l^{-D+2m}, & D=2n. \end{cases} \tag{2.10}$$

From Eq. (2.2), we have

$$\kappa = \begin{cases} \dfrac{l^{D-2}(D-2)}{16\pi G(n-1)}, & D=2n-1, \\ \dfrac{l^{D-2}}{16\pi G n}, & D=2n. \end{cases} \tag{2.11}$$

III. TOLMAN WORMHOLE

A. *D*-dimensional Tolman wormhole in Einstein theory

It is well known that the simplest wormhole is the Tolman wormhole [30] which is a Euclideanized version of the closed Tolman universe [31]. The D-dimensional Tolman wormhole solution of Einstein theory could be derived as follows. The Einstein equation reads

$$R_{ab} - \tfrac{1}{2} R g_{ab} = 8\pi G T_{ab}, \quad a,b=0,1,\ldots,D-1 , \tag{3.1}$$

where T_{ab} is the perfect fluid energy-momentum tensor,

$$T_{ab} = p g_{ab} - (\rho + p) U_a U_b , \tag{3.2}$$

where p is the pressure, ρ is the energy density, and U_a is the D velocity of the perfect fluid normalized by the condition $U_a U^a = 1$. Taking Eqs. (3.1) and (3.2) into account, the Einstein equation reduces to

$$\tfrac{1}{2}(D-1)(D-2)(\dot{a}^2 - 1) = -8\pi G a^2 \rho , \tag{3.3}$$

$$(D-2)a\ddot{a} + \tfrac{1}{2}(D-2)(D-3)\dot{a}^2 - \tfrac{1}{2}(D-2)(D-3) = 8\pi G a^2 p . \tag{3.4}$$

Now we take the state equation to be radiation dominated, or matter dominated; then the first integral gives

$$\dot{a}^2 = 1 - \left[\frac{a_0}{a}\right]^{D-2} \quad \text{for } \rho = (D-1)p ,$$
$$\dot{a}^2 = 1 - \left[\frac{a_0}{a}\right]^{D-3} \quad \text{for } p = 0 , \tag{3.5}$$

where $a_0 = a(0)$ is a constant which will be interpreted as the radius of the wormhole. For either case of the state equation in any dimension, when $\tau=0, \dot{a}=0, a=a_0$, we have $\ddot{a}>0$ which means $a_{\min}$ exists. In the limit of $\tau \to \infty$, we have the asymptotic solution

$$a^2(\tau) \approx \tau^2 . \tag{3.6}$$

Therefore, we have indeed found wormhole solutions which are a generalization of a Tolman wormhole in D-dimensional spacetime.

B. Lovelock-Tolman wormhole for the state equation $p=0$

Next we consider the generalized Tolman wormhole in the specialized Lovelock theory in which the Lovelock coefficients are chosen as in Sec. II. Taking Eqs. (2.6), (2.8), (2.9), and (3.2) into account, the equation of motion reduces to

$$(D-1)! \sum_m C_m (D-2m) \left[\frac{1-\dot{a}^2}{a^2}\right]^m = \rho , \tag{3.7}$$

$$(D-2)! \sum_m C_m (D-2m) \left[\frac{1-\dot{a}^2}{a^2}\right]^{m-1} \times \left[2m\frac{\ddot{a}}{a} - (D-2m-1)\frac{1-\dot{a}^2}{a^2}\right] = p . \tag{3.8}$$

Using Eqs. (3.7) and (3.8), we have $\rho a^{D-1} = \text{const}$ for the state equation to be matter dominated. In the case of odd dimensions, the first integral gives

$$\dot{a}^2 = 1 + \left[\frac{a}{l}\right]^2 - \left[\frac{a_0}{l}\right]^2 \left[\frac{8\pi G l^2 \rho_0}{(D-2)!(D-2)}\right]^{2/(D-1)} , \tag{3.9}$$

where $a_0 = a(0)$ is a constant, and $\rho_0 = \rho(0)$ is also a constant which will be interpreted as the energy density at $\tau=0$. From Eq. (3.9), we have

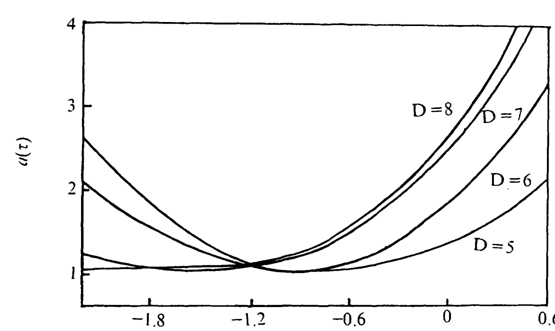

FIG. 1. The numerical solutions of the state equation $p=0$ case in the Lovelock theory, corresponding to $a_{\min}=l$ and $8\pi G l^2 \rho_0 = 200/9$.

$$a=\frac{l}{2}\left[\gamma e^{\tau/l}-\gamma^{-1}\left\{1-\left[\frac{a_0}{l}\right]^2\left[\frac{8\pi G l^2\rho_0}{(D-2)!(D-2)}\right]^{2/(D-1)}\right\}e^{-\tau/l}\right], \tag{3.10}$$

where

$$\gamma=\frac{a_0}{l}+\left\{1-\left[\frac{a_0}{l}\right]^2\left[\frac{8\pi G l^2\rho_0}{(D-2)!(D-2)}\right]^{2/(D-1)}+\left[\frac{a_0}{l}\right]^2\right\}^{1/2}. \tag{3.11}$$

Taking $\tau=\tau_*$ as

$$\tau_*=\frac{l}{2}\ln\frac{\left[\frac{a_0}{l}\right]^2\left[\frac{8\pi G l^2\rho_0}{(D-2)!(D-2)}\right]^{2/(D-1)}-1}{\gamma^2}, \tag{3.12}$$

we find the radius of the wormhole $a_{\min}=a(\tau_*)$:

$$a(\tau_*)=l\left\{\left[\frac{a_0}{l}\right]^2\left[\frac{8\pi G l^2\rho_0}{(D-2)!(D-2)}\right]^{2/(D-1)}-1\right\}^{-1/2}. \tag{3.13}$$

In the case of even dimensions, the first integral gives

$$\dot a^2=1+\left[\frac{a}{l}\right]^2-\left[\frac{a}{l}\right]^2\left[\frac{8\pi G l^2\rho_0}{(D-1)!}\left[\frac{a_0}{a}\right]^{D-1}\right]^{2/(D-2)}. \tag{3.14}$$

In principle, Eq. (3.14) can be integrated since this equation is a separable equation for first order. The numerical calculations are shown in Fig. 1. When $\tau=\tau_*$ and $\dot a(\tau_*)=0$, we can show $\ddot a>0$ which means $a_{\min}$ exists (see Appendix). The radius of wormhole $a_{\min}=a(\tau_*)$ is satisfied with the algebra equation as follows:

$$1+\left[\frac{a_{\min}}{l}\right]^2-\left[\frac{l}{a_{\min}}\right]^{2/(D-2)}\left[\frac{8\pi G l^2\rho_0}{(D-1)!}\left[\frac{a_0}{l}\right]^{D-1}\right]^{2/(D-2)}=0\,. \tag{3.15}$$

In the limit of $\tau\to\infty$, we have the asymptotic solution of Eq. (3.14):

$$a(\tau)\approx l\sinh\frac{\tau}{l}\,. \tag{3.16}$$

Therefore, we have indeed obtained wormhole solutions which are a generalized Tolman wormhole in D-dimensional spacetime.

C. Lovelock-Tolman wormhole for the state equation $\rho=(D-1)p$

Using Eqs. (3.7) and (3.8), we have $\rho a^D=$const for the state equation to be radiation dominated. In the case of odd dimension, the first integral gives

$$\begin{aligned}\dot a^2&=1+\left[\frac{a}{l}\right]^2-\left[\frac{a}{l}\right]^2\left[\frac{8\pi G l^2\rho_0}{(D-2)!(D-2)}\left[\frac{a_0}{a}\right]^{D}\right]^{2/(D-1)} \quad \text{for } D=2n-1\,,\\ \dot a^2&=1+\left[\frac{a}{l}\right]^2-\left[\frac{a}{l}\right]^2\left[\frac{8\pi G l^2\rho_0}{(D-1)!}\left[\frac{a_0}{a}\right]^{D}\right]^{2/(D-2)} \quad \text{for } D=2n\,.\end{aligned} \tag{3.17}$$

The equation of the radius of the wormhole is

$$\begin{aligned}&1+\left[\frac{a_{\min}}{l}\right]^2-\left[\frac{l}{a_{\min}}\right]^{2/(D-1)}\left[\frac{8\pi G l^2\rho_0}{(D-2)!(D-2)}\left[\frac{a_0}{l}\right]^{D}\right]^{2/(D-1)}=0 \quad \text{for } D=2n-1\,,\\ &1+\left[\frac{a_{\min}}{l}\right]^2-\left[\frac{l}{a_{\min}}\right]^{4/(D-2)}\left[\frac{8\pi G l^2\rho_0}{(D-1)!}\left[\frac{a_0}{l}\right]^{D}\right]^{2/(D-2)}=0 \quad \text{for } D=2n\,.\end{aligned} \tag{3.18}$$

In either case, when $\tau=\tau_*$ and $\dot a(\tau_*)=0$, we can show $\ddot a>0$ which means $a_{\min}$ exists and $a_{\min}=a(\tau_*)$ (see Appendix). In the limit of $\tau\to\infty$, we have the asymptotic solution $a(\tau)\approx l\sinh(\tau/l)$. Therefore, we have indeed obtained wormhole solutions which are also the generalized Tolman wormhole in D-dimensional spacetime.

IV. WORMHOLE SOLUTIONS WITH AN AXIONIC FIELD

The axion is a prediction of the most elegant solution to the strong-CP problem of quantum chromodynamics, Peccei-Quinn symmetry. If the axion does indeed exist, it has important astrophysical and cosmological implications. In this section we will discuss the wormhole solutions in Lovelock theory with a spherically axionic field. We consider the model which contains a $(D-1)$-form antisymmetric tensor field H. The action for axionic field is

$$S_{\mathrm{mat}}=\int f^2 H\wedge * H\ , \tag{4.1}$$

where $*$ is the Hodge dual, and f is the Peccei-Quinn scale. The $(D-1)$-form field is defined as $H=dB$ so that $dH=0$. In writing down the action we have omitted the possible topological term and surface term which are irrelevant to the solutions discussed below. The action for the model is

$$S=S_{\mathrm{Lov}}+S_{\mathrm{mat}}\ , \tag{4.2}$$

where S_{Lov} is defined in Eq. (2.1). The field equations could be derived by using the method of variation:

$$\sum_{m=0}^{(D-1)/2} C_m(D-2m)G_{(m)ab}=f^2[(D-1)H_{acd\cdots e}H_b^{cd\cdots e}-\tfrac{1}{2}g_{ab}H_{cd\cdots e}H^{cd\cdots e}]\ , \tag{4.3}$$

$$d*H=0\ . \tag{4.4}$$

If we take $(D-1)$-form field H to be

$$H=\frac{\mu a_0^{a-1}}{f^2 a^{D-1}}\varepsilon_{ij\cdots k}dx^i\wedge dx^j\wedge\cdots\wedge dx^k\ , \tag{4.5}$$

where μ is a constant, $i,j=1,\dots,D-1$, and a_0 is a constant which will be interpreted as the initial value. It is easy to prove that the H equation $dH=d*H=0$ is satisfied. Considering the metric (2.3) and Eq. (4.5), the equation of motion (4.3) could be reduced to

$$\sum_m C_m(D-2m)\left[\frac{1-\dot a^2}{a^2}\right]^m=\frac{\mu^2 a_0^{2D-2}}{f^2 a^{2D-2}}\ , \tag{4.6}$$

$$\sum_m C_m(D-2m)\left[\frac{1-\dot a^2}{a^2}\right]^{m-1}\left[2m\frac{\ddot a}{a}-(D-2m-1)\frac{1-\dot a^2}{a^2}\right]=\frac{(D-1)\mu^2 a_0^{2D-2}}{f^2 a^{2D-2}}\ . \tag{4.7}$$

In the case of odd dimensions, we have the first integral

$$\dot a^2=1+\left[\frac{a}{l}\right]^2\left\{1-\left[\frac{a_0}{a}\right]^4\left[\frac{8(D-1)\pi G\mu^2 l^2}{(D-2)f^2}\right]^{2/(D-1)}\right\}, \tag{4.8}$$

where $a_0=a(0)$ is a constant. From Eq. (4.8), we have

$$a(\tau)=\frac{l}{2}\left[\tilde\gamma e^{2\tau/l}+\left\{1+4\left[\frac{a_0}{l}\right]^4\left[\frac{8(D-1)\pi G\mu^2 l^2}{(D-2)f^2}\right]^{2/(D-1)}\right\}\tilde\gamma^{-1}e^{-2\tau/l}-2\right]^{1/2}, \tag{4.9}$$

where

$$\tilde\gamma=1+2\left[\frac{a_0}{l}\right]^2+2\left\{\left[\frac{a_0}{l}\right]^4+\left[\frac{a_0}{l}\right]^2-\left[\frac{a_0}{l}\right]^4\left[\frac{8(D-1)\pi G\mu^2 l^2}{(D-2)f^2}\right]^{2/(D-1)}\right\}^{1/2}. \tag{4.10}$$

If we take $\tau=\tau_*$, and

$$\tau_*=\frac{l}{4}\ln\left[\left\{1+4(a_0/l)^4\left[\frac{8(D-1)\pi G\mu^2 l^2}{(D-2)f^2}\right]^{2/(D-1)}\right\}\Big/\tilde\gamma^2\right]\ , \tag{4.11}$$

then $a(\tau_*)$ will be interpreted as the radius of the wormhole. We have

$$a_{\min}=l\left[\left\{\left[\frac{a_0}{l}\right]^4\left[\frac{8(D-1)\pi G\mu^2 l^2}{(D-2)f^2}\right]^{2/(D-1)}+\frac{1}{4}\right\}^{1/2}-\frac{1}{2}\right]^{1/2}. \tag{4.12}$$

In the case of even dimensions, the first integral gives

$$\dot a^2=1+\left[\frac{a}{l}\right]^2\left\{1-\left[\frac{8\pi G\mu^2 l^2}{f^2}\left[\frac{a_0}{a}\right]^{2D-2}\right]^{2/(D-2)}\right\}. \tag{4.13}$$

When $\tau=\tau_*$ and $\dot{a}(\tau_*)=0$, we can show $\ddot{a}(\tau_*)>0$ which means $\dot{a}_{\min}$ exists (see Appendix). The radius of the wormhole $a_{\min}=a(\tau_*)$ is satisfied with the algebra equation as follows:

$$1+\left[\frac{a_{\min}}{l}\right]^2-\left[\frac{l}{a_{\min}}\right]^{2D/(D-2)}\left[\frac{8\pi G\mu^2 l^2}{f^2}\left[\frac{a_0}{l}\right]^{2D-2}\right]^{2/(D-2)}=0 . \tag{4.14}$$

In the limit of $\tau\to\infty$, the asymptotic solution of Eq. (4.13) is $a(\tau)=l\sinh(\tau/l)$. So we indeed have obtained dimensionally continued wormhole solutions.

V. WORMHOLE SOLUTIONS WITH A CONFORMAL SCALAR FIELD

The action of a scalar field conformally coupled to the Hilbert-Einstein term can be written as

$$S_{\text{mat}}=\int d^Dx\sqrt{g}\,[\tfrac{1}{2}(\partial\varphi)^2+\varphi\varphi R] . \tag{5.1}$$

For a spherically symmetric scalar field, we take $\psi=a^{(D-2)/2}\varphi$; the field equation could be derived by using the method of variation:

$$\sum_{m=0}^{[(D-1)/2]} C_m(D-2m)G_{(m)ab}=\frac{1}{a^D}\left[\frac{(D-2)^2}{4}\psi^2-a^2\dot{\psi}^2\right] , \tag{5.2}$$

$$a\frac{d}{d\tau}(a\dot{\psi})-\frac{(D-2)^2}{4}\psi=0 . \tag{5.3}$$

The general solution of Eq. (5.3) can be written as

$$\psi(\eta)=\alpha a_0^D\sinh\left[\frac{D-2}{2}\eta\right]+\beta a_0^D\cosh\left[\frac{D-2}{2}\eta\right] , \tag{5.4}$$

where η is the conformal time, i.e., $d\eta=d\tau/a$. Using Eq. (5.4), the Lovelock-Einstein equation could be reduced to

$$(D-1)!\sum_m C_m(D-2m)\left[\frac{1-\dot{a}^2}{a^2}\right]^m=\frac{(\beta^2-\alpha^2)(D-2)^2a_0^D}{4a^D} , \tag{5.5}$$

$$(D-2)!\sum_m C_m(D-2m)\left[\frac{1-\dot{a}^2}{a^2}\right]^{m-1}\left[2m\frac{\ddot{a}}{a}-(D-2m-1)\frac{1-\dot{a}^2}{a^2}\right]=\frac{(\beta^2-\alpha^2)(D-2)^2a_0^D}{4(D-1)a^D} . \tag{5.6}$$

The first integral gives

$$\begin{aligned}\dot{a}^2&=1+\left[\frac{a}{l}\right]^2-\left[\frac{a}{l}\right]^2\left[\frac{2\pi Gl^2(\beta^2-\alpha^2)(D-2)}{(D-3)!}\left[\frac{a_0}{a}\right]^D\right]^{2/(D-1)} &&\text{for } D=2n-1 ,\\ \dot{a}^2&=1+\left[\frac{a}{l}\right]^2-\left[\frac{a}{l}\right]^2\left[\frac{2\pi Gl^2(\beta^2-\alpha^2)(D-2)^2}{(D-1)!}\left[\frac{a_0}{a}\right]^D\right]^{2/(D-2)} &&\text{for } D=2n .\end{aligned} \tag{5.7}$$

The radius of the wormhole solution is satisfied with algebra equation as follows:

$$\begin{aligned}&1+\left[\frac{a_{\min}}{l}\right]^2-\left[\frac{l}{a_{\min}}\right]^{2/(D-1)}\left[\frac{2\pi Gl^2(\beta^2-\alpha^2)(D-2)}{(D-3)!}\left[\frac{a_0}{l}\right]^D\right]^{2/(D-1)}=0 \quad\text{for } D=2n-1 ,\\ &1+\left[\frac{a_{\min}}{l}\right]^2-\left[\frac{l}{a_{\min}}\right]^{4/(D-2)}\left[\frac{2\pi Gl^2(\beta^2-\alpha^2)(D-2)^2}{(D-1)!}\left[\frac{a_0}{l}\right]^D\right]^{2/(D-2)}=0 \quad\text{for } D=2n .\end{aligned} \tag{5.8}$$

In either case, when $\tau=\tau_*$, and $\dot{a}(\tau_*)=0$, we can show $\ddot{a}>0$ which means $a_{\min}$ exists (see Appendix). When $\tau\to\infty$, the asymptotic solution is $a(\tau)\approx l\sinh(\tau/l)$. Therefore, we indeed have obtained dimensionally continued wormhole solutions in Lovelock theory with a conformal scalar field.

VI. CONCLUSIONS

The action of special Lovelock theory is, in odd dimensions, the Chern-Simons form for the anti–de Sitter group and, in even dimensions, the Euler density constructed with the Lorentz part of the anti–de Sitter curvature tensor. Adding these reasonable restrictions to the Lovelock coefficient, it is shown that the Lovelock corrections preserve the essential feature of Einstein gravity models of giving rise to an essentially unique wormhole solution. We have analyzed systematically the wormhole solutions to special Lovelock field equation with various matter contents, including the perfect fluid energy-momentum tensor, axionic field, and conformal scalar field. We found that the wormhole solutions indeed exist for these models. A discussion of the connection between the asymptotic forms and other results would be of interest. The wormhole solutions are well behaved in spite of the ill posed nature of the Cauchy problem in the generic Lovelock theory. It is not surprising. BTZ have shown that the solutions are well behaved for black holes and for cosmological models with geometry very similar to that of Eqs. (2) and (3) [28].

In odd-dimensional Tolman wormhole and axionic wormhole cases, the solutions can be described by element functions. In principle, other separable differential equations (3.14), (3.17), (4.13), or (5.7) can be easily integrated. In practice, the integration is not an element function. As an example, the numerical integrations are shown in Fig. 1 for $D=5$, 6, 7, and 8. Discussions could be easily generalized to other theories and further examples will be considered elsewhere.

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APPENDIX

Lemma. In the Lovelock theory of D-dimensional isotropic spacetime with the special coefficients (2.14) and various matter contents, including the perfect fluid energy-momentum tensor, axionic field, and conformal scalar field, the function $a(\tau)$ will have a minimum at τ_* according as the $\dot{a}(\tau_*)=0$.

The usual rule for determining minima is to find the roots of $\dot{a}(\tau_*)=0$ and $\dot{a}(\tau_*)>0$. We will show that $\dot{a}(\tau_*)=0$ implies $\ddot{a}(\tau_*)>0$. From Eqs. (2.7)–(2.11), we will have

$$\sum_{m=0}^{n-1}\binom{n-1}{m} l^{2m}\left[\frac{1-\dot{a}^2}{a^2}\right]^{m-1}\left[2m\frac{\ddot{a}}{a}-(D-2m-1)\left[\frac{1-\dot{a}^2}{a^2}\right]\right]\geq 0 \quad (D=2n-1)\,, \tag{A1}$$

or

$$\sum_{m=0}^{n-1}(D-2m)\binom{n}{m} l^{2m}\left[\frac{1-\dot{a}^2}{a^2}\right]^{m-1}\left[2m\frac{\ddot{a}}{a}-(D-2m-1)\left[\frac{1-\dot{a}^2}{a^2}\right]\right]\geq 0 \quad (D=2n)\,, \tag{A2}$$

for the various matter contents. In $D=2n-1$ case, we have

$$\begin{aligned}\sum_{m=0}^{n-1}2m\binom{n-1}{m} l^{2m}\left[\frac{1-\dot{a}^2}{a^2}\right]^{m-1}\frac{\ddot{a}}{a} &= l\frac{d}{dl}\left\{\sum_{m=0}^{n-1}\binom{n-1}{m} l^{2m}\left[\frac{1-\dot{a}^2}{a^2}\right]^{m-1}\frac{\ddot{a}}{a}\right\}\\ &=2l^2\left[1+l^2\left[\frac{1-\dot{a}^2}{a^2}\right]\right]^{n-2}\frac{\ddot{a}}{a}\end{aligned} \tag{A3}$$

and

$$\begin{aligned}\sum_{m=0}^{n-1}(D-2m-1)\binom{n-1}{m} l^{2m}\left[\frac{1-\dot{a}^2}{a^2}\right]^{m} &= \sum_{m}^{n-2}2(n-1)\binom{n-2}{m} l^{2m}\left[\frac{1-\dot{a}^2}{a^2}\right]^{m}+2(n-1)l^{2n-2}\left[\frac{1-\dot{a}^2}{a^2}\right]^{n-1}\\ &=2(n-1)\left\{\left[1+l^2\left[\frac{1-\dot{a}^2}{a^2}\right]\right]^{n-2}+l^{2n-2}\left[\frac{1-\dot{a}^2}{a^2}\right]^{n-1}\right\}.\end{aligned} \tag{A4}$$

When $\tau=\tau_*$ and $\dot{a}(\tau_*)=0$, using (A3) and (A4), we have

$$\ddot{a}(\tau_*)\geq\frac{a_{\min}}{l^2}\left\{\left[1+\left[\frac{l}{a_{\min}}\right]^2\right]^{n-2}+\left[\frac{l}{a_{\min}}\right]^{n-1}\right\}\Big/\left[1+\left[\frac{l}{a_{\min}}\right]^2\right]^{n-2}>0\,. \tag{A5}$$

Using the relation

$$(D-2m)\binom{n}{m}=2n\binom{n-1}{m}\,, \tag{A6}$$

the proof of the $D=2n$ case is similar to that of odd dimensions.

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