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Exact solution for scalar field collapse

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We give a class of exact spherically symmetric solutions for the Einstein-scalar field system. The solutions may be interpreted as inhomogeneous dynamical scalar field cosmologies. The spacetimes have a timelike conformal Killing vector field and are asymptotically conformally flat. They also have black- or white-hole-like regions containing trapped surfaces. The properties of the apparent horizons are described in detail.

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The gravitational collapse of distributions of matter is one of the most important research areas in general relativity. The essential question posed is whether and under what initial conditions a black hole or naked singularity forms in the collapse. One of the aims of studying such problems is to test the cosmic censorship conjecture [1], one form of which states that gravitational collapse produces black holes.

A model system for studying this question is provided by the Einstein equations minimally coupled to a massless scalar field. While this full system appears to be intractable, the simplified set of equations obtained by imposing spherical symmetry is easier to handle. Without any matter fields, the spherically symmetric metric does not contain any field degrees of freedom. Therefore, with a scalar field, the system is effectively a two-dimensional field theory and it can be described by a single two-dimensional nonlinear differential equation [2,3].

There are a number of exact solutions known for this system, almost all of which are either static or depend only upon the time coordinate [4,5]. The first nonstatic solutions were given by Roberts [6] (which are different from the one we give below). The equations have been studied in detail by Christodoulou [2] who established, among other things, that there exist regular solutions for arbitrarily long times for particular types of initial data.

The model has also been studied numerically and there are a number of interesting numerical results. The first results obtained by Goldwirth and Piran [7] indicated that there is a class of initial data that leads to black hole formation. More recently it has been shown by Choptuik [8] that, for large classes of initial data, there is critical behavior at the onset of black hole formation: the black hole mass $M_{\rm BH}$ is given by the equation $M_{\rm BH} = K|c - c_*|^\gamma$, where K is a constant, c is any one of the parameters in the initial data for the scalar field, c_* is a critical value of the parameter, and $\gamma \sim 0.37$ is an exponent independent of the shapes of the initial data. It has been shown [9] that the same mass formula is obtained for the axisymmetric collapse of gravitational radiation. Thus this "critical" behavior appears to be independent of not only the type of matter fields, but also the symmetries of the system.

It would be very useful to understand the universality of this result analytically. A modest approach is to attempt to find an exact solution describing scalar field collapse and to see if one can read off the critical behavior by calculating the mass of the black hole.

Here we describe an exact solution for scalar field collapse and discuss some of its properties. While the solution we present does not describe a realistic collapse corresponding to the asymptotically flat situation, it appears to be among the few exact nonstatic solutions known for this system and has some interesting properties.

The Einstein-scalar field equations we consider (in units $G = c = 1$) are

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \quad T_{\mu\nu} = \phi_{,\mu}\,\phi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha}\,\phi_{,\beta}\,, \quad (1)$$

which may be written in the form

$$R_{\mu\nu} = 8\pi \phi_{,\mu}\,\phi_{,\nu}\,. \quad (2)$$

The spherically symmetric solution we obtain is

$$\begin{aligned} ds^2 = {} & (at+b)\,[-f^2(r)dt^2 + f^{-2}(r)dr^2] \\ & + R^2(r,t)(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \quad (3)$$

where

$$\begin{aligned} f^2(r) &= (1 - 2c/r)^\alpha, \\ R^2(r,t) &= (at+b)r^2(1-2c/r)^{1-\alpha}. \end{aligned} \quad (4)$$

The scalar field is

$$\phi(r,t) = \pm \frac{1}{4\sqrt{\pi}} \ln[d\,(1-2c/r)^{\alpha/\sqrt{3}}\,(at+b)^{\sqrt{3}}], \quad (5)$$

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where a, b, c, d are constants, $\alpha = \pm\sqrt{3}/2$, and the overall sign of ϕ is independent of the sign of α. The parameter d in the scalar field is a trivial additive constant.

The only Killing vectors of the metric (3) are the three associated with spherical symmetry. There is also a conformal Killing vector field $V = \partial/\partial t$ such that

$$\mathcal{L}_V g_{\mu\nu} = \frac{a}{at+b} g_{\mu\nu}. \tag{6}$$

Asymptotically ($r \to \infty$) the metric is conformally flat. Since the spacetime is not asymptotically flat, it may be interpreted as an inhomogeneous scalar field cosmology.

The Ricci scalar derived from (3) is

$$\mathcal{R} = \frac{12ca^2(r-c) - 3a^2r^2}{2r^2(at+b)^3}\left(1 - \frac{2c}{r}\right)^{-2-\alpha} + \frac{2c^2(1-\alpha^2)}{(at+b)r^4}\left(1 - \frac{2c}{r}\right)^{-2+\alpha}, \tag{7}$$

which shows that curvature singularities are present at $r = 2c$, and at $t = -b/a$ for both values of α.

While (3) is the most general form of our solution, the ranges of the coordinates and the parameters are coupled. As a result there are a number of special cases which we now discuss.

I. $a \neq 0$, $c \neq 0$. The parameters b, c may be removed by coordinate transformations, but a cannot. We may set $b = 0$, $c = 1$, and $a = \pm 1$ without loss of generality. The range of the t coordinate depends on the sign of a, with $0 \leq t \leq \infty$ for $a = 1$, and $-\infty \leq t \leq 0$ for $a = -1$. As shown below these two cases correspond to white- and black-hole-like solutions, respectively, and simply reflect the choice of the arrow of time. There is a spacelike (cosmological) curvature singularity at $t = 0$ and a timelike singularity at $r = 2$ [which corresponds to $R = 0$ (4)]. The range of r is therefore $2 \leq r \leq \infty$.

II. $a \neq 0$, $c = 0$. The parameter a may be removed by a coordinate rescaling, and the metric is homogeneous and conformally flat with t as the conformal factor. There is a cosmological curvature singularity at $t = 0$ as in case I.

III. $a = 0$, $c \neq 0$. The parameter b may be removed by a coordinate transformation. The metric is now static and is one of a class of known solutions [4,5]. These known static solutions, however, have an *arbitrary* exponent α in the function $f(r)$ of Eq. (4), whereas the present case for our metric gives only the *fixed* values $\alpha = \pm\sqrt{3}/2$. There is a timelike curvature singularity at $r = 2c$, so the horizon is shrunk to a point. The metric is asymptotically flat.

The parameters a, c are similar to the parameter k in Friedmann-Robertson-Walker cosmologies, which discretely distinguishes the spatial curvatures of the metrics. Here a and c distinguish, respectively, static from nonstatic metrics, and homogeneous from inhomogeneous ones. The main qualitative features of the above results are summarized in Table I.

For comparison with recent numerical work [8,9], where the scalar field and its time derivative are specified as part of the initial data on a spacelike hypersurface, we note that the data for our solution are

$$\phi(r, t = t_0) = \frac{1}{4\sqrt{\pi}} \ln[d\,(1 - 2\,c/r)^{\alpha/\sqrt{3}}\,(at_0 + d)^{\pm\sqrt{3}}],$$
$$\dot{\phi}(r, t = t_0) = \frac{1}{4(at_0 + b)}\sqrt{\frac{3}{\pi}}. \tag{8}$$

The asymptotic behavior of ϕ is

$$\phi(r = \infty, t = t_0) = \frac{1}{4\sqrt{\pi}} \ln[d\,(at_0 + b)^{\pm\sqrt{3}}]. \tag{9}$$

These are not the initial data associated with the standard collapse situation [8,9], where the data are ingoing pulses with a specified amplitude and width. The data are in fact singular at $r = 2c$, which is why there is always a curvature singularity at this coordinate value.

The general metric (3) is not static so it is of interest to investigate the existence and properties of the apparent horizon(s), if they exist. This horizon is the three-surface on which outgoing or ingoing null rays are momentarily stationary. The presence of the horizon indicates that there are regions containing trapped surfaces, which correspond to black or white holes.

For dynamical spacetimes which are not asymptotically flat and contain a cosmological singularity (such as ours), it is not straightforward to determine the masses of any black holes. In the absence of a clearly defined concept of energy, a good measure of the "size" of the spherically symmetric black hole is given by the proper radius of the apparent horizon, $R_{\rm AH}$. In general this radius will vary from one spatial surface to the next, but it provides a definite mass at late times if $\lim_{t\to\infty} R_{\rm AH}(t) =$ constant. This provides one motivation for studying the evolution of $R_{\rm AH}$ for non-static spacetimes, regardless of their asymptotic properties.

TABLE I. Summary of spacetime properties. The parameter b and the value of α do not affect these properties. There is a spacelike singularity at $t = 0$ for the nonstatic cases.

		a: -1	a: 0	a: 1
c	1	Nonstatic future AH black hole timelike singularity at $r = 2$	Static timelike singularity at $r = 2$	Nonstatic past AH at $r = 2$ at $r = 2$
	0		Nonstatic conformally flat	

The apparent horizon surface is given by

$$g^{\alpha\beta} R_{,\alpha} R_{,\beta} = 0. \tag{10}$$

For our metric this gives the following equation for the radial coordinate of the apparent horizon $r(t)$ on a given time slice:

$$\frac{a}{at+b} = \frac{2}{r^2}[r - c(1+\alpha)](1 - 2c/r)^{\alpha-1}. \tag{11}$$

This equation has no nontrivial solution for $a = 0$ which corresponds to the static metric (case III above), whereas for $a \neq 0$ there is always an evolving apparent horizon. Since (11) is difficult to invert to get $r_{AH}(t)$, in the following we will instead consider it as an equation $t_{AH}(r)$ which gives the *time surface* on which a given value of r is the radial coordinate of the apparent horizon. It is not possible to write (11) in terms of the proper radial coordinate R, except for large r (or t), in which case $R_{AH} \sim 2t^{3/2}$.

The apparent horizon can in general be spacelike, null or timelike in different spacetime regions. This is easily determined by calculating the ratio of the slopes of the apparent horizon and the outgoing null ray. Since the null rays are given by

$$t_{N,r} \equiv \frac{dt}{dr} = \pm\left(1 - \frac{2c}{r}\right)^{-\alpha}, \tag{12}$$

this ratio for our metric (with $a \neq 0$) is

$$\frac{t_{AH,r}}{t_{N,r}} = 1 - \frac{(1-2/r)}{2[1-(1+\alpha)/r]^2}. \tag{13}$$

The magnitude of this ratio is always less than or equal to one. The apparent horizon is, therefore, always spacelike, except at particular r values where it is null. These values are $r = 2$ and $r = 2.16$ for $\alpha = \sqrt{3}/2$, and $r = 2$ for $\alpha = -\sqrt{3}/2$. As $r \to \infty$, this ratio tends to 1/2. At the apparent horizon the scalar field is not singular. Also, we see from (12) that for both values of α the light cones open up to a slope of ± 1 as $r \to \infty$ (since the metric is conformally Minkowski in this limit).

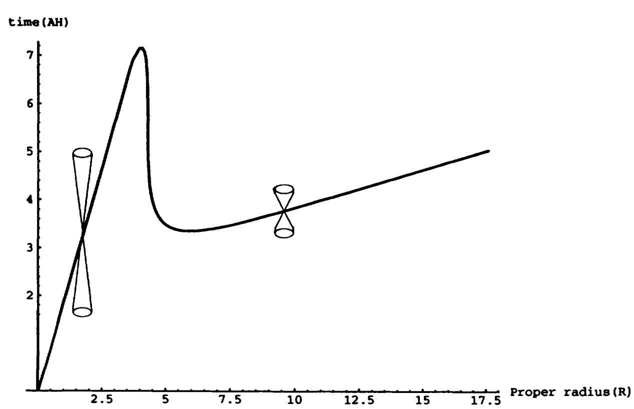

FIG. 1. The apparent horizons for $\alpha = +\frac{\sqrt{3}}{2}$ and $c = 1, b = 0$. The light cones (whose slopes are not to scale) illustrate the spacelike character of the horizon. The singularities are at $R = 0$ and $t = 0$. The trapped white-hole region is $t < t_{AH}$.

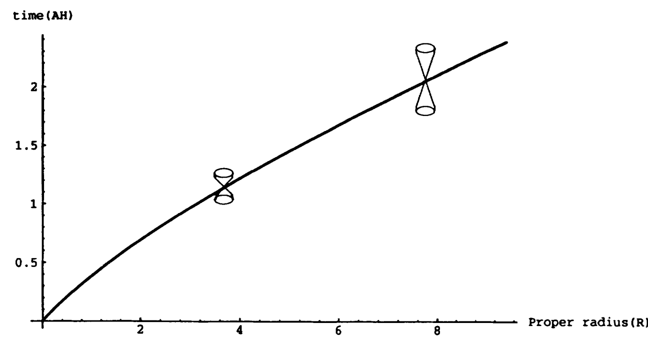

FIG. 2. The apparent horizon for $\alpha = -\frac{\sqrt{3}}{2}$ and $c = 1, b = 0$.

Figures 1 and 2 are plots of t_{AH} as a function of the proper radial coordinate R for $\alpha = \pm\sqrt{3}/2$ and $a = 1$ for the nonstatic metrics *I*. We now describe the horizons for each case.

$\alpha = -\sqrt{3}/2$. The light cones collapse to a horizontal line at $R = 0$. There is only one intersection of the spatial t surfaces with the apparent horizon curve, and hence only a single horizon on each surface.

$\alpha = \sqrt{3}/2$. The light cones are collapsed to a vertical line at $R = 0$. The behavior of the apparent horizon is more interesting. The apparent horizon curve intersects some spatial surfaces of constant t more than once and gives the appearance of "multiple" horizons for particular ranges of t. On such surfaces there are "domains" of alternative trapped, normal, and trapped regions as the coordinate R increases. In time sequence, a single horizon forms and grows monotonically. Soon after, there is a "pair creation" of horizons at $t \sim 3.3$ and $R \sim 5.5$. One of these shrinks and annihilates the first horizon at $t \sim 7.2$, and thereafter only one horizon remains. It is interesting that this pair creation and annihilation seems to occur when the ratio (13) is zero. The cause of the trapped and normal domains is a particularly nonuniform radial distribution of the scalar field energy density for this α value. Figure 3 shows a three-dimensional cross section

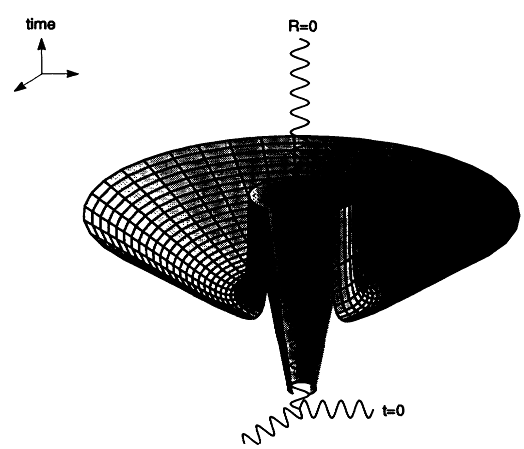

FIG. 3. A three-dimensional cross section of the apparent horizons of Fig. 1 (on a contracted proper time scale). The initial horizon extends down to $R = 0, t = 0$.

of the apparent horizon surface.

This "multiple" horizon situation is somewhat reminiscent of the Reissner-Nordström solution, except that for the latter the horizons are everywhere null, whereas the apparent horizon for our solution is everywhere spacelike except at $R = 0, 4.3$ where it is null. However, the analogy cannot be carried too far since our spacetime has no Cauchy horizons.

It is of interest to note a number of other features of the apparent horizon. By computing the expansions of the spacelike symmetry two-spheres

$$ds^2 = R^2(r,t)(d\theta^2 + \sin^2\theta d\phi^2) \tag{14}$$

along future pointing null directions orthogonal to the spheres, one can determine whether the apparent horizon is past or future, and inner or outer, and what region is trapped. (See for example Ref. [10] for a general discussion of apparent horizons.)

The expansions $\theta_\pm$ of the area two-form $\omega = R^2(r,t)\sin\theta \; d\theta \wedge d\phi$ of the two-spheres are defined by

$$\mathcal{L}_{l_\pm}\omega = \theta_\pm \omega, \tag{15}$$

where $\mathcal{L}$ denotes the Lie derivative and

$$l_+ := \frac{\partial}{\partial t} + f^2\frac{\partial}{\partial r} \quad l_- := \frac{\partial}{\partial t} - f^2\frac{\partial}{\partial r} \tag{16}$$

are the outgoing and ingoing future pointing null directions. For $a > 0$ (so that $0 \le t \le \infty$), we find

$$\theta_\pm = \frac{1}{t} \pm \frac{1}{t_{\rm AH}}, \tag{17}$$

with $t_{\rm AH}$ as given in Eq. (11). Thus it is the ingoing expansion θ_- that vanishes at the apparent horizon, while the outgoing expansion is $\theta_+ = 2/t_{\rm AH} > 0$ at this horizon. This implies that it is a *past horizon*. For a given value of r, the symmetry two-spheres are trapped surfaces for $t < t_{\rm AH}$, since this is the region where both the ingoing and outgoing light rays have positive expansions (the white-hole region). Similarly, the region $t > t_{\rm AH}$ is a normal region where the outgoing light expansion (θ_+) is positive and the ingoing one (θ_-) is negative.

We note also that if $\mathcal{L}_{l_+}\theta_-|_{\rm AH} < 0$ the horizon is an *outer* one (otherwise it is *inner*). For our metric (3), we find

$$\mathcal{L}_{l_+}\theta_-|_{\rm AH} = \frac{1}{t_{\rm AH}^2}\left[\frac{t_{{\rm AH},r}}{t_{N,r}} - 1\right]. \tag{18}$$

From (13) it follows that this is always less than or equal to zero. It is zero at those points where $t_{\rm AH}(r)$ is null. Thus, according to this criterion (which distinguishes for example, the inner and outer horizons of the Reissner-Nordström spacetime), the horizon is an *outer* one. It is important to emphasize that in spite of the appearance of "multiple" horizons on spatial surfaces, the *spacetime* apparent horizon curve is always *outer*: an observer on a timelike trajectory from the white-hole region will cross only one horizon which will appear to be an outer one.

Summarizing the above results, for $a = 1$, the apparent horizon is a past outer one and is spatial everywhere except at $R = 0$, where it is null. The scalar field flows from the past trapped region $t < t_{\rm AH}$ (white hole) into the untrapped region $t > t_{\rm AH}$. An observer in the untrapped region sees both the $t = 0$ initial singularity and the one at $R = 0$.

The time reversed case occurs for $a = -1$ ($-\infty \le t \le 0$). The horizon is now a future outer one and corresponds to a black-hole situation. The future singularity at $t = 0$ is covered by the spacelike horizon and the region is a black hole.

We have given a new exact nonstatic solution for the spherically symmetric Einstein-scalar field system. While this solution does not shed light on the scalar field collapse problem in the asymptotically flat case (since the solution is not asymptotically flat), it nevertheless provides an example of a spacetime with evolving apparent horizons. It would be of interest to seek other solutions of this type, perhaps with other matter fields, corresponding to realistic collapse situations.

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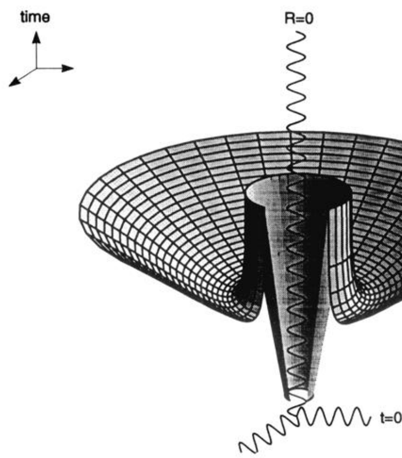

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