PHYSICAL REVIEW D VOLUME 50, NUMBER 6 15 SEPTEMBER 1994

Perfect fluid scalar-tensor cosmologies

John D. Barrow and José P. Mimoso*
Astronomy Centre, School of Mathematical and Physical Sciences, University of Sussex, Brighton BN1 9QH, United Kingdom
(Received 31 March 1994)

A method is investigated which enables exact solutions to be found for $k = 0$ Friedmann cosmological models with a perfect fluid satisfying the equation of state $p = (\gamma - 1)\rho$, where γ is constant and $0 \leq \gamma \leq 2$, in scalar-tensor gravity theories with an arbitrary form for the gravitational coupling function $\omega(\phi)$, which defines the theory. A number of explicit solutions are investigated for $p = 0$ universes and inflationary universes, including those for theories in which $\omega(\phi)$ has a power-law dependence on the scalar field ϕ. When $p = -\rho$ new varieties of inflation arise in which $a(t) \propto t^n \exp(H_0 t^m)$.

PACS number(s): 98.80.Hw, 04.50.+h, 98.80.Cq

I. INTRODUCTION

Scalar-tensor gravity theories [1–3] have become a focal point of interest in many areas of gravitational physics and cosmology. They provide the most natural generalizations of general relativity (GR) and thus provide a convenient set of representations for the observational limits on possible deviations from GR. They are required for a complete evaluation of the cosmological implications of any "fifth force" variation in the behavior of gravity in the weak-field limit [4,5] and associated ideas of "oscillating physics" [6]. The evolution of extra dimensions of space in quantum cosmologies produces behavior characteristic of scalar-tensor theories [7,8]. Any variation of the Newtonian gravitational "constant" with time may produce unusual physical effects if black holes are formed in the very early Universe [9]. It has also been shown that a scalar field responsible for time variation in the gravitational coupling can drive new forms of inflation [10,11].

In order to evaluate these effects in scalar-tensor gravity theories it is necessary to have exact cosmological solutions for the entire span of cosmological evolution. Only in this way is it also possible to constrain the theory by simultaneously imposing the observational limits arising from different epochs in cosmic history (primordial nucleosynthesis [12], contemporary cosmological expansion dynamics) and the weak-field solar system tests [5,13]. In the past this has been impossible because exact cosmological solutions of scalar-tensor gravity theories, characterized by a scalar coupling $\omega(\phi)$, have only been available in particular cases. In the most studied (special) case of Brans-Dicke theory [1], where $\omega(\phi)$ is constant, particular solutions are known for the Friedmann cosmological models with perfect fluid equations of state when $k = 0$. When $k \neq 0$ only vacuum, radiation, and stiff fluid solutions have been found. Recently, it has been shown by one of us [14] that Friedmann cosmological models of all k can be found in $\omega(\phi)$ theories by a suitable choice of variables, but this method works by exploiting the conformal invariance of the theory and is successful only when the energy-momentum tensor is trace-free, that is, for vacuum and radiation-filled models (incidentally, this method also applies to stiff fluid solutions because they can be reduced to the vacuum case [15]). No exact solutions appear to be known for $\omega(\phi)$ cosmologies with other perfect fluid equations of state, for example, that of zero pressure (although an approximate study of some inflationary models was given by Barrow and Maeda [11] and other qualitative studies have been made by Damour and Nordvedt [16]). In this paper we show how an appropriate choice of variables permits an integration of the gravitational field equations for $k = 0$ Friedmann models in $\omega(\phi)$ theories with a general $p = (\gamma - 1)\rho$ equation of state and γ constant. Two classes of solutions are of particular interest: those for dust ($\gamma = 1$) models, which describe the post-recombination history of the Universe to a very good approximation, and those with $\gamma = 0$, which describe an era of inflation dominated by a slowly evolving scalar field with $p = -\rho$. The exact solutions obtained in the latter case enable us to evaluate the forms of inflation that arise from a wide class of $\omega(\phi)$ gravity theories. The $p = 0$ solutions, in conjunction with the radiation solutions studied earlier [14], will allow us to construct complete histories encompassing both the radiation era and the dust-dominated era up to the present and to discover which variations of $G(\phi)$ are ruled out by a combination of primordial nucleosynthesis and solar system gravity tests.

In Sec. II we introduce the field equations for the scalar-tensor gravity theories. In Sec. III we give the equations defining cosmological models in $\omega(\phi)$ theories in terms of new variables, which reduce them to integrable form after the choice of one free "generating" function of one variable, which is equivalent to defining the theory by specifying $\omega(\phi)$. In Sec. IV we give a number of explicit dust ($p = 0$) solutions and discuss their asymptotic forms. In Sec. V solutions are given for the $p = -\rho$ case appropriate for simple inflationary universes. We

*Present address: Departamento de Física, Universidade de Lisboa, Campo Grande, Ed. C1, 1700 Lisboa, Portugal. Electronic address: fmimoso@neuron.cc.fc.ul.pt

find that new forms of inflation are possible. In Sec. VI we discuss the behavior of the cosmological models for general γ for some interesting forms of $\omega(\phi)$ where the asymptotic behaviors are simple. In Sec. VII we consider the radiation models in order to relate the methods investigated in this paper and in Ref. [14]. Finally, in Sec. VIII we provide a brief discussion of the results.

II. SCALAR-TENSOR GRAVITY THEORIES

Scalar-tensor gravity theories [2,3] can be derived from the Lagrangian

$$\mathcal{L}_\phi = \phi\mathcal{R} - \frac{\omega(\phi)}{\phi}\,\partial_a\phi\partial^a\phi + 16\pi G_N\mathcal{L}_m \,, \qquad (1)$$

where $\mathcal{R}$ is the Ricci curvature scalar of the spacetime, ϕ is a scalar field, $\omega(\phi)$ is a dimensionless coupling parameter, and, finally, $\mathcal{L}_m$ represents the Lagrangian for the matter fields. It is apparent that the scalar field plays a role which in Einstein's GR is taken by the gravitational constant. Since ϕ is now a dynamical variable, these theories exhibit a varying gravitational "constant." The archetypal case of Brans-Dicke theory arises when we specialize $\omega(\phi)$ to be a constant in the Lagrangian Eq. (1).

Taking the variational derivatives of the action Eq. (1) with respect to the two dynamical variables g_{ab} and ϕ yields the field equations

$$R_{ab} - \frac{1}{2}\,g_{ab}\,R = \frac{\omega(\phi)}{\phi^2}\left[\phi_{;a}\phi_{;b} - \frac{1}{2}\,g_{ab}\phi_{;c}\phi^{;c}\right] + \frac{1}{\phi}\,[\phi_{;ab} - g_{ab}\phi_{;c}{}^{;c}] + 8\pi G_N\,\frac{T_{ab}}{\phi}\,, \qquad (2)$$

$$\Box\phi = \frac{1}{2\omega(\phi)+3}\,[8\pi G_N\,T - \omega'(\phi)\phi_{;c}\phi^{;c}]\,, \qquad (3)$$

where $T \equiv T_c{}^c$ is the trace of the energy-momentum tensor $T_a{}^b$ of the matter content of space-time and G_N is a dimensionless normalization constant which fixes the present-day value of the gravitational constant G (in what follows we set units so that $G_N = 1$). Note that a further relation $T^{ab}{}_{;b} = 0$, establishing the matter conservation laws, holds true. This ensures that the principle of equivalence is satisfied. We see from Eq. (3) that the matter acts as the source of the scalar field ϕ, which helps generate the space-time curvature associated with the metric. Matter may create this field, but the latter cannot act back directly on the matter, which responds only to the metric [17].

The strength of the coupling between the scalar field and gravity is gauged by the parameter $\omega(\phi)$, which hereafter will be termed the *coupling function.* When ω approaches infinity ($\omega \to \infty$) the scalar-tensor theory approaches Einstein's general relativity provided that we also have $\omega'/\omega^3 \to 0$ in this limit [3]. This is necessary for light-bending, perihelion precession and radar echo-delay phenomena in the solar system to lie within the bounds imposed by observation.

III. FRIEDMANN UNIVERSES

Consider the Friedmann-Robertson-Walker universes with the metric in (t, r, θ, ψ) coordinates given by the usual line element with the curvature parameter k:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\psi^2)\right]. \qquad (4)$$

When the matter content of the universe is a perfect fluid obeying the general equation of state $p = (\gamma - 1)\rho$ with γ constant and $0 \leq \gamma \leq 2$, the field equations become (where an overdot denotes differentiation with respect to t)

$$H^2 + H\,\frac{\dot\phi}{\phi} - \frac{\omega(\phi)}{6}\,\frac{\dot\phi^2}{\phi^2} + \frac{k}{a^2} = \frac{8\pi}{3}\,\frac{\rho}{\phi}\,, \qquad (5)$$

$$\ddot\phi + \left[3H + \frac{\dot\omega(\phi)}{2\omega(\phi)+3}\right]\dot\phi = \frac{8\pi\rho}{2\omega(\phi)+3}\,(4-3\gamma)\,, \qquad (6)$$

$$\dot H + H^2 + \frac{\omega(\phi)}{3}\,\frac{\dot\phi^2}{\phi^2} - H\,\frac{\dot\phi}{\phi} = -\frac{8\pi\rho}{3\phi}\,\frac{(3\gamma-2)\,\omega+3}{2\omega(\phi)+3} + \frac{1}{2}\,\frac{\dot\omega}{2\omega(\phi)+3}\,\frac{\dot\phi}{\phi}\,, \qquad (7)$$

$$\dot\rho + 3\gamma\,H\,\rho = 0\,, \qquad (8)$$

where the Hubble parameter $H \equiv \dot a/a$. These equations differ from the corresponding equations of Brans-Dicke theory through the presence of additional terms involving $\dot\omega$ in Eqs. (6) and (7) and reduce to those of GR when ϕ is a constant.

It is important to appreciate that the solutions of these field equations are defined by four integration constants while the GR solutions only depend on three [18]. In addition to the present values of $a(t_0)$, $\dot a(t_0)$, and $\phi(t_0) \propto 1/G(t_0)$, we also need $\dot\phi(t_0)$ [or $\rho(t_0)$]. In Brans-Dicke theory this extra freedom was eliminated by some authors by imposing that $\dot\phi a^3$ should vanish for a approaching the initial singularity at $a = 0$ [1,19]. This ensures that the solutions are of a power-law type, with exponents which are mildly different from their general relativistic analogues, and represent universes where the matter fields dominate the free ϕ field during the entire expansion. Such solutions are to be contrasted with the general ($k = 0$) Brans-Dicke solutions found by Gurevich *et al.* [20] where this restrictive assumption is not made. The general solutions exhibit two distinct regimes: an early period during which the expansion is dominated by the scalar field and approximates the vacuum solution of

Eqs. (5)–(7) and a late period during which the solutions asymptote towards the matter-dominated power-law behavior. In what follows no restrictive assumption will be made regarding the boundary conditions satisfied by $\phi(t)$ at $t = 0$ and so our analysis will be kept fully general.

The system of equations (5)–(8) allows considerable simplification in the cases where the trace of the energy-momentum tensor vanishes and the $\omega(\phi)$ theories are conformally related to general relativity [11]. In these cases, the wave equation (6) is sourceless and the general relativity solutions always arise as a particular ($\phi =$ const) case of the general solutions. By exploiting this situation, exact vacuum and radiation cosmological solutions for scalar-tensor gravity theories for all values of k have been found by one of us [14].

To address the remaining nonvacuum fluid cases ($\gamma \neq 4/3$) we resort to a generalization of the method of integration used by Gurevich *et al.* [20] in their study of Brans-Dicke (BD) theory to the more complicated cases where ω depends on ϕ. Our procedure will be applied to the $k = 0$ models.

We introduce a new time variable η via

$$dt = a^{3(\gamma-1)} \sqrt{\frac{2\omega+3}{3}}\, d\eta \tag{9}$$

(we assume $2\omega + 3 > 0$) and two new dynamical variables

$$x \equiv \left[\phi a^{3(1-\gamma)} \frac{d}{d\eta} a^3\right] , \tag{10}$$

$$y \equiv \left[a^{3(2-\gamma)} \frac{d}{d\eta}\phi\right] . \tag{11}$$

The $k = 0$ field equations (6) and (7) reduce to

$$y' = M\,(4 - 3\gamma) \tag{12}$$

and

$$x' = 3M\,[(2-\gamma)\omega + 1] + \frac{3}{2}\left(\frac{2}{3}x + y\right) \frac{\omega'}{2\omega+3} , \tag{13}$$

where the prime denotes differentiation with respect to η and M is defined by $8\pi\rho = 3Ma^{-3\gamma}$. In addition to these equations, the Friedmann equation (5) yields the constraint

$$\left(\frac{2}{3}x + y\right)^2 = \left(\frac{2\omega+3}{3}\right) \left[y^2 + 4M\,\phi\, a^{3(2-\gamma)}\right] . \tag{14}$$

It is straightforward to integrate Eqs. (12) and (13). The solutions are

$$y = M(4-3\gamma)\,(\eta - \eta_1) \quad , \tag{15}$$

$$x = \frac{3}{2}\left[-y + \sqrt{2\omega+3}\left(C + M(2-\gamma) \times \int_{\eta_1}^{\eta} \sqrt{2\omega+3}\, d\bar{\eta}\right)\right] , \tag{16}$$

where η_1 and C are integration constants.

Now we differentiate y using the definition (11) and

$$3\frac{a'}{a}\frac{\phi'}{\phi} = \frac{(\phi')^2}{\phi^2}\; 3\frac{a'}{a}\frac{\phi}{\phi'} = \frac{(\phi')^2}{\phi^2}\frac{x}{y} \tag{17}$$

to get

$$\left(\frac{\phi'}{\phi}\right)' + \left[\frac{3\gamma-4}{2} + \frac{f'(\eta)}{\eta-\eta_1}\right] \left(\frac{\phi'}{\phi}\right)^2 = \frac{1}{\eta-\eta_1}\left(\frac{\phi'}{\phi}\right) , \tag{18}$$

where we have defined the function

$$f(\eta) \equiv \int_{\eta_1}^{\eta} \frac{3(2-\gamma)}{2M(4-3\gamma)} \sqrt{2\omega(\phi)+3} \times \left[C + M(2-\gamma)\int_{\eta_1}^{\bar{\eta}} \sqrt{2\omega(\phi)+3}\, d\bar{\bar{\eta}}\right] d\bar{\eta} \, . \tag{19}$$

By introducing another function $g(\eta)$ we can absorb $f(\eta)$ into

$$g(\eta) \equiv f(\eta) + \frac{3\gamma-4}{4}(\eta-\eta_1)^2 + D, \tag{20}$$

where D is the integration constant arising from solving the Bernoulli equation (18). The solutions to Eq. (18) can be cast into the particularly simple form

$$\ln\left(\frac{\phi}{\phi_0}\right) = \int_{\eta_1}^{\eta} \frac{\eta-\eta_1}{g(\eta)} d\eta \tag{21}$$

and

$$a^3 = a_0^3 \left(\frac{g}{\phi}\right)^{\frac{1}{2-\gamma}} , \tag{22}$$

which follows from Eq. (17). In terms of $f(\eta)$ the behavior of the coupling $\omega(\phi)$, which defines the theory, is given by

$$2\omega(\phi(\eta)) + 3 = \frac{4-3\gamma}{3(2-\gamma)^2} \frac{(f')^2}{\left[f + \frac{4-3\gamma}{3(2-\gamma)^2} f_0\right]} , \tag{23}$$

where f_0 is another arbitrary constant. The $\omega(\phi)$ dependence is obtained by solving Eq. (21) with respect to ϕ, whenever this is possible. In practice a theory can be chosen by specifying the generating function $g(\eta)$ from which $\phi(\eta)$ follows from Eq. (21) and hence $a(\eta)$ from Eq. (22), $f(\eta)$ from Eq. (20), $\omega(\phi)$ from Eq. (23), and $\eta(t)$ from Eq. (9) if all the integrals can be performed.

It is worth noticing that the constant η_1, which was introduced in Eq. (15), can be set equal to zero without loss of generality. This merely amounts to a translation of the origin of the time variable. Henceforth, we use this freedom and take $\eta_1 = 0 = t$.

The various constants[1] C, D, and f_0 are not independent. In fact, evaluating the Friedmann constraint Eq. (14) we derive

$$D = \frac{4-3\gamma}{3(2-\gamma)^2}\, f_0 \, , \tag{24}$$

$$a_0^{3(2-\gamma)}\phi_0 = M\,(4-3\gamma) \, . \tag{25}$$

Consistency between Eqs. (15), (21), and (22) imposes that $\phi_0 = 1$, and so we see that $\phi = 1$ defines η_1. From the requirement that the BD theory be recovered when $\omega = \omega_0$ is a constant we obtain the further condition

$$C^2 = \left(\frac{2M(4-3\gamma)}{3(2-\gamma)}\right)^2 f_0 \, ; \tag{26}$$

hence $f_0 \geq 0$ and setting one of C, D, or f_0 to zero implies the simultaneous vanishing of the others unless $\gamma = 4/3$. However, if this is done, it restricts the solutions to be matter (rather than ϕ) dominated at early times. For instance, the solutions which arise in the BD case are of the form $\phi \propto \eta$, $a^{3(2-\gamma)} \propto \eta$, which lead to Nariai's [19] power-law solutions upon integration of the time transformation (9). To retain the regime of scalar-field domination it is necessary to keep one of the integration constants nonvanishing and to express the other two in terms of the first. In what follows we shall use f_0 [which can be absorbed into $f(\eta)$ if desired] to parametrize the solutions. The nonvanishing of the integration constants alters the expression yielding $a(\eta)$ and affects the situations where $f(\eta) - (3\gamma-4)\eta^2/4$ is small compared to f_0.

The BD case provides a point of reference from which to infer the asymptotic behavior of other models where $\omega(\phi)$ is not constant. It arises when both $f(\eta)$ and $g(\eta)$ are quadratic polynomials in η since

$$f_{\rm BD}(\eta) = \frac{3(2-\gamma)}{2M(4-3\gamma)}\sqrt{2\omega_0+3} \times \left[C\,\eta + \frac{M(2-\gamma)}{2}\,\eta^2\,\sqrt{2\omega_0+3}\right] \, . \tag{27}$$

If a theory is characterized by a function $f(\eta)$ [or equivalently by a choice of $g(\eta)$], which grows faster than η^2 when η increases, then $2\omega(\phi)+3$ increases with time, while $2\omega(\phi)+3$ decreases if $f(\eta)$ grows slower than $f_{\rm BD}$.

To facilitate comparison with the BD solutions of Gurevich *et al.* [20] we give their solutions using their time variable ξ. Since in this case ω is a constant, our time variable η is simply proportional to ξ [$d\eta = \sqrt{3/(2\omega_0+3)}\,d\xi$]. For $\gamma \neq 4/3$ one has

$$a(\xi) = a_0\,(\xi-\xi_1)^{\frac{\omega_0}{3(\sigma \mp \sqrt{1+2\frac{\omega_0}{3}})}}\,(\xi-\xi_2)^{\frac{\omega_0}{3(\sigma \pm \sqrt{1+2\frac{\omega_0}{3}})}} \, , \tag{28}$$

$$\phi(\xi) = \phi_0\,(\xi-\xi_1)^{\frac{(1 \mp \sqrt{1+2\frac{\omega_0}{3}})}{(\sigma \mp \sqrt{1+2\frac{\omega_0}{3}})}}\,(\xi-\xi_2)^{\frac{(1 \pm \sqrt{1+2\frac{\omega_0}{3}})}{(\sigma \pm \sqrt{1+2\frac{\omega_0}{3}})}} \, , \tag{29}$$

and for $\gamma = 4/3$,

$$a(\xi) = a_0\,(\xi-\xi_1)^{\frac{1}{2}(1 \mp \frac{1}{\sqrt{1+2\frac{\omega_0}{3}}})}\,(\xi-\xi_2)^{\frac{1}{2}(1 \pm \frac{1}{\sqrt{1+2\frac{\omega_0}{3}}})} \, , \tag{30}$$

$$\phi(\xi) = \phi_0\,(\xi-\xi_1)^{\mp\frac{1}{\sqrt{1+2\frac{\omega_0}{3}}}}\,(\xi-\xi_2)^{\pm\frac{1}{\sqrt{1+2\frac{\omega_0}{3}}}} \, . \tag{31}$$

In these solutions $\xi_{1,2}$ are the roots of $g_{\rm BD}(\eta)$ defined by Eqs. (20) and (27) and $\sigma \equiv (2-\gamma)\omega_0 + 1$.

We see that these solutions (28)–(31) exhibit two branches which differ in the sign of ϕ'. Futhermore, assuming without loss of generality that $\xi_1 > \xi_2$, they have a curvature singularity at ξ_1, where $a = 0$ and $\rho \to \infty$. When approaching this singularity (from $\xi > \xi_1$) the solutions asymptote towards the vacuum solutions of O'Hanlon and Tupper [21]. Similarly, when ξ becomes much larger than $\xi_1 - \xi_2$, the solutions approach the power-law behavior of the solutions derived by Nariai [19] under the assumption $\dot{\phi}a^3 \to 0$ when $a \to 0$ (note that this condition amounts to requiring that $\Delta \equiv \xi_1 - \xi_2 \to 0$, or equivalently that $D \to 0$). This asymptotic behavior reveals an important feature of the solutions. At early times they are dominated by the scalar field (vacuum) energy while at late times matter dominates.

In addition, it is important to notice that the Friedmann constraint equation imposes a relation between the integration constants a_0, ϕ_0, and ρ_0:

$$\phi_0 \propto \frac{8\pi\rho_0 a_0^{3(2-\gamma)}\,\beta_0}{2\omega+3} \, , \tag{32}$$

where β_0 is the coefficient of the $(\eta-\eta_1)^2$ term in $g(\xi)$ given by Eq. (20). Because the proportionality constant in this expression is positive, the sign of ϕ_0 is defined by the sign of β_0. Since

$$\beta_0 = (3/\omega_0)\,(\sigma^2 - 1 - 2\omega_0/3) \, ,$$

the sign of β_0 changes when

$$\gamma = \gamma_*(\omega_0) = 2 + \frac{1-\sqrt{(2\omega_0+3)/3}}{\omega_0} \, , \tag{33}$$

which is always bounded by $\gamma_*(\omega_0 \simeq +\infty) \simeq 2$. If we require $\gamma \geq 0$, this sets a lower bound of $\omega_0 \geq -6/5$. Thus, for $\gamma > \gamma_*$, the solutions have a singularity ($a = 0$) in the future for the fast branch [that with a minus square root in the exponent of the $(\xi - \xi_1)$ factor], where ϕ approaches zero. Gravitation becomes effectively repulsive, but this

---

[1]The integration constants we have introduced define at η_1 the following quantities: a, $a^3\dot{\phi}$, and $\phi d(a^3)/dt$. In fact, we have

$$a^3 = DM(4-3\gamma) \, ,$$

$$a^3\dot{\phi} = \lim_{\eta\to\eta_1} \sqrt{3}M(4-3\gamma)\,(\eta-\eta_1)\,/\sqrt{2\omega(\phi)+3} \, ,$$

$$\phi(a^3)^{\cdot} = 3\sqrt{3}C/2 \, .$$

Thus we see that as $\eta \to \eta_1$ (which we set $\eta_1 = 0$), $a^3\dot{\phi}$ depends on whether $\sqrt{2\omega+3}$ vanishes and, in that case, also depends on how it approaches zero. We further see that the vanishing of a is defined by D being zero.

only occurs when $\gamma > 4/3$, because only for these values of γ is the sign of the right-hand side of the scalar-field equation (12) reversed. A similar result holds for the solutions of the more general scalar-tensor theories. From Eq. (25), we see that $\phi_0 < 0$ when $\gamma > 4/3$.

Let us now consider a number of cases of interest which illustrate the behavior of the solutions of general scalar-tensor theories for the $\gamma < 4/3$. The dust and $p = -\rho$ models stand out as cases of particular relevance and we shall then consider them separately.

IV. PERFECT FLUID FRIEDMANN UNIVERSES

All known exact cosmological solutions to scalar-tensor gravity theories require the energy-momentum tensor to be trace-free [14] or the equation of state to be $p = \rho$. It is of considerable interest to study solutions with other equations of state. The post-recombination history of the universe is well approximated by a $p = 0$ state and pressure-free cosmological solutions of scalar-tensor theories other than Brans-Dicke would enable entire cosmological histories to be constructed. Strong limits could be imposed on deviations from general relativity by considering phenomena such as baryogenesis, primordial nucleosynthesis, galaxy formation, and the present-day expansion dynamics in conjunction with weak-field tests of gravitation theories. It is also of great interest to examine the class of $\omega(\phi)$ theories which can create inflationary expansion ($\ddot{a} > 0$). So far this has only been examined in Brans-Dicke models by means of exact solutions [10] or through approximate analyses in a class of $\omega(\phi)$ theories by Barrow and Maeda [11].

We shall consider a number of representative $\omega(\phi)$ theories created by choices of the generating function $g(\eta)$. This choice is sufficient to produce an exact solution of the field equations.

A. Case $g(\eta) = h\eta^2 \ln\eta$

We consider first perfect fluid solutions with $3\gamma < 4$ generated by the choice

$$g(\eta) = h\eta^2 \ln\eta, \quad h > 0, \text{const} . \tag{34}$$

If we substitute this into the system of equations (21)–(23) then we obtain the following form for the cosmological solution:[2]

$$\phi(\eta) = \phi_0 \ln^{\frac{1}{h}} \eta , \quad \phi_0 \text{ const} \tag{35}$$

$$a^{3(2-\gamma)}(\eta) = g(\eta)\,\phi^{-1}(\eta) = a_0^{3(2-\gamma)}\, \frac{h}{\phi_0}\, \eta^2 \ln^{\frac{h-1}{h}} \eta, \quad a_0 \text{ const} \tag{36}$$

$$2\omega + 3 = \frac{4-3\gamma}{3(2-\gamma)^2}\, \frac{\left[2h\ln\eta + h + 2 - \frac{3\gamma}{2}\right]^2}{\left[h\ln\eta + 1 - \frac{3\gamma}{4}\right]} . \tag{37}$$

The $t(\eta)$ relation is

$$t = \int a^{3(\gamma-1)} \sqrt{\frac{2\omega+3}{3}}\, d\eta; \tag{38}$$

hence

$$t = \lambda \int \eta^{\frac{2(\gamma-1)}{2-\gamma}} \left[\frac{2h\,(\ln\eta)^{1+\left(\frac{h-1}{h}\right)\left(\frac{\gamma-1}{2-\gamma}\right)} + \left(h+2-\frac{3\gamma}{2}\right)(\ln\eta)^{\left(\frac{h-1}{h}\right)\left(\frac{\gamma-1}{2-\gamma}\right)}}{\left(h\ln\eta - \frac{3\gamma}{4} + 1\right)^{\frac{1}{2}}}\right] d\eta , \tag{39}$$

where λ is given by

$$\lambda \equiv a_0^{3(\gamma-1)}\, \frac{\sqrt{4-3\gamma}}{3(2-\gamma)} \left(\frac{h}{\phi_0}\right)^{\frac{\gamma-1}{2-\gamma}} . \tag{40}$$

The solutions (35)–(40) admit further mathematical simplifications if $\gamma = 1$, $\gamma = 0$, or $h = 1$. If we consider the general behavior of Eqs. (35)–(40) as $\eta \to \infty$, so that $\phi \to \infty$ according to Eq. (35), then since $h > 0$, we obtain a late-time asymptote for scalar-tensor theories with power-law variation of $\omega(\phi)$. We have

$$t \simeq 2\lambda h^{\frac{1}{2}} \int \eta^{\frac{2(\gamma-1)}{2-\gamma}} (\ln\eta)^{\frac{\gamma h+2-2\gamma}{2h(2-\gamma)}}\, d\eta , \tag{41}$$

$$2\omega(\phi) + 3 \simeq \frac{4h}{3}\, \frac{(4-3\gamma)}{(2-\gamma)^2} \left(\frac{\phi}{\phi_0}\right)^h . \tag{42}$$

Thus we have to leading order as $\eta \to \infty$, for $0 < \gamma < 4/3$,

$$t \propto \frac{\lambda(2-\gamma)}{\gamma}\, h\eta^{\frac{\gamma}{2-\gamma}} , \tag{43}$$

$$a(t) \propto t^{2/3\gamma} (\ln t)^{\frac{h-1}{h}} , \tag{44}$$

$$\phi(t) \propto (\ln t)^{1/h} , \tag{45}$$

$$\rho = \rho_0\, a^{-3\gamma} , \quad \rho_0 \text{ const} . \tag{46}$$

There is a logarithmic approach to the Friedmann models of GR with the same equations of state. The deviations from GR are relatively mild for the form of $\omega(\phi)$ produced by the choice (34) and this can be seen explicitly by considering the time variation that is created in the Newtonian gravitational coupling $G_N(t) \propto \phi^{-1} \propto (\ln t)^{-1/h}$ as $t \to \infty$. The deviations produced by theories defined by Eq. (37) permit agreement with the solar system tests of

[2] Although $\phi_0 = 1$, we shall keep reference to ϕ_0 in our results to make explicit their dimensional consistency.

GR since the correction terms they add to the GR predictions are all of order $\omega'(\phi)\,\omega^{-3} \sim \phi^{-1-2h}$ and tend to zero as $(\ln t)^{-(1+2h)/h}$ as $t \to \infty$. Hence these theories can produce acceptable weak-field consequences.

It is clear from Eqs. (39)–(46) that, for general $\gamma < 4/3$, a simple special case occurs when $h = 1$. Also, it is necessary to choose h so as to ensure the positivity of $a(t)$ at early times. This is guaranteed if $(h-1)/h$ is an even integer, for example.

B. Case $g(\eta) = E\,\eta^n$

Another interesting choice of generating function is given by

$$g(\eta) = E\,\eta^n\,, \quad E > 0,\ n > 2 \text{ const}\,. \tag{47}$$

The restriction $n > 2$ has been imposed in order to recover solutions which do not approach Brans-Dicke theory at late times. In this case we obtain the general solutions

$$\phi(\eta) = \phi_0 \exp\left[\frac{\eta^{2-n}}{E(2-n)}\right]\,, \tag{48}$$

$$a^{3(2-\gamma)}(\eta) = a_0^{3(2-\gamma)}\,\frac{E}{\phi_0}\,\eta^n \exp\left[\frac{\eta^{2-n}}{E(n-2)}\right]\,, \tag{49}$$

$$2\omega(\eta) + 3 = \frac{(4-3\gamma)}{3(2-\gamma)^2}\,\frac{[nE\eta^{n-2} - (3\gamma-4)/2]^2}{[E\eta^{n-2} - (3\gamma-4)/4]}\,. \tag{50}$$

When $\eta \to \infty$, we obtain

$$t \propto \eta^{\frac{n\gamma}{2(2-\gamma)}}\,. \tag{51}$$

Hence, at late times, we have

$$\phi(t) \propto \exp\left[-C\,t^{\frac{2(2-n)(2-\gamma)}{n\gamma}}\right]\,,\ C \text{ const} \tag{52}$$

$$a(t) \propto t^{\frac{2}{3\gamma}}\,, \tag{53}$$

$$\omega(\phi) \propto \frac{(4-3\gamma)}{(2-n)}\,\frac{1}{\ln(\phi/\phi_0)}\,. \tag{54}$$

Thus we see that the solutions approach the GR solutions. This could be expected since the coupling ω asymptotes to infinity at late times since $\phi \to \phi_0$. This behavior naturally guarantees that these theories survive the scrutiny of weak-field solar system tests. In fact, $\omega'(\phi)/\omega^3 \sim \ln(\phi/\phi_0)/\phi \to 0$ as $t \to \infty$.

C. Case $g(\eta) = \eta^2\,(\eta^n + \alpha^n)$

If we choose

$$g(\eta) = \eta^2\,(\eta^n + \alpha^n)\,, \quad \alpha \text{ const} \tag{55}$$

then we obtain the solutions

$$\phi(\eta) = \phi_0 \left(\frac{\eta^n}{\eta^n + \alpha^n}\right)^{\frac{1}{n\alpha^n}}\,, \tag{56}$$

$$a^{3(2-\gamma)}(\eta) = \frac{a_0^{3(2-\gamma)}}{\phi_0}\,\eta^{2-1/\alpha^n}\,(\eta^n + \alpha^n)^{1+\frac{1}{n\alpha^n}}\,, \tag{57}$$

$$2\omega(\phi) + 3 = \frac{4-3\gamma}{3(2-\gamma)^2}\,\frac{[(1+n)\eta^n + \alpha^n - (3\gamma-4)/2]^2}{\eta^n + \alpha^n - (3\gamma-4)/4}\,. \tag{58}$$

For large η, we have

$$\phi(\eta) \sim \text{const}\,, \tag{59}$$

$$a(\eta) \sim \eta^{2+n}\,, \tag{60}$$

and

$$2\omega + 3 \propto \frac{(\phi/\phi_0)^{n\alpha^n}}{1 + (\phi/\phi_0)^{n\alpha^n}}\,. \tag{61}$$

In the same limit, we obtain

$$t \propto \eta^{1+\frac{n}{2}}, \tag{62}$$

which means that, beside ϕ being constant, we also recover $a(t) \propto t^{2/(3\gamma)}$ and hence approach GR in the late time limit for this choice of $g(\eta)$.

V. INFLATIONARY MODELS

We consider exact solutions for $k = 0$ Friedmann models when $\gamma = 0$. This corresponds to a matter source which is created by a potential-dominated scalar field. It is the standard matter source generating a wide variety of inflationary models. We recall that in Brans-Dicke models the $\gamma = 0$ source produced an example of power-law inflation [10] in contrast to the exponential, de Sitter behavior obtained in general relativity with the same matter source.

A. Case $g(\eta) = h\eta^2 \ln\eta$

Again, we consider the form of $g(\eta)$ defined by Eq. (34). The exact form of the solution is

$$\phi(\eta) = \phi_0 \ln^{1/h}\eta\,, \tag{63}$$

$$a^6(\eta) = a_0^6\, h\phi_0^{-1}\,(\ln\eta)^{\frac{h-1}{h}}\,, \tag{64}$$

$$2\omega + 3 = \frac{\left[2h\,(\phi/\phi_0)^h + h + 2\right]^2}{3\left[h\,(\phi/\phi_0)^h + 1\right]}. \tag{65}$$

If we set

$$u = h\ln\eta + 1\,, \tag{66}$$

then

$$t = \frac{\sqrt{\phi_0}}{3}h^{-\frac{2h+1}{2h}} \int \left[2u^{\frac{1}{2}}\,(u-1)^{\frac{1-h}{2h}}\,du + hu^{-\frac{1}{2}}\,(u-1)^{\frac{1-h}{2h}}\right] du\,. \tag{67}$$

There are simple integrable cases whenever $\frac{(1-h)}{2h}$ is a non-negative integer. For example, when $h = 1$ we have

$$t = \frac{2\sqrt{\phi_0}}{3}\left[2(\ln\eta+1)^{3/2} + (\ln\eta+1)^{1/2}\right] , \quad (68)$$

$$\phi(\eta) = \phi_0 \ln\eta , \quad (69)$$

$$a^6(\eta) = \frac{\eta^2}{\phi_0} . \quad (70)$$

So $t \in (-\infty, +\infty)$ for $\eta \in (0, +\infty)$.

Rather than list the complicated catalogue of exact solutions that can be generated by the allowed choices of h, it is more instructive to derive the general form of the solutions for all h as $\eta \to \infty$. These display some new varieties of inflation that can arise in scalar-tensor theories. We have

$$t \propto (\ln\eta)^{\frac{1+2h}{2h}} , \quad (71)$$

$$a \propto \eta^{\frac{1}{3}} (\ln\eta)^{\frac{h-1}{6h}} ; \quad (72)$$

Hence

$$a \propto t^{\frac{h-1}{3(2h+1)}} \exp\left[A\, t^{\frac{2h}{2h+1}}\right],\ A \text{ const} \quad (73)$$

with

$$\phi(t) \propto t^{\frac{2}{2h+1}} , \quad (74)$$

$$2\omega(\phi) + 3 \sim \frac{4h}{3}\left(\frac{\phi}{\phi_0}\right)^h . \quad (75)$$

These solutions display new varieties of inflationary universe. When $h = 1$, we have $a(t) \propto \exp\left[At^{2/3}\right]$. This is the particular example of the "intermediate inflation" proposed in [22,23]. It is especially interesting that this form of the scale factor, which arises when $h = 1$, is precisely that which generates the exact Zel'dovich-Harrison spectrum for the density and gravitational wave fluctuations produced during inflation to first order in perturbation theory [24].

To establish the nature of inflation in models with $h \neq 1$ we need to examine the general conditions under which a scale factor evolving as

$$a(t) \propto t^\alpha \exp\left[At^\beta\right] , \quad A > 0 \quad (76)$$

is inflationary as $t \to \infty$. Inflation occurs if $\ddot{a} > 0$ and we find that, as $t \to \infty$,

$$\frac{\ddot{a}}{a} \sim \frac{A^2}{9} t^{2\alpha+\beta-2} > 0 . \quad (77)$$

So inflation will always occur (and continue indefinitely unless other physics intervenes or γ changes with time), but the acceleration increases with time (ultimately producing a scalar curvature singularity to the future) if $2\alpha + \beta - 2 > 0$, that is, in our model, if $h > 7$.

From Eq. (44) we see that when $h = 1$ the inflationary models which arise when $0 < \gamma < 2/3$ are the same as the power-law inflationary models of general relativity for these equations of state, but when $h \neq 1$ they give rise to a logarithmically moderated form of power-law inflation.

B. Case $g(\eta) = E\,\eta^n$

For this choice of the generating function we derive a solution

$$\phi(\eta) = \phi_0 \exp\left[\frac{\eta^{2-n}}{E(2-n)}\right] , \quad (78)$$

$$a^6(\eta) = a_0^6 \frac{E}{\phi_0}\eta^n \exp\left[\frac{\eta^{2-n}}{E(n-2)}\right] , \quad (79)$$

$$2\omega(\eta) + 3 = \frac{1}{3}\frac{\left(nE\eta^{n-2}+2\right)^2}{\left(E\eta^{n-2}+1\right)} . \quad (80)$$

So, when $\eta \to \infty$, we obtain

$$t \propto \ln\eta . \quad (81)$$

Hence the late-time behavior of the solutions is

$$\phi(t) \propto \exp\left[-\frac{1}{E(n-2)}\exp\left[-C\,t\right]\right] , \quad (82)$$

$$a(t) \propto \exp\left[-\frac{C}{n-2}\,t\right] . \quad (83)$$

This shows that the exponential inflationary behavior is rapidly attained since $\phi(t)$, and hence G, approaches its asymptotic constant value very quickly.

VI. THE DUST CASE

The importance of the dust models arises from the fact that they provide a very good description of the present thermodynamic state of matter. The only known exact cosmological solutions are for zero-curvature Brans-Dicke models. In what follows we give a number of solutions which cover a wide range of realistic models.

A. Case $g(\eta) = h\eta^2 \ln\eta$

Inserting this generating function into the general solution (35)–(40) derived above, we obtain

$$\phi(\eta) = \phi_0 \ln^{\frac{1}{h}}\eta , \quad (84)$$

$$a^3(\eta) = \frac{ha_0^3}{\phi_0}\eta^2 \ln^{\frac{h-1}{h}}\eta , \quad (85)$$

and

$$2\omega + 3 = \frac{4\left[2h(\phi/\phi_0)^h + h + 1/2\right]^2}{3\left[h(\phi/\phi_0)^h + 1/4\right]} . \quad (86)$$

From Eq. (84) we see that ϕ increases with η and from Eq. (86) it follows that $2\omega(\phi) + 3$ asymptotically approaches a power dependence on ϕ, with $2\omega + 3 \propto \phi^h$.

A special case where simplifications arise is when $h = 1$. We obtain

$$\phi(\eta) = \phi_0 \ln \eta \,, \tag{87}$$

$$a^6(\eta) = \frac{\eta^2}{\phi_0} \,. \tag{88}$$

Defining $u \equiv h \ln \eta + 1/4$, we derive

$$t = \frac{2}{3} \int \frac{2u + h}{\sqrt{u}} \exp\left(\frac{u - 1/4}{h}\right) du \,. \tag{89}$$

Now we consider the asymptotic behavior of these solutions at large η. We immediately have $u \sim h \ln \eta$, which implies

$$t \propto \eta \,. \tag{90}$$

Since $a \propto \eta^{2/3} (\ln \eta)^{(h-1)/3h}$, we obtain the asymptotic behavior

$$a(t) \propto t^{\frac{2}{3}} \, (\ln t)^{\frac{(h-1)}{3h}} \,, \tag{91}$$

$$\phi(t) \propto \ln^{\frac{1}{h}} t \,, \tag{92}$$

and also

$$2\omega(\phi) + 3 \propto \frac{4h}{3} \left(\frac{\phi}{\phi_0}\right)^h \,. \tag{93}$$

Note that this result for the late-time behavior is consistent with the results derived for general γ. It corresponds to a mild deviation from GR behavior.

B. Case $g(\eta) = E\,\eta^n$

With this generating function the solutions are

$$\phi(\eta) = \phi_0 \exp\left(\frac{\eta^{2-n}}{E(2-n)}\right) \,, \tag{94}$$

$$a^3(\eta) = \frac{E}{\phi_0}\, \eta^n \exp\left(-\frac{\eta^{2-n}}{E(2-n)}\right) \,. \tag{95}$$

At late times we have $a \propto \eta^{\frac{n}{3(2-\gamma)}} \propto t^{\frac{2}{3\gamma}}$, $\phi \propto \exp\left[-Ct^{\frac{2(2-n)(2-\gamma)}{n\gamma}}\right]$, where C is a constant, and

$$2\omega + 3 \propto \frac{(4 - 3\gamma)}{(2-n)\ln\left(\frac{\phi}{\phi_0}\right)}. \tag{96}$$

VII. THE RADIATION MODELS

Exact solutions for radiation models ($3\gamma = 4$) of all curvatures can more easily be found by exploiting the conformal invariance of scalar-tensor theories [14]. However, it is instructive to see how the method considered in this paper applies to radiation models.

From Eq. (12) we see that $y = y_0$ =const when $\gamma = 4/3$. Repeating the derivations of Sec. III, using this result, we obtain

$$\frac{\phi'}{\phi} = \frac{1}{D + f(\eta)} \,, \tag{97}$$

where $f(\eta)$ is now defined by

$$f(\eta) \equiv \int_0^\eta \frac{\sqrt{2\omega+3}}{y_0} \left[C + \frac{2M}{3} \int_0^{\bar\eta} \sqrt{2\omega+3}\, d\bar{\bar\eta}\right] d\bar\eta. \tag{98}$$

If we further define

$$g(\eta) = f(\eta) + D \tag{99}$$

the solutions take forms similar to those given by Eqs. (21) and (22) with $\gamma = 4/3$, with

$$\ln\left(\frac{\phi}{\phi_0}\right) = \int_0^\eta \frac{d\bar\eta}{g(\bar\eta)} \,, \tag{100}$$

$$a^2 = a_0^2 \left(\frac{g(\eta)}{\phi}\right) \,, \tag{101}$$

and

$$2\omega + 3 = \frac{3y_0}{4M} \frac{f'2}{\left(f + \frac{3y_0}{4M} f_0 + D\right)} \,. \tag{102}$$

From the Friedmann equation (14) we obtain the constraints

$$y_0 = \phi_0\, a_0^{2/3} = \frac{4MD}{3f_0 - 1} \tag{103}$$

upon the integration constants y_0, ϕ_0, a_0, D, and f_0. Consistency between Eqs. (100) and (101) and the definition of y further require that $y_0 = a_0^2 = \phi_0^{3/2}$.

In order to understand how the present method of integration compares with that arising from the conformal invariance of the radiation models [14], note that our time variable η relates to conformal time $\tilde\eta$ [defined as usual by $dt = a(\tilde\eta)\, d\tilde\eta$] as

$$d\tilde\eta = \sqrt{\frac{2\omega+3}{3}}\, d\eta. \tag{104}$$

Hence substitution of η by $\tilde\eta$ in (97) yields

$$\sqrt{2\omega+3}\, \frac{d\phi}{\phi} = \frac{d\tilde\eta}{g(\tilde\eta)} = \frac{d\tilde\eta}{\phi\, a^2}, \tag{105}$$

which is precisely one of the equations in Ref. [14]. When restricted, the $k = 0$ models, the other fundamental equation of that work [14], establishes that

$$\frac{d^2 g}{d\tilde\eta^2} = 2M. \tag{106}$$

This yields the $\tilde\eta$ behavior of the generating function. However, the use of this latter result in the present case is restricted to the situations where the relation $\eta = \eta(\tilde\eta)$ can be explicitly solved.

We now consider some specific choices of the generating functions $g(\eta)$ which transpose to the radiation models the particular solutions investigated above.

A. Case $g(\eta) = h\eta \ln \eta$

This choice of the generating function $g(\eta)$ yields

$$\phi = \phi_0 \ln^{\frac{1}{h}} \eta \,, \tag{107}$$

$$a^2 = a_0^2 \, h \, \eta \ln^{\frac{h-1}{h}} \eta \,, \tag{108}$$

and

$$2\omega + 3 = \frac{3y_0 h}{4M} \frac{(\ln \eta + 1)^2}{\eta \ln \eta + \frac{3y_0}{4M}\left(\frac{f_0}{h}\right)} \,. \tag{109}$$

B. Case $g(\eta) = E\,\eta^n$

With this choice for $g(\eta)$ we obtain the solutions

$$\phi = \phi_0 \exp\left(\frac{\eta^{1-n}}{E(1-n)}\right) \quad n \neq 1, \tag{110}$$

$$a^2 = a_0^2 \left(\frac{E}{\phi_0}\right) n \neq 1 \, \eta^n \exp\left(\frac{\eta^{1-n}}{E(n-1)}\right) \,, \tag{111}$$

and

$$2\omega + 3 = \frac{3y_0 E}{4M} \frac{n^2 \eta^{2(n-1)}}{\eta^n - \left(\frac{D}{E}\right)} \,. \tag{112}$$

VIII. DISCUSSION

In this paper we have shown how to derive exact cosmological solutions for the Friedmann $k = 0$ models with a perfect fluid characterized by the equation of state $p = (\gamma - 1)\rho$, where γ is a constant and $0 \leq \gamma < 2$, in general scalar-tensor gravity theories. By an appropriate choice of variables we have shown that the solutions can be defined in terms of a generating function $g(\eta)$, which amounts to the specification of the $\omega(\phi)$ dependence of the theories. We have provided a number of specific examples of dust solutions and inflationary universe theories for theories with power-law $\omega(\phi)$ dependence. The $p = -\rho$ inflationary models provide examples of a new form of inflation in which the scale factor evolves in proportion to $t^n \exp(H_0 t^m)$, $0 < m < 1$. The derivation of $p = 0$ scalar-tensor cosmological models will allow complete histories to be constructed in such theories and these will be studied elsewhere. The availability of new exact solutions allows cosmological tests of gravitation theories to be combined with weak-field and laboratory limits on allowed deviations from general relativity to constrain the permitted depertures of ω^{-1} and $\omega' \omega^{-3}$ from zero.

ACKNOWLEDGMENTS

The authors are grateful to David Wands for useful discussions and acknowledge the use of the Starlink computer system at Sussex.

[1] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

[2] P. G. Bergmann, Int. J. Theor. Phys. **1**, 25 (1968); R. V. Wagoner, Phys. Rev. D **1**, 3209 (1970).

[3] K. Nordvedt, Astrophys. J. **161**, 1059 (1970).

[4] G. W. Gibbons and B. F. Whiting, Nature (London) **291**, 636 (1981).

[5] C. M. Will, *Theory and Experiment in Gravitation* (Cambridge University Press, Cambridge, England, 1993).

[6] C. T. Hill, P. J. Steinhardt, and M. S. Turner, Phys. Lett. B **252**, 343 (1990); F. S. Accetta and P. J. Steinhardt, Phys. Rev. Lett. **67**, 298 (1991).

[7] P. G. O Freund, Nucl. Phys. **B209**, 146 (1982).

[8] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory: 2* (Cambridge Univeristy Press, Cambridge, England, 1987); T. Appelquist, A. Chodos, and P. G. O. Freund, *Modern Kaluza-Klein Theories* (Addison-Wesley, New York, 1987); J. A. Casas, J. Garcia-Bellido, and M. Quirós, Nucl. Phys. **B361**, 713 (1991).

[9] J. D. Barrow, Phys. Rev. D **46**, 3227 (1992).

[10] D. La and P. J. Steinhardt, Phys. Rev. Lett. **62**, 376 (1989); C. Mathiazhagen and V. B. Johri, Class. Quantum Grav. **1**, L29 (1984); P. J. Steinhardt and F. S. Ascetta, Phys. Rev. Lett. **64**, 2470 (1990).

[11] J. D. Barrow and K. Maeda, Nucl. Phys. **B341**, 294 (1990).

[12] G. Greenstein, Astrophys. Space Sci. **2**, 155 (1968); J. D. Barrow, Mon. Not. R. Astron. Soc. **184**, 677 (1978); R. Dominguez-Tenreiro and G. Yepes, Astrophys. J. **177**, 2 (1987); A. Serna, R. Dominguez-Tenreiro, and G. Yepes *ibid.* **391**, 433 (1992).

[13] R. D. Reasenberg *et al.*, Astrophys. J. **234**, L219 (1979).

[14] J. D. Barrow, Phys. Rev. D **47**, 5329 (1993).

[15] J. P. Mimoso and D. Wands (unpublished).

[16] T. Damour and K. Nordvedt, Phys. Rev. Lett. **70**, 2217 (1993); Phys. Rev. D **48**, 3436 (1993).

[17] K. S. Thorne and C. M. Will, Astrophys. J. **163**, 595 (1971).

[18] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[19] H. Nariai, Prog. Theor. Phys. **40**, 49 (1968).

[20] L. E. Gurevich, A. M. Finkelstein, and V. A. Ruban, Astrophys. Space Sci. **22**, 231 (1973); D. Lorenz-Petzold, *ibid.* **98**, 249 (1984).

[21] J. O'Hanlon and B. O. J. Tupper, Nuovo Cimento **7**, 305 (1972); D. Lorenz-Petzold, Astrophys. Space Sci. **98**, 101 (1984).

[22] J. D. Barrow, Phys. Lett. B **235**, 40 (1989).

[23] J. D. Barrow and P. S. Saich, Phys. Lett. B **249**, 406 (1990).

[24] J. D. Barrow and A. R. Liddle, Phys. Rev. D **47**, R5219 (1993).