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Two-loop neutrino masses and the solar neutrino problem

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The addition of m singlet right-handed neutrinos to the standard model leads to radiatively generated mass corrections for the $\mathrm{SU}(2)_L$ doublet neutrinos. For those neutrinos which are massless at the tree level after this addition, this implies a small mass generated at the two-loop level via $W^\pm$ exchange. We calculate these mass corrections exactly by obtaining an analytic form for the general case of n doublets and m singlets. As a phenomenological application, we study the $m = 1$, $n = 3$ case in detail, for which there are two massive and two massless neutrinos at the tree level. After calculating the loop-induced masses, the full mass spectrum is examined in the light of experimental data from the four solar neutrino experiments and the COBE satellite. Requiring compatibility with these experiments enables us to constrain the parameter space and to obtain a handle on the scale of new physics, as typified by the mass of the heavy singlet. We also obtain the interesting result that the final mass eigenvalues, corrected for loop effects, may all be significantly different from the seesaw value of the lighter of the two neutrinos which have tree level masses, thus demonstrating the importance of the loop effects considered here.

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I. INTRODUCTION

It is fair to say that the problem of understanding the origin of fermion masses is one of the most perplexing questions facing particle physics today. The standard model [1] can reproduce the observed fermion masses via electroweak symmetry breaking and the Higgs mechanism, but provides no explanation for their values. When such an understanding is obtained, one of the issues that it must clarify is the smallness of neutrino masses (if, indeed, neutrinos are massive) relative to those of the other fermions. An attractive explanation for this observed feature of the fermion mass spectrum is the seesaw mechanism [2]. It postulates the existence of right-handed neutrinos with masses of the order of the next energy threshold and uses this in combination with the Higgs mechanism to generate light (Majorana) neutrino masses via an effective dimension five operator.

Given our present ignorance of the origins of mass and the lack of experimental pointers towards any particular mechanism, it is important to keep an open mind on the smallness of neutrino mass. In this paper, we explore, via detailed calculation, the issue of radiatively generated neutrino masses, since this is also a natural way in which masses small compared to those of other fermions may be generated.

Any such effort needs to invoke physics beyond the standard model. In view of the extraordinary and demonstrated robustness of the model to experimental tests over the last 20 years, we have thought it reasonable to make the simplest possible extension to the standard theory and study its effect on neutrino masses via radiative corrections, i.e., the addition of m $\mathrm{SU}(2)_L \bigotimes \mathrm{U}(1)_Y$ singlet right-handed neutrinos. *A priori*, there is no connection between their number and that of the doublet neutrinos; hence, the simplest case corresponds to $m = 1$, i.e., the addition of one right-handed singlet neutrino to the standard model [3].

The gauge group structure of the weak sector remains unchanged as a consequence of this extension, but Majorana mass terms incorporating the scale of new physics are now allowed. We do not speculate on their origin, but only note that it would require invoking an additional global symmetry (such as a conserved lepton number) to set these to zero. The doublet neutrinos acquire radiative (and, in some cases, tree-level) masses due to the presence of the singlets, as we discuss below. The radiative masses arise (via mixing) due to a two-loop mechanism [4,5] involving the exchange of $W^\pm$ bosons. In Sec. II and the Appendix, we calculate, exactly and in analytic form, the two-loop masses acquired by the initially massless doublet neutrinos. Our calculation is general and valid for any number of doublet and singlet neutrinos, but in order to obtain phenomenologically useful information, we focus, in Sec. III, on the $n = 3$, $m = 1$ case, for which we calculate the mass spectrum fully. In this

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instance there are two massive and two massless neutrinos at the tree level. Prior to the incorporation of loop effects, the lighter tree-level mass is simply due to the seesaw effect generated by the heavy singlet. An interesting result of our calculation is that once loop corrections to the three doublet neutrinos are calculated and the full (tree + loop) matrix diagonalized, the seesaw mass value can be significantly modified. In other words, in the corrected mass spectrum, none of the eigenvalues need to be close to the seesaw value.

Even this simplest ($m = 1$) extension of the standard model introduces four new parameters into the theory. On the issue of neutrino masses, it is nonaccelerator experiments that provide information on the cutting edge. Hence we have chosen to examine the results for the $m = 1$ case in the light of (a) the Mikheyev-Smirnov-Wolfenstein (MSW) [6] solution to the solar neutrino deficit seen by the Kamiokande [7], GALLEX [8], SAGE [9], and Homestake [10] neutrino detectors and (b) the implications for hot dark matter (neutrinos) from the recent Cosmic Background Explorer (COBE) observations on the anisotropy of the microwave background [11]. Invoking this experimental information restricts the parameter space and consequently, in the context considered in this paper, permits a handle on the range of the mass scale characterizing physics beyond the standard model. We show that doublet neutrino masses compatible with both (a) and (b) above can result naturally from such physics at the several hundred GeV scale. Masses compatible also with the atmospheric neutrino deficit [12] cannot, however, be naturally realized in the scenario considered here without making special assumptions.

II. RADIATIVE GENERATION OF NEUTRINO MASSES

In this section we give a description of an exact general procedure for calculating two-loop neutrino masses applicable to any extension of the standard model which incorporates singlet right-handed neutrinos. (We remark below on the reason why a one-loop mass does not arise in the situation considered here, where *only* right-handed handed neutrinos are added to the existing particle spectrum.) After setting up the generic integral that needs to be calculated we describe the procedure for evaluating it exactly in the Appendix.

The lepton sector of the extension considered here has, in general, $n(\geq 3)$ doublet fields $[\nu'_{iL} l_{iL}]^T$ and m singlet fields $(\nu'_{AL})^c = (\nu'^c_A)_R$. (Here $i = 1,...,n, A = 1,...,m$, and $\nu^c \equiv C\bar{\nu}^T$ is the charge conjugate spinor.) In addition, one has the charged lepton $\mathrm{SU}(2)_L$ singlet fields l_{iR}. The primes on the neutrino fields denote weak eigenstates as opposed to physical particle states. Without any loss of generality, we have assumed that the weak eigenstates l_i are the same as the corresponding mass eigenstates; i.e., the charged lepton mass matrix is diagonal.

As noted in the Introduction, in addition to the Dirac mass terms, the most general Lagrangian consistent with the gauge symmetry of the standard model also contains possible Majorana mass terms for neutrinos of the form $m_{AB}\overline{(\nu'_{AL})^c}\nu'_{BL}$. In the minimal model under consideration here, such terms must be bare mass terms, but in a more involved model they could arise, for instance, due to the vacuum expectation value of a Higgs singlet. To facilitate discussion, we combine all the left-handed neutrinos into a $(n+m)$-dimensional vector in the flavor space denoting it by $\nu'_{\alpha L}$. The most general mass term is thus given by

$$\mathcal{L}_m = \sum_{i=1}^{n} \mu_i \bar{l}_{iL} l_{iR} + \sum_{\alpha,\beta=1}^{n+m} \overline{(\nu'_{\alpha L})^c} \mathcal{M}_{\alpha\beta} \nu'_{\beta L} + \text{H.c.} \quad (2.1)$$

Here $\mathcal{M}$ is a complex symmetric[1] $(n+m)\times(n+m)$ matrix of the form

$$\mathcal{M} = \begin{pmatrix} O_{n\times n} & D_{n\times m} \\ D^T_{m\times n} & M_{m\times m} \end{pmatrix} \quad (2.2)$$

with D and M denoting the Dirac and the Majorana mass terms, respectively. The first block is identically zero in the absence of a nontrivial vacuum expectation value for a $\mathrm{SU}(2)_L$-triplet Higgs field. (This restriction is imposed not only by our philosophy of minimal extension, but more importantly, by m_W/m_Z—the observed ratio of the gauge boson masses.) $\mathcal{M}$ can be diagonalized by a biunitary transformation of the form

$$V^T \mathcal{M} V = \widehat{\mathcal{M}} = \text{diag}(m_\alpha). \quad (2.3)$$

The mass eigenstates (ν_α) are then easily identified to be

$$\nu_L = V^\dagger \nu'_L. \quad (2.4)$$

The relevant piece of the weak Lagrangian is then given by

$$\mathcal{L}_{\text{wk}} = J_\mu W^\mu, \quad (2.5)$$

where

$$\begin{aligned} J^+_\mu &= \frac{ig}{\sqrt{2}} \sum_{i=1}^{n} \bar{l}_i \gamma_\mu P_L \nu'_i = \frac{ig}{\sqrt{2}} \sum_{i=1}^{n} \sum_{\alpha=1}^{n+m} K_{i\alpha} \bar{l}_i \gamma_\mu P_L \nu_i, \\ J^3_\mu &= \frac{ig}{2c_W} \sum_{i=1}^{n} (\bar{l}_i \gamma_\mu P_L l_i + \bar{\nu}'_i \gamma_\mu P_L \nu'_i) \quad (2.6) \\ &= \frac{ig}{2c_W} \left[\sum_{i=1}^{n} \bar{l}_i \gamma_\mu P_L l_i + \sum_{\alpha,\beta=1}^{n+m} (K^\dagger K)_{\alpha\beta} \bar{\nu}_\alpha \gamma_\mu P_L \nu_\beta \right]. \end{aligned}$$

Here $c_W \equiv \cos\theta_W$, where θ_W is the Weinberg angle, $g = e/\sin\theta_W$, $P_L \equiv (1-\gamma_5)/2$, and

$$K = \begin{pmatrix} I_{n\times n} & 0 \\ 0 & 0 \end{pmatrix} V \quad (2.7)$$

is the $(n+m)$-dimensional analogue of the quark sector Cabibbo-Kobayashi-Maskawa matrix. Note that though

[1]That $\mathcal{M}$ has to be symmetric is evident from the charge conjugation property of fermion bilinears.

$KK^\dagger = \mathrm{diag}(I_{n\times n},0), (K^\dagger K)_{\alpha\beta} \neq \delta_{\alpha\beta}$. Thus we do have flavor-changing neutral currents (FCNC's) in the neutrino sector.

Having set up the general formalism, we now concentrate on the case where $n > m$. There exist then $n - m$ neutrinos that are strictly massless at the tree level. We calculate the changes to such a spectrum accruing from quantum corrections.

Before proceeding, in view of the fact that there exist massive neutrinos and also FCNC's in the neutrino sector, it is appropriate at this point to remark on the possibility of one-loop graphs with a Z or a Higgs exchange introducing a nontrivial correction to the neutrino mass matrix. However, it can be easily seen that it is possible to rotate the neutrino states such that only m of them have Yukawa couplings to the Higgs boson. Thus, only those doublet states that are massive at the tree level obtain a Higgs induced mass at the one-loop level. In addition, since the flavor-changing Z couplings have the same mixing parameters as the flavor-changing Yukawa couplings, the one-loop Z exchange diagrams do not contribute to the masses of the $n - m$ neutrinos which are massless at the tree level. This reasoning applies at all orders to any diagram where all virtual particles are neutral. Hence the relevant diagram to compute is that given in Fig. 1.

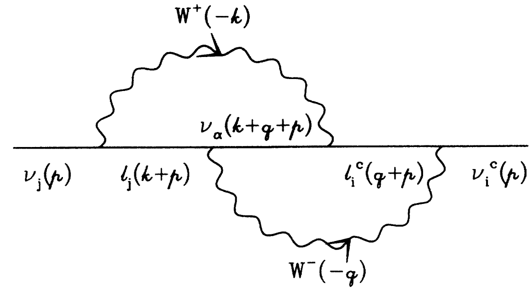

FIG. 1. The two-loop diagram which gives rise to the mass corrections considered in this paper.

We shall work in the weak interaction basis for the external neutrinos and the mass basis for all the virtual particles. Furthermore, we shall concentrate only on the first $n\times n$ block of $\mathcal{M}$, i.e., on the generation of Majorana mass terms for the doublet neutrinos. In the unitary gauge, the correction to the neutrino propagator is then given by

$$i\Sigma_{ij}^{(2)}(p) = \left(\frac{ig}{\sqrt{2}}\right)^4 \sum_{\alpha=1}^{n+m} \int \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\gamma_\mu P_R \frac{i}{\not p+\not q-\mu_i} K^\dagger_{\alpha i}\gamma_\nu P_R \frac{i}{\not p+\not q+\not k-m_\alpha} K^\dagger_{\alpha j}\gamma_\sigma P_L \frac{i}{\not p+\not k-\mu_j}\gamma_\lambda P_L$$
$$\times \frac{-i(g^{\mu\sigma}-q^\mu q^\sigma/m_W^2)}{q^2-m_W^2}\frac{-i(g^{\nu\lambda}-k^\nu k^\lambda/m_W^2)}{k^2-m_W^2}. \tag{2.8}$$

The mass correction is given by $\mathcal{M}_{ij}^{(2)} = \Sigma_{ij}^{(2)}(p=0)$, and after some algebra this leads to

$$\mathcal{M}_{ij}^{(2)} = \frac{g^4}{4}\sum_{\alpha=1}^{n+m} m_\alpha K^\dagger_{\alpha j}K^\dagger_{\alpha i}\int \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{(k+q)^2 k\cdot q}{\mathcal{D}_{ij;\alpha}}\left[\left(4+\frac{k^2q^2}{m_W^4}\right)-4\frac{q^2+k^2}{m_W^2}\right] \tag{2.9}$$

where

$$\mathcal{D}_{ij;\alpha} = (k+q)^2\{(k+q)^2-m_\alpha^2\}(q^2-\mu_i^2)(q^2-m_W^2)(k^2-m_W^2)(k^2-\mu_j^2). \tag{2.10}$$

We see that the mass corrections would be identically zero if $m_\alpha = 0$, $\forall\alpha$. This ought to be so as any mass renormalization must be proportional to the bare mass terms. The integral above has a naive degree of divergence of 4. However, note that

$$\sum_{\alpha=1}^{n+m} m_\alpha K^\dagger_{\alpha j}K^\dagger_{\alpha i} = \mathcal{M}_{ij} = 0, \tag{2.11}$$

and hence,

$$\sum_{\alpha=1}^{n+m} m_\alpha K^\dagger_{\alpha i}K^\dagger_{\alpha i}\frac{(k+q)^2}{(k+q)^2-m_\alpha^2} = \sum_{\alpha=1}^{n+m}\frac{K^\dagger_{\alpha i}K^\dagger_{\alpha i}m_\alpha^3}{(k+q)^2-m_\alpha^2}. \tag{2.12}$$

This clearly is analogous to the Glashow-Iliopoulos-Maiani (GIM) mechanism in the quark sector. Even on substitution of Eq. (2.12) in Eq. (2.9), the integral in the latter is still formally divergent. Notice, however, that this is but an artifact of the unitary gauge and is not a real divergence [13]. In fact, by invoking identities similar to Eq. (2.12) or equivalently, by working in the Feynman gauge, one obtains

$$\mathcal{M}_{ij}^{(2)} = g^4 \sum_{\alpha=1}^{n+m} m_\alpha^3 K_{\alpha j}^\dagger K_{\alpha i}^\dagger \left[4 + 4\frac{\mu_i^2+\mu_j^2}{m_W^2} + \frac{\mu_i^2\mu_j^2}{m_W^4}\right] \Lambda(\mu_i^2, m_W^2, m_\alpha^2, 0, \mu_l^2, m_W^2) \tag{2.13}$$

where

$$\Lambda(m_1^2,m_2^2,m_3^2,m_4^2,m_5^2,m_6^2) \equiv \int \frac{d^4k d^4q k\cdot q}{(q^2+m_1^2)(q^2+m_2^2)\{(k+q)^2+m_3^2\}\{(k+q)^2+m_4^2\}(k^2+m_5^2)(k^2+m_6^2)} \tag{2.14}$$

is an Euclidean integral evaluated in the Appendix.

The expression in Eq. (2.13) thus represents the Majorana mass generated for the doublet neutrino at the two-loop level. In operator language, it arises from terms of the form

$$\overline{(L_{iL})^c} L_{jL}\phi\phi S, \tag{2.15}$$

where L_{iL} represents the doublet lepton fields, ϕ is the usual Higgs field, and S represents the lepton-number-violating operator (whether a Higgs singlet or a bare mass term). We note that this five-dimensional effective operator for the radiative masses is the same as that for the conventional seesaw mechanism. The difference between the two resides in the scale of mass generation. Two-loop radiative masses compatible with the solar and COBE data can arise from right-handed neutrinos at the several hundred GeV scale, as we show below, whereas the seesaw mechanism generates similar valued masses via heavy neutrinos at the grand unified scale.

We also note that though the corrections ostensibly are proportional to m_α^3 [Eq. (2.13)], the actual dependence is linear (apart from logarithmic corrections) due to suppressions hidden in Λ. As m_α becomes larger and terms of the order $(\mu_i/m_\alpha)^2$ become negligible, the correction goes as $\Sigma_\alpha K_{\alpha i}^\dagger K_{\alpha j}^\dagger m_\alpha$, which is simply the (ij)th element of the tree-level mass matrix, and hence zero for the cases of interest here.

Finally, we remark that a complex $\mathcal{M}$ in Eq. (2.1) obviously leads to a complex diagonalizing matrix V and hence to CP-violating processes in general. However, for the $m = 1$ case which we study here, we perform all numerical calculations assuming a real neutrino mass matrix.

III. APPLICATION: THE SOLAR NEUTRINO DEFICIT AND COBE DATA

In order to make a connection to experiment and phenomenology, we now specialize to the $n = 3$ and $m = 1$ case and examine the two-loop mass corrections in the context of (a) the MSW solution [6] to the solar neutrino deficit reported by various detectors [7–10] and (b) recent COBE [11] data and its implications for neutrinos as dark matter.

The solar deficit is the only long-standing possible evidence for physics beyond the standard model, and the MSW mechanism is its most popular resolution. In its essence, the mechanism requires neutrinos to be massive (and nondegenerate), allowing the interaction eigenstate ν_e (assumed to comprise predominantly of the lightest mass eigenstate) to oscillate to ν_μ or ν_τ due to the difference in the forward scattering potential seen by the two states in their passage through solar matter. It thus identifies a range of vacuum mixing angle and mass squared difference values which are compatible with the deficit observed by the various detectors. Figure 2, excluding curves labeled (a), (b), and (c), is taken from Ref. [14] and shows the familiar two-flavor mixing MSW solution space, where θ is the Cabibbo mixing angle and Δm^2 is the difference of the squares of the two neutrino masses, which, in the present context, reflect the two-loop quantum correction calculated here.

COBE data on the anisotropy of the microwave back-

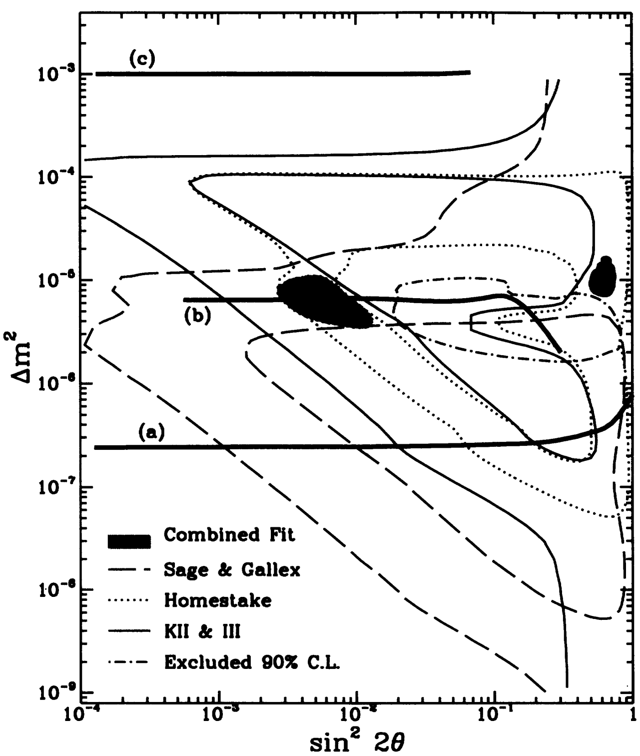

FIG. 2. The MSW solution space for the solar neutrino deficit, from Ref. [14]. Superimposed on it are the three curves (a), (b), and (c) which represent sample calculations using our results. All neutrino masses reflect corrections at the two-loop level. Each curve shows the mass squared differences and mixings for the two lightest neutrinos for fixed values of the masses of the other two heavier neutrinos. Curve (a) corresponds to a singlet mass of 100 GeV and a ν_τ mass of ≈ 8.6 eV. Curve (b) corresponds to a singlet mass of 400 GeV and a ν_μ mass of 7 eV. Finally, curve (c) represents a singlet mass of 1 TeV and a ν_μ mass of ≈ 9.8 eV.

ground, while not making a definitive statement on the nature of dark matter, seem to suggest that it may have both hot and cold components, with the former being a neutrino (since it is the only known hot dark matter candidate) with a mass of ≈ 10 eV.

We use both of the above considerations to restrict the rather large parameter space available to us.

In the scenario with one additional singlet, we have two massive and two massless neutrinos at the tree level. The lighter of the two massive neutrinos has a mass generated by the seesaw effect of the heavy singlet. The two massive ones acquire both one-loop and two-loop corrections, which are small (as expected) compared to the corresponding entries of the tree-level input mass matrix. The other two massless neutrinos acquire masses at the two-loop level. Once the full (tree + loop) mass matrix is diagonalized, however, it is fully possible for the loop corrected masses to depart from the "ball park" scale set by the seesaw mass. This underscores the importance of the loop corrections.

The two tree-level masses and all the radiative corrections are expressible in terms of four input mass parameters for the matrix $\mathcal{M}$. For various plausible (fixed) values of m_α, (the singlet mass, signifying the scale of new physics) and the added constraint that one other neutrino have a mass in the 10 eV range, we obtain a one parameter set of curves (see Fig. 2) which denotes the intersection of the "two-loop space" with the MSW solution space. Note that restricting ourselves to the two-dimensional MSW space imposes a third constraint, i.e., that the ν_e mixes predominantly with only one other state. Curve (a) in Fig. 2 corresponds to a singlet mass of 100 GeV and a ν_τ mass of ≈ 8.6 eV. The two-loop masses and mixings of ν_e and ν_μ are then such that they span the MSW space as shown. Curve (b) corresponds to a singlet mass of 400 GeV, and a ν_μ mass of ≈ 7 eV. ν_τ and ν_e then acquire radiative masses and mixings that span the solution space as shown. For $\sin^2 2\theta$ greater than $\approx 3\times 10^{-1}$, ν_τ becomes lighter than ν_e, and MSW oscillations occur between antineutrino rather than neutrino states, and are thus not relevant. We note that (b) passes through the (small-angle, nonadiabatic) MSW region that is compatible with all detectors and also represents a value of m_{ν_μ} (7 eV) that provides a good fit to COBE data in the context of a hot plus cold dark matter scenario. Finally, curve (c) represents a singlet mass of 1 TeV and a ν_μ mass of ≈ 9.8 eV, and terminates where it does because for larger mixing angles the ν_e becomes heavier than the ν_τ. Note that the determination of which flavor the ν_e oscillates to is made by examining the mixing (diagonalizing) matrix of the full (i.e., tree + loop) mass matrix. A (reasonable) assumption built into the results is that ν_e is the lightest state.

We stress that these curves represent a phenomenological exercise more than anything else to demonstrate that our calculations can make connection with experiment when the full parameter space, which is quite large, is constrained by imposing physically and empirically well-motivated restrictions.

We note that the singlet mass values chosen by us (100 GeV, 400 GeV, and 1 TeV) are not in conflict with accelerator [15] or cosmological [16] bounds on these particles.

We also note that the well-known atmospheric neutrino anomaly [12], in conjunction with the solar neutrino and COBE data, seems to point towards highly degenerate neutrinos of mass ~ 2.5 eV each. It is not possible to obtain (at least, not naturally) such a spectrum via the radiative mass generation scheme considered in this paper, without special assumptions. This is the reason why we have based our phenomenological calculations on solar and COBE data only.

IV. CONCLUSIONS

We have explicitly obtained an analytic form for the radiative two-loop masses acquired by doublet neutrinos in models where right-handed singlets are present. We have made an effort to keep our calculation general, and the expression for the mass correction that we obtain may have applications in other models with right-handed neutrinos. We have calculated these masses (for the one singlet case) in the light of experimental data from solar neutrino detectors and from COBE, within the confines of the MSW solution to the solar deficit. In doing so our objective has been to identify a scale of new physics that can lead, in a simple way, to naturally small masses for neutrinos which have physically meaningful values, without requiring drastic changes in the presently known particle spectrum or gauge group structure.

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APPENDIX A: EVALUATION OF Λ_{123456}

In this section we discuss the exact evaluation of the fundamental finite two-loop four-dimensional integral underlying the mechanism. As a first step, though, we consider the more general two-loop Euclidean space integral Λ_{123456} defined by

$$\Lambda_{123456} = \int_{pq} \frac{p\cdot q}{(p^2+m_1^2)(p^2+m_2^2)[(p+q)^2+m_3^2][(p+q)^2+m_4^2](q^2+m_5^2)(q^2+m_6^2)} \tag{A1}$$

which we will evaluate analytically and then specialize to the case we are concerned with. For reasons which we explain below we choose to calculate Eq. (A1) in d dimensions where

$$\int_k = \frac{\mu^{4-d}}{(2\pi)^3}\int d^d k \tag{A2}$$

and μ is an arbitrary mass parameter introduced to ensure the coupling constant remains dimensionless in our d-dimensional manipulations. The subscripts on Λ_{123456} correspond to the masses m_i^2 of the integral and we note that the function has certain obvious symmetries, $\Lambda_{123456} = \Lambda_{213456} = \Lambda_{563412}$, which ought to be preserved in the final expression. The strategy to evaluate Eq. (A1) is to use partial fractions to obtain a sum of two-loop integrals with three propagators and then to substitute for the value of each of these subintegrals, which have been considered by other authors in different contexts before, [17–20]. For instance, if we define

$$J_{ijk} = \int_p\int_q \frac{p\cdot q}{(p^2+m_i^2)(q^2+m_j^2)[(p+q)^2+m_k^2]}, \tag{A3}$$

then Eq. (A1) is built out of a sum of eight such integrals where its only symmetry is $J_{ijk} = J_{jik}$. Rewriting the numerator of Eq. (A3) one finds

$$J_{ijk} = \tfrac{1}{2}[I_iI_j - I_jI_k - I_kI_i - (m_k^2 - m_i^2 - m_j^2)I_{ijk}] \tag{A4}$$

where

$$I_i = \int_p \frac{1}{(p^2+m_i^2)}, \tag{A5}$$

$$I_{ijk} = \int_p\int_q \frac{1}{(p^2+m_i^2)[(p+q)^2+m_j^2](q^2+m_k^2)} \tag{A6}$$

and the latter function is totally symmetric, corresponding to a two-loop vacuum graph (i.e., zero external momentum). The integral I_{ijk} has been considered in [17,18] and a single integral representation of it exists, [19–21]. For our purposes, however, we have chosen to use the elegant formula given in [21] since it is explicitly symmetric in the masses. Although Λ_{123456} is itself ultraviolet finite the subintegrals, Eqs. (A3) and (A4), are divergent and therefore require regularization. In [20,21] dimensional regularization was introduced to control these infinities, which is why we choose to calculate Eq. (A1) in d dimensions, so that I_{ijk} involves double and simple poles in ϵ where $d = 4 - 2\epsilon$. Therefore, in the final result these must cancel for all m_i^2. As a first step, it is trivial to observe that in the partial fraction decomposition of Eq. (A1) the I_iI_j type terms, which are also divergent, formally cancel to leave only the I_{ijk} terms. To proceed we recall the important properties of I_{ijk} which have been discussed in more detail in [21]. In d dimensions the exact value, for arbitrary $(\text{mass})^2$, x, y, and z, is

$$I(x,y,z) = I(2a,0,0) + \Gamma'[F(\tfrac{1}{2}c - y) + F(\tfrac{1}{2}c - z) - F(x - \tfrac{1}{2}c)], \tag{A7}$$

where

$$\begin{aligned} I_{ijk} &= I(m_i^2, m_j^2, m_k^2),\\ \Gamma' &= \frac{(\mu^2)^{4-d}}{(4\pi)^d}\Gamma(2-\tfrac{1}{2}d)\Gamma(1-\tfrac{1}{2}d),\\ a &= \tfrac{1}{2}[x^2+y^2+z^2-2xy-2yz-2zx]^{1/2},\\ c &= x+y+z \end{aligned} \tag{A8}$$

and

$$F(w) = \int_a^w ds \frac{1}{(s^2-a^2)^{(4-d)/2}}. \tag{A9}$$

The result (A7) is valid in the region of (x,y,z) space where $a^2 \geq 0$. For the case when $a^2 < 0$, then the solution is, with $b^2 = -a^2$,

$$I(x,y,z) = -I(2b,0,0)\sin(\tfrac{1}{2}\pi d) + \Gamma'[G(\tfrac{1}{2}c - x) + G(\tfrac{1}{2}c - y) + G(\tfrac{1}{2}c - z)] \tag{A10}$$

where

$$G(w) = \int_0^w ds \frac{1}{(s^2+b^2)^{(4-d)/2}} \tag{A11}$$

and, for example,

$$I(x,0,0) = \frac{\Gamma(2-\frac{1}{2}d)\Gamma(3-d)\Gamma^2(\frac{1}{2}d-1)x^{d-3}}{(4\pi)^d\Gamma(\frac{1}{2}d)(\mu^2)^{d-4}} \tag{A12}$$

which is clearly singular in four dimensions. To obtain the finite part of Λ_{123456} each part of $I(x,y,z)$ needs to be expanded in powers of ϵ to the order 1 term and the poles in ϵ canceled. The nontrivial part of this exercise is the ϵ expansion of the $F(w)$ and $G(w)$ integrals. These have been given in [21] and we record that, to the ϵ-finite term,

$$\begin{aligned}(r\pi)^4 I(x,y,z) = &-\frac{c}{2\epsilon^2} - \frac{1}{\epsilon}\left[\frac{3c}{2} - L_1\right] - \tfrac{1}{2}\{L_2 - 6L_1 + \xi(x,y,z)\\ &+c[7+\zeta(2)] + (y+z-x)\overline{\ln}y\overline{\ln}z\\ &+(z+x-y)\overline{\ln}z\overline{\ln}x + (y+x-z)\overline{\ln}y\overline{\ln}x\},\end{aligned} \tag{A13}$$

where $\zeta(n)$ is the Riemann zeta function, $L_i = x\overline{\ln}^i x + y\overline{\ln}^i y + z\overline{\ln}^i z$, $\overline{\ln}x = \ln(x/\hat{\mu}^2)$, $\hat{\mu}^2 = 4\pi e^{-\gamma}\mu^2$ and γ is Euler's constant, and for $a^2 > 0$,

$$\xi(x,y,z) = 8a[M(\phi_z) + M(\phi_y) - M(-\phi_x)] \tag{A14}$$

where

$$M(t) = -\int_0^t d\phi \ln \sinh\phi \tag{A15}$$

and the angles ϕ_x are defined by

$$\phi_x = \operatorname{arccoth}\left[\frac{\frac{1}{2}c - x}{a}\right]. \tag{A16}$$

For $a^2 < 0$, then

$$\xi(x,y,z) = 8b[L(\theta_x) + L(\theta_y) + L(\theta_z) - \tfrac{1}{2}\pi \ln 2], \tag{A17}$$

where the θ_x angles are given by

$$\theta_x = \arctan\left[\frac{\frac{1}{2}c - x}{b}\right] \tag{A18}$$

and $L(t)$ is the Lobachevskij function,

$$L(t) = -\int_0^t d\theta \ln \cos\theta. \tag{A19}$$

Equation (A17) can also be rewritten as

$$\xi(x,y,z) = 8b[\tilde{L}(\theta_z) + \tilde{L}(\theta_y) - \tilde{L}(-\theta_x)] \tag{A20}$$

where $\tilde{L}(t) = \int_s^{\pi/2} d\theta \ln\cos\theta$ in order to make the obvious analytic continuation across $a^2 = 0$ more apparent. It is worth noting that essentially Eq. (A1) has been reduced to a single simple function, Eq. (A19), whose properties are well known. We have used the following identities in order to write an efficient program to calculate Λ_{123456} for a range of physical mass values. For instance [22],

$$\begin{aligned} &L(t) = -L(-t) \quad \text{for } -\tfrac{1}{2}\pi \le t \le \tfrac{1}{2}\pi, \\ &L(t) = L(\tfrac{1}{2}\pi - t) + (t - \tfrac{1}{4}\pi)\ln 2 - \tfrac{1}{2}L(\tfrac{1}{2} - 2t) \quad \text{for } 0 \le t \le \tfrac{1}{4}\pi, \\ &L(t) = \pm L(\pi \pm t) \mp \pi \ln 2. \end{aligned} \tag{A21}$$

Therefore, when the argument of the Lobachevskij function is known, the identities of Eq. (A21) mean that one need only write a routine to evaluate $L(t)$ numerically in the range $[0, \frac{1}{2}\pi)$. For example, if $0 \le \lambda < 2\pi$ then, for any integer n,

$$L(2\pi n + \lambda) = 2\pi n \ln 2 + L(\lambda) \tag{A22}$$

and so on.

Returning to the partial fraction form of Λ_{123456} with the result for I_{ijk}, the c and L_i terms of the ϵ expansion cancel in the final expression, and we can therefore take the limit back to four dimensions, $\epsilon \to 0$. Consequently, we end up with the following analytic expression:

$$\begin{aligned} \Lambda_{123456} = -\frac{1}{4(4\pi)^4(m_1^2 - m_2^2)(m_3^2 - m_4^2)(m_5^2 - m_6^2)} \Bigg\{ &(m_3^2 - m_1^2 - m_5^2)\left[\xi_{135} - m_1^2 \ln\left(\frac{m_1^2}{m_3^2}\right)\ln\left(\frac{m_1^2}{m_5^2}\right)\right. \\ &\left. - m_3^2 \ln\left(\frac{m_3^2}{m_1^2}\right)\ln\left(\frac{m_3^2}{m_5^2}\right) - m_5^2 \ln\left(\frac{m_5^2}{m_1^2}\right)\ln\left(\frac{m_5^2}{m_3^2}\right)\right] \\ &-(m_3^2 - m_1^2 - m_6^2)\left[\xi_{136} - m_1^2 \ln\left(\frac{m_1^2}{m_3^2}\right)\ln\left(\frac{m_1^2}{m_6^2}\right) - m_3^2 \ln\left(\frac{m_3^2}{m_1^2}\right)\ln\left(\frac{m_3^2}{m_6^2}\right) - m_6^2 \ln\left(\frac{m_6^2}{m_1^2}\right)\ln\left(\frac{m_6^2}{m_3^2}\right)\right] \\ &-(m_4^2 - m_1^2 - m_5^2)\left[\xi_{145} - m_1^2 \ln\left(\frac{m_1^2}{m_4^2}\right)\ln\left(\frac{m_1^2}{m_5^2}\right) - m_4^2 \ln\left(\frac{m_4^2}{m_1^2}\right)\ln\left(\frac{m_4^2}{m_5^2}\right) - m_5^2 \ln\left(\frac{m_5^2}{m_1^2}\right)\ln\left(\frac{m_5^2}{m_4^2}\right)\right] \\ &+(m_4^2 - m_1^2 - m_6^2)\left[\xi_{146} - m_1^2 \ln\left(\frac{m_1^2}{m_4^2}\right)\ln\left(\frac{m_1^2}{m_6^2}\right) - m_4^2 \ln\left(\frac{m_4^2}{m_1^2}\right)\ln\left(\frac{m_4^2}{m_6^2}\right) - m_6^2 \ln\left(\frac{m_6^2}{m_1^2}\right)\ln\left(\frac{m_6^2}{m_4^2}\right)\right] \\ &-(m_3^2 - m_2^2 - m_5^2)\left[\xi_{235} - m_2^2 \ln\left(\frac{m_2^2}{m_3^2}\right)\ln\left(\frac{m_2^2}{m_5^2}\right) - m_3^2 \ln\left(\frac{m_3^2}{m_2^2}\right)\ln\left(\frac{m_3^2}{m_5^2}\right) + m_5^2 \ln\left(\frac{m_5^2}{m_2^2}\right)\ln\left(\frac{m_5^2}{m_3^2}\right)\right] \\ &+(m_3^2 - m_2^2 - m_6^2)\left[\xi_{236} - m_2^2 \ln\left(\frac{m_2^2}{m_3^2}\right)\ln\left(\frac{m_2^2}{m_6^2}\right) - m_3^2 \ln\left(\frac{m_3^2}{m_2^2}\right)\ln\left(\frac{m_3^2}{m_6^2}\right) - m_6^2 \ln\left(\frac{m_6^2}{m_2^2}\right)\ln\left(\frac{m_6^2}{m_3^2}\right)\right] \\ &+(m_4^2 - m_2^2 - m_5^2)\left[\xi_{245} - m_2^2 \ln\left(\frac{m_2^2}{m_4^2}\right)\ln\left(\frac{m_2^2}{m_5^2}\right) - m_4^2 \ln\left(\frac{m_4^2}{m_2^2}\right)\ln\left(\frac{m_4^2}{m_5^2}\right) - m_5^2 \ln\left(\frac{m_5^2}{m_2^2}\right)\ln\left(\frac{m_5^2}{m_4^2}\right)\right. \\ &-(m_4^2 - m_2^2 - m_6^2)\left[\xi_{246} - m_2^2 \ln\left(\frac{m_2^2}{m_4^2}\right)\ln\left(\frac{m_2^2}{m_6^2}\right) - m_4^2 \ln\left(\frac{m_4^2}{m_2^2}\right)\ln\left(\frac{m_4^2}{m_6^2}\right) - m_6^2 \ln\left(\frac{m_6^2}{m_2^2}\right)\ln\left(\frac{m_6^2}{m_4^2}\right)\right]\Bigg\} \end{aligned} \tag{A23}$$

where $\xi_{ijk} = \xi(m_i^2, m_j^2, m_k^2)$ and it is evaluated according to Eqs. (A14) or (A17) depending on whether the particular a^2 is positive or negative. A further check on our manipulations to obtain Eq. (A23) is the absence of the arbitrary mass μ which was required at intermediate steps to have logarithms whose arguments were dimensionless quantities.

Although it may appear that the final result is singular in certain cases through denominator factors like $(m_1^2 - m_2^2)$ when $m_1^2 = m_2^2$, the expression within the square brackets also vanishes. Moreover, if one sets $m_2^2 = m_1^2 + \delta$, where δ is small, and expands in powers of δ then in the limit as $\delta \to 0$ a nonzero nonsingular function of the independent mass remains. Further, there is no difficulty with singularities when one or more masses is zero. To illustrate this point explicitly we consider the integral Λ_{123056} where the zero subscript means the corresponding mass of Eq. (A1) is zero. Its form can readily be deduced from Eq. (A23) by taking the $m_4^2 \to 0$ limit. However, to do this the behavior of $\xi(x,y,z)$ in the $z \to 0$ limit is required since Eq. (A23) has terms like $\ln m_4^2$ which are potentially infinite in the limit we require. It is easy to deduce from the explicit representation, Eq. (A14), that

$$\xi(x,y,z) \sim (x-y)\left[2\mathrm{Li}_2\left(1-\frac{y}{x}\right) + \ln\left(\frac{x}{y}\right)\ln\left(\frac{x}{z}\right)\right] \tag{A24}$$

as $z \to 0$. Thus a little algebra leads to the compact expression

$$\begin{aligned}\Lambda_{123056} = &-\frac{1}{4(4\pi)^4(m_1^2-m_2^2)m_3^2(m_5^2-m_6^2)}\\ &\times[(m_3^2-m_1^2-m_5^2)\xi_{135}-(m_3^2-m_1^2-m_6^2)\xi_{136}\\ &-(m_3^2-m_2^2-m_5^2)\xi_{235}+(m_3^2-m_2^2-m_6^2)\xi_{236}-\rho(m_3^2,m_1^2,m_5^2)+\rho(m_3^2,m_1^2,m_6^2)\\ &+\rho(m_3^2,m_2^2,m_5^2)-\rho(m_3^2,m_2^2,m_6^2)+\lambda(m_1^2,m_5^2)-\lambda(m_1^2,m_6^2)-\lambda(m_2^2,m_5^2)+\lambda(m_2^2,m_6^2)],\end{aligned} \tag{A25}$$

with

$$\rho(x,y,z) = (x-y-z)\left[x\ln\left(\frac{x}{y}\right)\ln\left(\frac{x}{z}\right) + y\ln\left(\frac{y}{x}\right)\ln\left(\frac{y}{z}\right) + z\ln\left(\frac{z}{x}\right)\ln\left(\frac{z}{y}\right)\right], \tag{A26}$$

and

$$\lambda(x,y) = (x+y)\left[2(x-y)\mathrm{Li}_2\left(1-\frac{y}{x}\right) - y\ln\left(\frac{x}{y}\right)\right], \tag{A27}$$

where $\mathrm{Li}_2(t)$ is the dilogarithm function. Its properties have been discussed extensively in [23] but we make use of the following ones heres:

$$\begin{aligned}\mathrm{Li}_2(-t)+\mathrm{Li}_2(-1/t) &= -\zeta(2)-\tfrac{1}{2}\ln^2 t \quad \text{for } t>0\\ \mathrm{Li}_2(t)+\mathrm{Li}_2(1-t) &= \zeta(2)-\ln t\ln(1-t),\end{aligned} \tag{A28}$$

and its integral representation is [23]

$$\mathrm{Li}_2(t) = -\int_0^t \frac{ds}{s}\ln(1-s) \tag{A29}$$

where $\mathrm{Li}_2(1) = \zeta(2) = \pi^2/6$.

Finally, another check on our overall expression Eq. (A23) is the comparison with the earlier result of [5] where only m_3^2 and m_4^2 are nonzero, i.e., Λ_{003400}, which was evaluated by an independent method. We can easily deduce an expression for Λ_{003400} from Eq. (A25) by using the relation (A24) or by returning to the I_{ijk} representation of Eq. (A1) and taking the appropriate limits in that case. Useful for the former approach are the properties of the dilogarithm function [23], while in the latter instance we made use of the Taylor expansion of the I_{ijk} about the zero mass and, in particular,

$$\left.\frac{\partial^2 I(x,y,z)}{\partial y\partial z}\right|_{y=z=0} = \frac{\Gamma^2(\frac{1}{2}d-2)\Gamma(4-\frac{1}{2}d)\Gamma(5-d)x^{d-5}}{(4\pi)^d(\mu^2)^{d-4}\Gamma(\frac{1}{2}d)} \tag{A30}$$

whose ϵ expansion is easy to determine. Consequently, we find

$$\Lambda_{003400} = -\frac{1}{(4\pi)^4(m_3^2 - m_4^2)} \ln\left(\frac{m_3^2}{m_4^2}\right). \tag{A31}$$

This is in total agreement with the explicit calculation of [5] and is a necessary nontrivial check that we have the overall normalization of our integral correct, in terms of signs and factors of 2π.

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