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QCD evolution equations for high energy partons in nuclear matter

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We derive a generalized form of Altarelli-Parisi equations to describe the time evolution of parton distributions in a nuclear medium. In the framework of the leading logarithmic approximation, we obtain a set of coupled integro-differential equations for the parton distribution functions and equations for the virtuality ("age") distribution of partons. In addition to parton branching processes, we take into account fusion and scattering processes that are specific to QCD in medium. A detailed balance between gain and loss terms in the resulting evolution equations correctly accounts for both real and virtual contributions which yields a natural cancellation of infrared divergences.

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I. INTRODUCTION

The future ultrarelativistic heavy ion collider experiments at the BNL Relativistic Heavy Ion Collider (RHIC) and the CERN Large Hadron Collider (LHC) are expected to exhibit new phenomena associated with "QCD in medium," i.e., with the microscopic dynamics of quarks and gluons in the hot, ultradense environment that may be created in the central collision region of these reactions [1,2]. Recently, considerable progress has been made in a better understanding of the space-time structure of parton interactions during the early stage of these reactions [3–5]. The conclusion emerging from different independent investigations [6–9] is that, for RHIC energies and beyond, most of the entropy and transverse energy is presumably produced already during very early times (within the first 2 fm after the nuclear contact) by frequent, mostly inelastic, semihard parton collisions involving typical momentum transfers of only a few GeV.

The underlying notion is that the early stage of nuclear collisions at sufficiently high energies can well be described in terms of the space-time evolution of many internetted parton cascades [10], based on renormalization group improved perturbative QCD [11] and relativistic kinetic theory [12]. This physical picture is motivated by the successful "semihard QCD" description of high energy hadronic interactions [13]. The application to ultrarelativistic nuclear collisions, in particular heavy ion reactions [14–16], assumes that the colliding nuclei may be viewed as two coherent clouds of spacelike partons with small virtualities that materialize into "real" excitations due to primary parton-parton scatterings. This primary parton production is expected to result in a large initial particle and energy density in the central collision region, which increases further by subsequent intense gluon bremsstrahlung, secondary scatterings, and rescatterings.

At some point, however, when the parton density becomes so large that the quanta begin to overlap in phase space, recombination (fusion) processes become relevant and the density must saturate towards its limiting value. It is realized that these semihard processes play the major role for the nuclear dynamics at collider energies. The copiously produced minijets [17] cannot be considered as isolated rare events, but are embedded in complicated multiple cascade-type processes, as has been discussed more recently in a number of works [6,5,18]. At the same time it is found [19,20] that color correlations among the initial partons randomize so rapidly as the beam particles interpenetrate, that the long range color field effectively vanishes on a space-time scale of a small fraction of a fm. Thus the short range character of the interactions implies that perturbative QCD can and must be used and that, for example, the string picture does not apply anymore.

However, one is still far from a complete and detailed picture, as is reflected by the considerable theoretical uncertainty in perturbative QCD predictions for global observables in nucleus-nucleus (AA) collisions at collider energies, such as particle multiplicities and transverse energy production. The inability to extrapolate accurately from of pp ($p\bar{p}$) data to heavy ion AA collisions is due to the current lack of better knowledge about the details of important nuclear and dense medium effects. It is neither surprising nor satisfying that numerical simulations with QCD based Monte Carlo models such as HIJING [21], DTUJET [22], or the parton cascade model (PCM) [5,23] agree very well in describing pp collisions at collider energies, but differ in their predictions, for, e.g., charged particle multiplicities, in heavy ion AA collisions by a factor of 2 or more.

The central question is therefore the following: How is "QCD in medium" modified as compared to "QCD in vacuum"? In trying to gain a more quantitative knowledge about the microscopic parton dynamics in medium, the most urgent questions concern (i) the initial conditions regarding the parton substructure of large nuclei,

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in particular the small x region and the magnitude of nuclear shadowing effects; (ii) the role of color screening and color diffusion; (iii) the impact of the Landau-Pomeranchuk-Migdal effect; and (iv) the space-time dependence of parton interactions with regard to the influence of the characteristic interaction times of parton scatterings and the formation times for gluons emitted in bremsstrahlung processes.

Understanding "QCD in medium" is also one of the most interesting experimental challenges for RHIC and LHC. We hope that this can be achieved by analyzing the characteristic space-time structure of parton interactions during the very early stage in pA and AA collisions, e.g., by measuring the production of particles emerging from these early times, such as dileptons, direct photons, and strange and charmed particles.

Recently it was pointed out by McLerran and Venugopalan [24] that a consistent perturbative calculation of parton structure functions at small values of the Bjorken variable x becomes possible when one considers the limit of a very thick nuclear target. A projectile propagating at high energy through such a target "sees" a very large area density of valence quarks and hence experiences an effective screening of color interactions in the transverse direction. The condition for the applicability of perturbative QCD then is that the screening distance is much shorter than the confinement scale Λ^{-1}. This is satisfied for a sufficiently thick target at sufficiently small x.

In this paper we want to make use of this insight to explore the evolution of a parton cascade inside nuclear matter under conditions where perturbative QCD applies because medium-induced effects, such as color screening and rescattering, provide dynamical cutoffs on a scale short compared to Λ^{-1}. In principle, our approach applies to the propagation of fast partons in any kind of dense medium, be it a thermalized quark-gluon plasma or ground state nuclear matter. The equations derived here can therefore be applied to jet quenching in a QCD plasma [25] as well as to the fragmentation cascade of a quark after deep-inelastic scattering in an infinitely large nucleus [26]. To keep the discussion specific, we will consider the following idealized problem. Beginning with a prescribed initial distribution of fast partons injected into infinitely extended nuclear matter by some highly localized process of space-time extent, $(Q_0^2)^{-1/2} \ll \Lambda^{-1}$, we want to follow the evolution of the parton distribution in laboratory time as it propagates through the medium.

Our plan is to introduce a simplified version of the parton cascade approach of Refs. [4,5], which we propose as a supplementary diagnostic tool for a more transparent analysis of the aforementioned aspects of QCD in medium. We will reduce the complex space-time structure of multiple connected parton cascades to the problem of the diffusion of quarks and gluons in dense nuclear matter, for which we can apply an analytical treatment. Our rationale is to ignore all processes irrelevant in the present context; therefore we will neglect here the quantitatively important effects of color screening, long range color correlations, parton shadowing, etc. Those aspects may be incorporated in a future extension of this work.

Let us describe in some more detail the essence of this work. We attempt to etablish a connection between the semiclassical probabilistic picture of parton evolution in the leading logarithmic approximation (LLA) [27–29] and the time development of parton cascades in six-dimensional phase space within the framework of non-equilibrium kinetic theory [4,5]. To do so, we need to clarify two fundamental issues: First we need to relate the Altarelli-Parisi-Lipatov (APL) evolution equations [30,31], which determine the change of the parton number densities under variation of the variables rapidity y and transverse momentum $k_\perp$, or $x \approx \exp(y)$ and $Q^2 \approx k_\perp^2$, with the Boltzmann equation, which controls the time evolution of the phase-space densities in both momentum and coordinate space. Second, we must relate the experimentally accessible number densities q_i, $\bar{q}_i$, and g (quarks and antiquarks of flavor i, and gluons) with the single-particle phase-space densities F_{q_i}, $F_{\bar{q}_i}$, and F_g, respectively.

The first point, the connection between the APL evolution equations and the Boltzmann equation, has been previously suggested, tentatively by Durand and Putikka [32] and explicitly by Collins and Qiu [33], however, in the formal context of hadron structure which is rather different from our considerations of parton cascades in nuclear matter. Although both these authors succeeded in a rederivation of the APL equations on the basis of a complete probabilistic picture, they stopped short in actually establishing the correspondence between QCD evolution and the space-time development of the parton distributions. Nevertheless, the alternative approach of Refs. [32,33] showed that the APL equations can be derived in a statistical manner similar to the Boltzmann equation, by taking into account both the gain *and* the loss of partons due to successive $1 \to 2$ branchings in (x, Q^2) space. This self-contained detailed balance eliminates the necessity of calculating vertex and wave function renormalization explicitly because the loss terms naturally take over this role. As a consequence, the resulting evolution equations are free of divergences and satisfy the constraints imposed by momentum and quark number conservation automatically.

Concerning the second point, we note that the measured parton number densities $a_N(x, Q^2)$, where $a \equiv q_i, \bar{q}_i, g$, give the probability for finding a quark, antiquark, or gluon, inside a nucleon with fraction $x = k_z/P$ of the longitudinal nucleon momentum P and with virtuality Q^2, or transverse momentum $k_\perp^2$. Here $Q^2 \approx k_\perp^2$ sets the scale of hardness that is identified with the momentum transfer of an interaction of the parton with a weakly interacting probe (e.g., a virtual photon) that measures the nucleon substructure. At present, the parton number densities are experimentally accessible only in a space-time integrated way and therefore must be interpreted as instantaneous distributions of partons inside a nucleon. This interpretation, however, has a justified operational meaning only in a reference frame in which the nucleon moves close to the speed of light, because in such a frame time dilation slows the internal motion of the partons such that the statistical picture can be applied [34]. On the other hand, in statistical many-particle systems the phase-space distribution $F_a(E, \mathbf{k}; \mathbf{r}, t)$ is the probability density for finding a parton of species a in a phase-space element d^3kd^3r at time t. Evidently F_a contains explicit additional information about the space-time structure of the initial state nucleons or nuclei, which is only present in an averaged way in the measured parton number densities.

On the basis of this knowledge, we will here extend the probabilistic approach to the space-time evolution of partons in nuclear matter. To do so, we will first relate the Q^2 evolution to the development with time t, and second, we will include not only the $1 \to 2$ branching processes, but also the reverse $2 \to 1$ fusion processes and in addition $2 \to 2$ scattering processes. The latter two types of processes indirectly also give rise to additional stimulated branchings. Stimulated emission, fusion, and scattering processes are naturally absent in vacuum, but in medium they are indispensible ingredients for obtaining a complete set of transition amplitudes and a self-consistent parton evolution.

The remainder of this paper is organized as follows. In Sec. II we set up the physical scenario of parton cascade evolution inside nuclear matter and introduce the framework of kinetic description including a consistent treatment of off-shell propagation of partons. We will relate the quark and gluon number densities, as measured by the structure functions, with the time-dependent phase-space distributions of off-shell partons and will translate the QCD evolution of the parton number densities into the space-time development of parton cascades. On the basis of this connection, the rates for branching, fusion, and scattering processes of the partons interacting with the medium are derived. In Sec. III we generalize the considerations and obtain a set of coupled equations for the space-time evolution of quarks, antiquarks, and gluons. Finally, in Sec. IV we summarize our results and give some possible future perspectives.

II. DESCRIPTION OF PARTON SHOWERS IN NUCLEAR MATTER

A. Physical picture of parton showers in medium

The physical situation that we have in mind is illustrated in Fig. 1. We consider a parton cascade initiated by a high energy timelike quark or gluon that has been produced at some point of time t_0 inside a heavy nucleus due to an external interaction: for instance, by a collision of a proton with the nucleus, in which case the parton is produced by a scattering with another parton of the incoming proton and is provided with a timelike virtuality $Q^2 \approx p_\perp^2$, where $p_\perp^2$ is determined by the momentum transfer in the scattering; or by deep inelastic scattering, where the parton is struck by a spacelike virtual photon with invariant mass squared $q^2 < 0$ and it acquires a timelike virtuality $Q^2 \approx |q^2|$. In any case, this *primary* parton then propagates through the partonic matter of the nucleus and initiates of a shower of *secondary* partons. The attractive feature of such a scenario is that it provides good control of the initial conditions: the primary parton is produced with a well defined

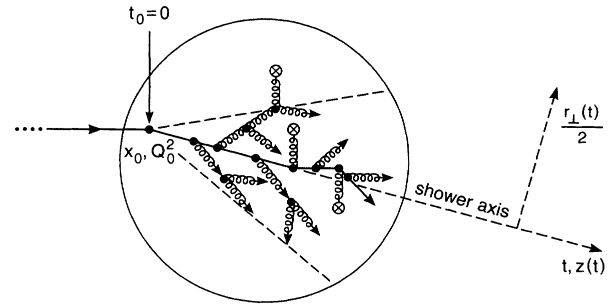

FIG. 1. Visualization of a parton cascade evolving in the quark-gluon matter of a nucleus. A primary parton that has been produced at some point of time t_0 with a timelike virtuality $Q_0 2$ initiates a shower of secondary partons by gluon bremsstrahlung and multiple interactions with the nuclear medium.

four-momentum, determined by the momentum transfer of the triggering interaction with the external particle. This is to be contrasted with a nucleus-nucleus collision, in which case there are many comoving nucleons and the initial state contains a rather complicated mixture of initially produced partons with a rather broad momentum distribution.

To set a definite physical situation we will, from now on, consider a proton-nucleus (pA) collision [35]. In order to apply the parton picture one has to go into a frame where both the projectile proton and the target nucleus are moving very fast, so that both the proton and the nucleons in the nucleus can be resolved into individual partons [36]. The description of a nucleon as an instantaneous distribution of partons at any time requires probing the nucleon over time durations and spatial distances small on the scale of internal motions of the partons. This condition is satisfied in any frame of reference in which the nucleon moves almost with the speed of light, because the time dilation effect slows the internal motions such that the nucleon can be described as a simple quantum-mechanical ensemble of quasireal partons that do not mix with vacuum fluctuations (except for the slowest gluons and sea quarks). It is convenient to choose the *nucleon-nucleon center-of-mass frame* (c.m.$_{NN}$) in which each nucleon has the same value of longitudinal momentum P (see Fig. 2),

$$P_z^{(p)} = +P, \quad P_z^{(A)} = -AP \ , \ \mathbf{P}_\perp^{(p)} = \mathbf{P}_\perp^{(A)} = \mathbf{0} \ , \quad (1)$$

so that

$$\sqrt{s} = \sqrt{4AP^2 + M_N^2(1+A^2)} \quad (2)$$

and

$$\sqrt{s_{NN}} = 2\,P \ , \quad (3)$$

where A is the nuclear mass number, M_N is the nucleon mass, and $P/M_N \gg 1$ is assumed, a requirement which is certainly satisfied at the colliding beam accelerators RHIC and LHC. For example, at RHIC, the maximum

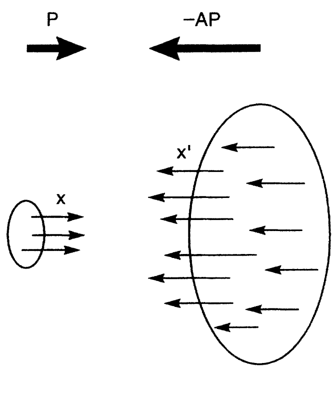

FIG. 2. Illustration of a pA collision in the c.m.$_{NN}$ frame. The incident proton "sees" the nucleus as a Lorentz contracted ensemble of virtual gluons and quarks [5,35]. Similarly, because of the symmetry of the reference frame, the proton itself appears to the nucleus as an incident distribution of individual partons, smeared out along the longitudinal direction. The longitudinal momenta of the partons from the proton are taken as $p_z = xP$ and the nuclear partons have $p'_z = -x'P$.

P is 250 GeV for $p+p$, 125 GeV for $p+{}^{16}$O, and 100 GeV for $p+{}^{197}$Au [37]. At LHC one has generally a factor of 30 larger energy available [38].

We assume that the primary quark or gluon originates from the proton structure function, carrying longitudinal momentum $k_{z0} = x_0 P$, and is provided with an initial timelike virtuality $Q_0^2 > 0$ by a hard scattering with one of the nuclear partons. Hence this primary parton is so to say injected into the nucleus and probes the nuclear environment. It will subsequently produce a number of secondary partons by various interactions with the nuclear background matter: The incident primary parton can either (i) radiate bremsstrahlung gluons, (ii) collide with partons of the nuclear background matter, or (iii) absorb these nuclear partons. The so produced secondary partons will subsequently undergo the same type of interactions. That is, they themselves will lose energy momentum either by exciting partons bound in the nucleus off which they scatter on their way or by radiating gluon bremsstrahlung, or, in the case of gluons, materializing through quark-antiquark pair production. At every new step the number of particles increases and their average energy and longitudinal momentum decrease until the growing density enhances reverse absorption (fusion) processes that may yield a detailed balance or until eventually all the energy momentum of the primary particle can be considered as completely dissipated. We will refer to this event as a *parton shower* or *parton cascade.* We emphasize that we will explicitly distinguish between the *shower partons*, on the one hand, and the *nuclear partons*, on the other hand, which are coherently bound in the wave function of the nucleus. Furthermore, it should be clear from the above selection of processes (i)–(iii) that we consider here only *interactions between the shower partons and the nuclear partons*, and not among the shower partons or the nuclear partons themselves. In other words, we consider here the evolution of a single cascade and therefore only account for interactions of the cascade with the medium but neglect interactions between possible simultaneous cascades (we will briefly consider a self-interacting cascade at the end of Sec. III).

We will describe the longitudinal evolution of the parton shower along the *shower axis* (z axis), which we define parallel to the direction of momentum of the initiating primary parton. It is convenient to parametrize the four-momenta $k \equiv k^\mu = (E, k_z, \mathbf{k}_\perp)$ of the shower partons such that, for the primary parton,

$$k_0 = \left(x_0 P + \frac{Q_0^2}{2x_0 P};\ x_0 P,\ \mathbf{0} \right) , \tag{4}$$

whereas for the jth secondary parton,

$$k_j = \left(x_j P + \frac{Q_j^2 + k_{\perp j}^2}{2x_j P};\ x_j P,\ \mathbf{k}_{\perp j} \right) , \tag{5}$$

where $k_{\perp j}^2 < Q_j^2 \ll Q_0^2 \ll P^2$ is assumed and all rest masses are neglected. It is important to realize that energy and momentum are independent variables since we are dealing with off-shell particles of virtuality Q^2 with a continuous mass spectrum.

The evolution of a many parton system is described by the change of *parton number densities* of quarks q_i and antiquarks $\bar{q}_i$ of flavor i, or gluons g, which are defined as

$$a(x, k_\perp^2, Q^2; t) = \int_0^{Q^2} dQ'^2 \frac{dN_a(t)}{dx dk_\perp^2 dQ'^2} \qquad (a \equiv q_i, \bar{q}_i, g) , \tag{6}$$

where

$$N_a(t) = \int d^3 r \int d^4 k\, F_a(E, \mathbf{k}; t, \mathbf{r}) \equiv \int d^4 k\, f_a(E, \mathbf{k}; t) \tag{7}$$

is the number of partons of type a present at time t with F_a denoting the corresponding phase-space density of off-shell partons in the volume $d^4 k d^3 r$ around $k^\mu = (E, \mathbf{k})$ and $\mathbf{r}$, and f_a in the second line representing the spatially integrated energy-momentum distribution. Note that we introduced an explicit time dependence in the parton number densities (6) and also treat Q^2 and $k_\perp^2$ as independent variables, in contrast to the usual identification $Q^2 \approx k_\perp^2$. Furthermore we stress that virtuality Q^2 and time t are actually correlated variables, as we will show. Therefore the parton densities are in fact functions of Q^2 as well as t, and we have to describe their Q^2 evolution as well as their time development. On the other hand, the experimentally accessible parton densities are related to the functions (6) by

$$a_N(x, Q^2) = \int_{-\infty}^{\infty} dt \int_0^{\infty} dk_\perp^2\, a(x, k_\perp^2, Q^2; t) . \tag{8}$$

Consequently the invariant mass spectrum in Q^2 measures the degree of excitation of the shower particles, which will be related to their "age" in the cascade, whereas the variable $x = k_z/P$ measures the change of the longitudinal momenta of the cascade partons, and the transverse momentum distribution in $k_\perp^2$ reflects the diffusion perpendicular to this axis. According to our chosen geometry, the primary parton has no transverse momentum at all, $k_{\perp 0} = 0$, but its mass is off shell by an initial virtuality Q_0^2. As the parton shower develops in space-time due to branching, fusion, and scattering processes, the distribution in x will shift to smaller values, the distribution in $k_\perp^2$ will broaden, and in the average the virtuality Q^2 of the secondary partons will decrease. How fast such a cascade evolves will depend on the medium properties, which will be reflected in the age distribution of shower particles. The denser the medium the slower the aging process. In Sec. II C we will derive a quantitative formulation of this evolution; however, as a motivation let us already here draw a qualitative picture. Consider for the moment the evolution of a cascade in vacuum, i.e., in the absence of a surrounding medium, so that scattering and fusion processes will not be present. In this case, the cascade evolves solely by successive branchings. In the LLA the evolution of timelike virtualities Q^2 is subject to the ordering condition

$$Q_0^2 \gg Q_1^2 \gg \dots \gg Q_j^2 \gg Q_{j+1}^2 \gg \dots \gg \mu_0^2 \ , \quad (9)$$

where μ_0 sets an invariant mass scale at which the perturbative description of the branching cascade fails. This strongly ordered decrease of virtualities is valid in the kinematic region where $p_{zj}^2 = (x_j P)^2 \gg Q_j^2 > k_{\perp j}^2$, as we already assumed after Eq. (5). It eliminates complicated quantum-mechanical interference effects in successive branchings. In a certain intermediate branching in the cascade $k_{j-1} \to k_j + k_j'$, the four-momentum k_{j-1} is given by (5) and energy-momentum conservation at the vertex uniquely relates the momenta of the daughters k_j and k_j'. In particular, conservation of longitudinal momentum implies

$$x_{j-1} = x_j + x_j' \ , \quad (10)$$

transverse momentum conservation requires

$$\begin{aligned} \mathbf{k}_{\perp j} &= z_j\, \mathbf{k}_{\perp j-1} + \mathbf{p}_{\perp j}, \\ \mathbf{k}_{\perp j}' &= (1 - z_j)\, \mathbf{k}_{\perp j-1} - \mathbf{p}_{\perp j}, \end{aligned} \quad (11)$$

and energy conservation yields

$$p_{\perp j}^2 \approx z_i(1 - z_j)\, Q_{j-1}^2 - (1 - z_j) Q_j^2 - z_j Q_j'^2 \ , \quad (12)$$

where $z_j = x_j / x_{j-1}$ and $p_{\perp j}^2$ is the squared intrinsic transverse momentum generated with respect to the $\mathbf{k}_{j-1}$ direction. For simplicity, we assumed here a symmetric distribution in azimuthal angle. Hence both the longitudinal and (on average) the squared transverse momentum are additive, and the value of $k_{\perp j}^2$ with respect to the shower axis is determined by the virtualities and the ratios of longitudinal momenta of mother and daughters. Each branching generates a $p_\perp$ kick, so that the cascade evolves as a random walk of partons in $k_\perp$ space [13] and at the same time the partons in the cascade become increasingly slower and closer to the mass shell.

In the presence of nuclear matter this simple evolution is modified by scattering and fusion processes. However, the effect of these interactions with the medium can be incorporated in a rather straightforward manner: In between scatterings or fusions, the successive branchings still determine the evolution of the cascade, but instead of evolving undisturbed all the way down to μ_0^2 [cf (7)], at each vertex of interaction with the medium the branching cascade is terminated prematurely and the interacted parton acquires a new virtuality. This sets a new starting point from which the parton continues to branch until the next scattering or fusion, or until it eventually reaches the minimum virtuality μ_0^2. We will show that this modified evolution can be cast in terms of a rejuvenation of the cascade, in the sense that each interaction of shower partons with the medium "resets the clock" by an amount that depends on the hardness of the interaction. We will describe this mechanism by introducing the concept of the age of the cascading partons.

At this point we would like to comment on some peculiar kinematic properties associated with our choice of the c.m.$_{NN}$ frame for the description of parton cascades. We adopt the convention that the momentum along the beam direction of a shower parton is $k_z = xP$, while that of a nuclear parton in the target nucleus is $k_z' = x'P = -|x'|P$ (Fig. 2). Neglecting nuclear shadowing, the initial structure functions of the target nucleus are ($x' < 0$)

$$F_i^{(A)}(x', Q^2) = \frac{A}{\pi R_A^2}\, F_i^{(N)}(|x'|, Q^2) \ , \quad (13)$$

where the index i labels the type of parton, and the functions

$$F_i^{(N)}(x, Q^2) = e_i^2 [x\, q_i(x, Q^2) + x\, \bar{q}_i(x, Q^2)] \quad (14)$$

denote the nucleon structure functions with

$$\sum_i \int_0^1 dx'\, F_i^{(N)}(x', Q^2) = 1 \ . \quad (15)$$

One must keep in mind that all nuclear partons have negative momenta along the z axis and hence their x' values are negative. In the c.m.$_{NN}$ frame we have the interesting feature that, although the distribution of the parton shower particles is, initially (at $t = 0$)

$$F_i(x, Q^2) = x\, \delta(x - x_0)\, \delta(Q^2 - Q_0^2)\, \delta_{i i_0} \quad (x_0 > 0) \ , \quad (16)$$

if we assume a specific primary parton of type i_0 with x_0 and Q_0^2, the distribution $F_i(x, Q^2, t)$ will, with progressing time, eventually take on nonzero values not only in the range $0 < x < x_0$, but also for *negative* values of x. This simply means that, in the chosen frame of reference, the parton cascade initiated by the projectile parton is stopped, and finally swept to the left by the bulk matter of the target.

There are three different types of processes contributing to the momentum degradation of the showering partons. A *branching* of a parton results in two partons that carry smaller momentum fractions $x_1 = zx$ and

$x_2 = (1-z)x$. Note that x_1 and x_2 have the same sign as x, i.e., the direction of propagation of partons in the c.m.$_{NN}$ frame is never reversed by branching processes. A *scattering* between two partons (x_1, Q_1^2) and (x_2, Q_2^2) can occur only between partons propagating in opposite directions, i.e., if $x_1 x_2 < 0$. Neglecting the transverse momentum, the partonic center-of-mass energy squared is

$$\hat{s} = 2\left(|x_1 x_2| - x_1 x_2\right) P^2 + Q_1^2 \frac{|x_1| + |x_2|}{|x_1|} + Q_2^2 \frac{|x_1| + |x_2|}{|x_2|} \approx 4|x_1 x_2| P^2. \quad (17)$$

For partons moving in the *same* direction, the contribution proportional to P^2 vanishes and hence their center-of-mass energy is too small to allow application of the parton picture of perturbative QCD interactions. On the other hand, two partons moving in the same direction in the c.m.$_{NN}$ frame may undergo *fusion*. The invariant mass of the composed parton is given by the same expression as above, except that now $x_1 x_2 > 0$ and hence

$$M^2 = \hat{s} = Q_1^2 \left(1 + \frac{x_2}{x_1}\right) + Q_2^2 \left(1 + \frac{x_1}{x_2}\right) . \quad (18)$$

For partons moving in *opposite* directions, the virtuality of the fused state would be comparable to its energy and momentum in the c.m.$_{NN}$ frame, violating the basic assumptions underlying the probabilistic parton picture $[Q^2 \ll (xP)^2]$. If we neglect interactions among cascading partons, as we will do in the following, we therefore have three types of events: (i) a shower parton (x, Q^2) can branch, with $xx_1, xx_2 > 0$; (ii) a shower parton (x, Q^2) can scatter off a parton from the medium, if $x > 0$; and (iii) a shower parton (x, Q^2) can fuse with a medium parton, if $x < 0$.

The visualization of the multiplication of particles as a parton shower developing inside the nuclear matter of the target nucleus that we sketched in this section is closely related to the picture of the parton cascade evolution that underlies the PCM [5]. However, the PCM is much more ambitious and complex, but considerably less transparent, as it takes into account all kinds of mutual interactions, as well as various nuclear and medium effects. In the present paper we shall instead make a number of simplifications and approximations in order to illuminate the essentials of the space-time structure of parton evolution in nuclear matter.

B. Time evolution versus Q^2 evolution

As stated, our next step is to find the connection between the time evolution of the parton number densities $a(x, k_\perp^2, Q^2; t)$, Eq. (6), and the well known Q^2 evolution of the experimentally observable parton densities in a nucleon $a_N(x, Q^2)$, Eq. (8). Originally, the Q^2 dependence of the structure functions was investigated using the method of operator-product expansion. Later Altarelli and Parisi [30], and independently Lipatov [31], derived a set of integro-differential equations for the Q^2 evolution in the leading logarithmic approximation of QCD. These equations are formulated in momentum space with no reference to the space-time structure of the parton evolution. Altarelli and Parisi determined the Q^2 evolution in momentum space by using "old fashioned" perturbation theory that involved squared S-matrix elements with asymptotic free states integrated over all space and time up to the infinite future. This is reasonable for the evolution *in vacuum*, where a parton cascade simply develops by successive branchings, undisturbed by external fields. However, our objective is to extend this approach to the evolution of such parton showers *in medium*, i.e., inside nuclear matter where, in addition to branchings, the shower particles are likely to undergo multiple interactions with the nuclear partons, so that the integration cannot be extended beyond previous and future interaction points.

To state our goal clearly, we want to derive from elementary principles a kinetic equation for the time-dependent parton number densities (6) that (i) relates the space-time evolution of parton cascades to the Q^2 evolution of virtualities, i.e., the *evolution of off shell particles* in time; (ii) accounts for conventional branching processes (QCD *in vacuum*), as well as for stimulated branchings, fusions, and scatterings in nuclear matter (QCD *in medium*); and (iii) correctly balances the *gain* and *loss* of partons in phase space for each of the branching, fusion, and scattering subprocesses.

Having set up the problem, we first need to ask the question of how the transition amplitudes or probabilities of intermediate parton states change if they are restricted to finite time intervals. Let us start by illustrating with rather general considerations how the time dependence in the LLA parton evolution is connected with the change of the parton virtualities, without making use of the specific form of the interaction matrix elements. We will, for the moment, consider only the branching gluons since they form the dominant component of the parton shower [8]. A detailed derivation, including quarks and antiquarks, as well as the effect of fusion processes and scattering processes that are characteristic for the medium, will be addressed in the following subsections.

The issue of time-dependent interaction amplitudes is most conveniently addressed by using time-dependent perturbation theory in the interaction picture [41], where the transition amplitudes are given by the matrix elements of the time evolution operator. We start with the first order transition amplitude $w^{(0)}(t)$ for a gluon in the initial state $|i\rangle$ to be scattered into the state $|f\rangle$ by an external interaction and assume that the final state becomes real, i.e., gets on mass shell, without further decay [see Fig. 3(a)]:

$$w^{(0)}(t) = (-i) \int_{t_i=0}^{t} dt_0 \, V_0 \, e^{i\omega_{fi} t_0} = \frac{V_0}{\omega_{fi}} \left(1 - e^{i\omega_{fi} t}\right) , \quad (19)$$

where $\omega_{fi} = E_f - E_i$. Note that the invariant matrix element $V_0 \equiv \langle f|\hat{V}|i\rangle$, which causes the production of the state $|f\rangle$ at time t_0, has no explicit dependence on time.

Now we consider the second order process [see Fig. 3(b)], where the triggering interaction V_0 at t_0 produces a certain intermediate virtual state $|a\rangle$ that subsequently decays (branches) into the two-gluon final state $|f\rangle = |bc\rangle$ at time t_1 according to the decay matrix element $V_1(E_a)$. The transition amplitude $w^{(1)}(E_a,t)$ depends therefore on the energy of the intermediate state and is given by:

$$w^{(1)}(E_a,t) = (-i)^2 \int_{t_i=0}^{t} dt_1\, V_1(E_a) e^{i\omega_{fa}t_1} \times \int_{t_i=0}^{t_1} dt_0 V_0 e^{i\omega_{ai}t_0}. \qquad (20)$$

We are interested in the total transition probability, irrespective of the energy of the intermediate virtual state $|a\rangle$. Hence we need to integrate over the continuous spectrum dE_a. In doing so we define $\tau \equiv t_1 - t_0$ to be the lifetime of the virtual state and use τ and t_0 as variables in the integration

$$\begin{aligned} w^{(1)}(t) &= \int dE_a\, w^{(1)}(E_a,t) \\ &= (-i)^2 \int dE_a \int_0^t d\tau\, V_1(E_a)\, e^{i\omega_{fa}\tau} \int_0^{t-\tau} dt_0\, V_0\, e^{i\omega_{fi}t_0} \\ &= i\,\frac{V_0}{\omega_{fi}} \int dE_a\, V_1(E_a) \int_0^t d\tau\, e^{i\omega_{fa}\tau} \left(1 - e^{i\omega_{fi}(t-\tau)}\right) . \end{aligned} \qquad (21)$$

Note that the last term in parentheses does not depend on E_a. Now we divide (21) by (19) to obtain the relative amplitude of the decay of the virtual state and to get information about its average lifetime:

$$R(t) = \frac{w^{(1)}(t)}{w^{(0)}(t)} = i \int dE_a V_1(E_a) \int_0^t d\tau e^{i\omega_{fa}\tau} \left(\frac{1 - e^{i\omega_{fi}(t-\tau)}}{1 - e^{i\omega_{fi}t}}\right). \qquad (22)$$

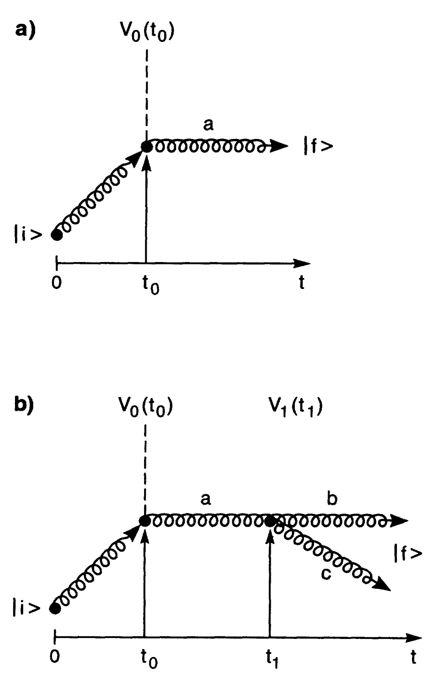

FIG. 3. (a) Diagram of the first order transition amplitude $w^{(0)}(t)$ for an initial gluon state $|i\rangle$ to be converted into a state $|f\rangle$ by means of an external interaction $V_0(t_0)$ with the final gluon being on mass shell. (b) Diagram of the amplitude $w^{(1)}(t)$ for the second order process where an intermediate virtual state $|a\rangle$ is produced that subsequently decays into the two-gluon final state $|f\rangle = |bc\rangle$ at time t_1 according to the decay matrix element $V_1(E_a)$.

In the limit $t \to \infty$ we have $E_f \to E_i$ or $\omega_{fi} \to 0$, and hence, for $\tau \ll t$ (i.e., if the virtual state lives a short time compared to the overall observation time),

$$R(t) \underset{t\to\infty}{\longrightarrow} R = i \int dE_a\, V_1(E_a) \int_0^\infty d\tau\, e^{i\omega_{fa}\tau} , \qquad (23)$$

because $\exp(i\omega_{fi}\tau) \to 1$ in this limit. In order to analyze the τ dependence let us define the function $\rho(\tau)$ through

$$R = i \int_0^\infty d\tau\, \rho(\tau) , \qquad (24)$$

so that

$$\rho(\tau) = \int d\varepsilon\, V_1(E_f + \varepsilon)\, e^{-i\varepsilon\tau} , \qquad (25)$$

where $\varepsilon = -\omega_{fa} = E_a - E_f$ characterizes the magnitude of virtuality of the intermediate state. Let us assume that V_2 limits the virtuality to a range $|\varepsilon| \lesssim \varepsilon_0$. In fact, this case corresponds to the ordering of virtualities (9) in the LLA, which implies, for a timelike parton cascade, strongly decreasing virtualities of subsequent intermediate states, where for a given intermediate state an upper limit of virtuality is constrained by the kinematics (12) at the vertex of production of this virtual state. For the purpose of lucidity, we make a simple Lorentzian ansatz

$$V_1(E_a) = \frac{\mathcal{V}}{1 + \varepsilon^2/\varepsilon_0^2} , \qquad (26)$$

with constant $\mathcal{V}$. Then we get, from (24),

$$\rho(\tau) \;=\; \mathcal{V} \int_{-\infty}^{\infty} d\varepsilon \; \frac{e^{-i\varepsilon\tau}}{1 \,+\, \varepsilon^2/\varepsilon_0^2} \;=\; \pi \, \mathcal{V} \, \varepsilon_0 \, e^{-\tau\,\varepsilon_0} \; . \quad (27)$$

As a result the probability

$$|\rho(\tau)|^2 \;=\; \pi^2 \, \varepsilon_0^2 \, |\mathcal{V}|^2 \, e^{-2\,\tau\,\varepsilon_0} \quad (28)$$

expresses that the *average* lifetime τ of the virtual state $|a\rangle$ with virtuality ε in Fig. 3(b) is determined by the typical virtuality ε_0. In general it is impossible to assign a lifetime to a particular virtuality ε because these amplitudes interfere coherently in the Fourier integral (25).

If we parametrize the four-momenta $k^\mu = (E; k_z, \mathbf{k}_\perp)$ of the particles a, b, and c in Fig. 3(b) as in (4) and (5),

$$k_a = \left(x_a P + \frac{Q_a^2}{2x_a P};\; x_a P,\; \mathbf{0} \right) ,$$
$$k_b = \left(x_b P + \frac{Q_b^2 + p_\perp^2}{2x_b P};\; x_b P,\; \mathbf{p}_\perp \right) ,$$
$$k_c = \left(x_c P + \frac{Q_c^2 + p_\perp^2}{2x_c P};\; x_b P,\; -\mathbf{p}_\perp \right) , \quad (29)$$

with $x_a = x_b + x_c$ and P denoting the longitudinal momentum defined in (1) and $p_\perp^2$ the relative transverse momentum squared generated in the branching, then

$$\varepsilon_0 \;\equiv\; \varepsilon_a \;=\; \frac{Q_a^2}{2|x_a|P} \quad (30)$$

sets the upper limit for the virtualities of the daughter partons with Q_b^2 and Q_c^2 determined as in Eq. (12). Hence Eq. (28) implies that the transition probability $|\rho(\tau)|^2$ is appreciable only for those final states $|f\rangle = |bc\rangle$ that satisfy

$$\tau \;\lesssim\; \tau_a \;=\; \frac{1}{2\,\varepsilon_0} \;=\; \frac{|x_a|P}{Q_a^2} \; . \quad (31)$$

C. Derivation of the time dependence of fragmenting parton cascades

Let us extend these considerations to the evolution of a parton cascade in the LLA with many intermediate virtual states. Recall the situation that we sketched in Sec. II A in which the cascade is initiated by a primary parton, say a gluon, with virtuality Q_0^2 at time t_0. By successive gluon emissions (branchings) the cascade evolves with strongly ordered decreasing virtualities $Q_i^2 \gg Q_{i+1}^2$ from Q_0^2 down to μ_0^2 [cf. Eq. (9)], so that interference terms can be neglected in this approximation. This is illustrated in Fig. 4(a). Consequently, each branching occurs in the average at a certain time $t_j(Q_j^2)$, with $t_j \ll t_{j+1}$, and the evolution stops at $t_f(\mu_0^2)$. In analogy to (31) the average lifetime of the jth gluon g_j is given by

$$\tau_j \;=\; \frac{|x_j|P}{Q^2} \; . \quad (32)$$

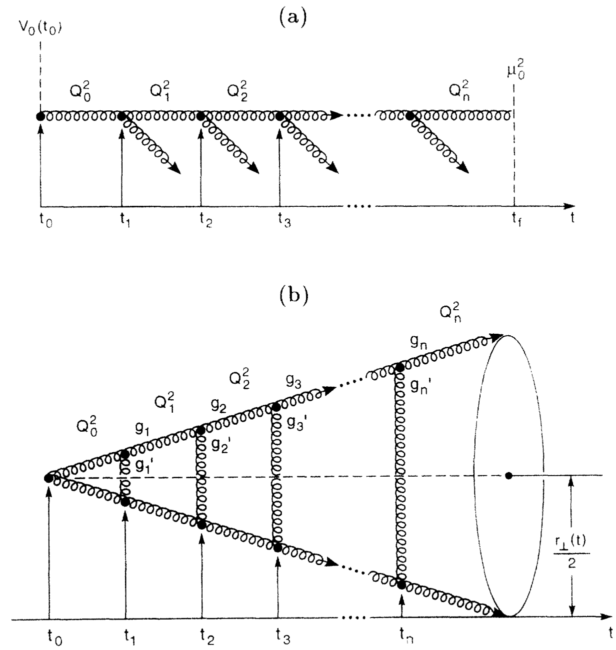

FIG. 4. (a) Evolution of a gluon cascade in terms of successive branchings the LLA with many intermediate virtual states. The cascade is initiated by a primary gluon with virtuality Q_0^2 at time t_0 and develops downwards with strongly ordered decreasing virtualities $Q_i^2 \gg Q_{i+1}^2$ from Q_0^2 to μ_0^2, corresponding to the time evolution from t_0 to t_f with $t_i \ll t_{i+1}$ and $t_f(\mu_0^2)$. (b) Corresponding longitudinal and lateral spread of the multigluon wave function as time progresses. The diffusion in $r_\perp$ direction results in a linear (with time) growing cross section of the cascade when evolving in a medium, as inside a nucleus.

Over this time the two gluons g_j and g_j' separate by

$$\Delta r_{\perp j} \;=\; v_{\perp j}\, \tau_j \;=\; \frac{k_{\perp j}}{Q_j^2} \lesssim \frac{1}{Q_j} \; , \quad (33)$$

where $v_{\perp j} = k_{\perp j}/E_j \approx k_{\perp j}/(x_j P)$ and $k_{\perp j} \equiv |\mathbf{k}_{\perp 1}|$. These expressions are in agreement with the uncertainty principle. Thus, after n gluon emissions the typical passed time is $t = \sum_{j=1}^{n} \tau_j \simeq |x_n|P/Q_n^2$ (where $Q_n^2 \ll Q_{n-1}^2$), and consequently the mean squared transverse spread is approximately $r_\perp^2(t) = [\Delta r_\perp/2]^2 \approx 2t/P$. From this simple consideration we can conclude that such a parton cascade, when embedded inside nuclear matter, develops a transverse cross section $\pi r_\perp^2(t)$ that grows linearly with time, as illustrated in Fig. 4(b). That is, with progressing time the interaction with the surrounding medium becomes increasingly probable, so that scattering and fusion processes need to be taken into account. In this case, the sequence of successive branchings is terminated naturally when the nth gluon collides or fuses with another parton of the medium. The gluon is then reexcited by the momentum transfer of the interaction and it can subsequently start a new cascade sequence of branchings, and the game repeats.

We proceed now by deriving a quantitative formulation of this picture. At first we will consider the time evolution in Q^2 and x; the lateral shower development with respect to the transverse momentum $k_\perp^2$ will be discussed subsequently. We will study the evolution of the gluon longitudinal momentum distribution

$$g(x,t) \;=\; \int dk_\perp^2 \; g(x,k_\perp^2,t) \;\; , \tag{34}$$

that is, the zeroth moment in $k_\perp^2$ of the full gluon distribution $g(x,k_\perp^2,t)$, defined by Eq. (6). Now let us introduce the integrated *timelike Sudakov factor*, or nonbranching probability,

$$T(t) \;\equiv\; T(Q_0^2,\mu_0^2;t_0,t) \;\; , \tag{35}$$

which is the probability that *no* branching occurs whatsoever between Q_0^2 and μ_0^2 within the time span between t_0 and t_0+t. Note that the Sudakov factor represents a rectangle in the (t,Q^2) plane, so there are in principle infinitely many possibilities to reach the point (t_0,Q_0^2) to the point (t,μ_0^2). However, it is not clear how to incorporate this property in a probabilistic description [42]; therefore we assume that the propagator property $T(Q_0^2,\mu_0^2;t_0,t) = T(Q_0^2,Q_1^2;t_0,t_1)T(Q_1^2,\mu_0^2;t_1,t)$, corresponding to a "classical path," holds on the average. This propagator property, which we will denote in shorthand as $T(t)=T(t_1)T(t-t_1)$, will be used in the following. We will set the clock at $t_0=0$, so that the time dependence resides solely in t. In a diagrammatic analysis of the branching cascade, the Sudakov factor arises from loop diagrams which restore unitarity [33].

The probability that a branching of a parton actually does occur between Q^2 and Q^2+dQ^2 and between t and $t+dt$ is given by

$$\Psi(z,Q^2;t)\,dz\,dQ^2\,dt \;=\; \psi_Q(z,Q^2)dz\,dQ^2\;\psi_t(t)\,dt. \tag{36}$$

Here the distribution in Q^2,

$$\psi_Q(z,Q^2)\,dz\,dQ^2 \;=\; \frac{\alpha_s(Q^2)}{2\pi Q^2}\,dQ^2\;\gamma_{a\to bc}(z)\,dz, \tag{37}$$

is the usual Altarelli-Parisi branching kernel [30] in which $\gamma_{a\to bc}(z)$ is the longitudinal momentum distribution of the two daughter partons b and c, characteristic for the type of branching,

$$\begin{aligned}\gamma_{q\to qg}(z) &= \frac{4}{3}\left(\frac{1+z^2}{1-z}\right) ,\\ \gamma_{q\to gq}(z) &= \frac{4}{3}\left(\frac{1+(1-z)^2}{z}\right) ,\\ \gamma_{g\to gg}(z) &= 6\left(\frac{z}{1-z}+\frac{1-z}{z}+z(1-z)\right) ,\\ \gamma_{g\to q\bar{q}}(z) &= \frac{1}{2}\left[z^2+(1-z)^2\right] ,\end{aligned} \tag{38}$$

and the running QCD coupling strength in one-loop order is, as usual,

$$\alpha_s(Q^2) \;=\; \frac{12\pi}{(33-2f)\ln(Q^2/\Lambda^2)} \;\; , \tag{39}$$

where Λ is the QCD renormalization scale and f is the number of quark flavors that can be probed at scale Q^2.

The time of branching is distributed according to a normalized distribution

$$\psi_t(t)\,dt \;=\; \frac{Q^2}{|x|P}\,f\left(\frac{Q^2}{|x|P}\,t\right)\,dt \;\; , \tag{40}$$

where $Q^2/(|x|P)$ represents the lifetime of the gluon due to its virtuality in the laboratory frame and the normalization is such that $\int_0^\infty d\xi f(\xi) = 1$. For example, for exponential decay, we have

$$f\left(\frac{Q^2}{|x|P}\,t\right) \;=\; \exp\left(-\frac{Q^2}{|x|P}\,t\right) \;\; . \tag{41}$$

We will now derive an evolution equation for the gluon number density that correlates the virtualities of the partons in the cascade with the times of branching by calculating the Sudakov factor. In order not to burden the discussion unnecessarily in this section, we restrict the cascade to containing gluons only and we solely consider interactions with gluons in the nuclear medium. We therefore have to evolve a single function, the gluon distribution $g(x,t)$. To do so, we expand the evolution of $g(x,t)$ into an infinite sum of contributions,

$$g(x,t) \;=\; \sum_{n=0}^{\infty} g^{(n)}(x,t) \;\; , \tag{42}$$

represented by Feynman diagrams for n successive branchings at Q_n^2 and t_n, as depicted in Fig. 5. The *initial gluon distribution* at Q_0^2 and $t_0=0$ is denoted as $g_0(x)$.

(i) If no branching occurs between Q_0^2 and μ_0^2 during a time t, the only change is that the probability of finding gluons that have not branched at all decreases; this is expressed by the Sudakov factor (34), such that (cf. Fig. 5)

$$g^{(0)}(x,t) \;=\; T(t)\;g_0(x) \;\; . \tag{43}$$

(ii) The probability that a single branching occurs somewhere between Q_0^2 and μ_0^2, within the time interval t, is given by the probability that no branching occurs between Q_0^2 and Q_1^2 up to t_1 [denoted as $T(t_1) = T(Q_0^2,Q_1^2;0,t_1)$], times the probability for a branching at Q_1^2 and t_1 [i.e., $\Psi(z_1,Q_1^2;t_1)$] convoluted with the gluon distribution g_0, times the probability that no further branching occurs between Q_1^2 and μ_0^2, in the time interval between t_1 and t [denoted as $T(t-t_1) = T(Q_1^2,\mu_0^2;t_1,t)$]. This has to be integrated over all possible intermediate Q_1^2 and t_1 (cf. Fig. 5). Accordingly we have

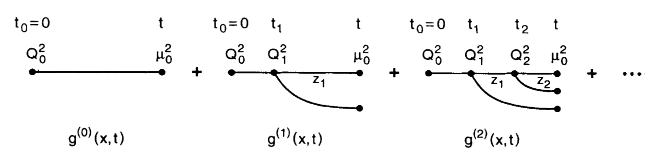

FIG. 5. Graphic representation of the time evolution of the gluon distribution in terms of an infinite sum of contributions involving n successive branchings $g(x,t) = \sum_n g^{(n)}(x,t)$.

$$g^{(1)}(x,t) = \int_0^t dt_1 \int_{\mu_0^2}^{Q_0^2} dQ_1^2\, T(t_1) \int_0^1 dz_1\, \Psi(z_1,Q_1^2;t_1)\, T(t-t_1)\, \frac{1}{z_1}\, g_0\left(\frac{x}{z_1}\right)$$
$$= T(t) \int_{\mu_0^2}^{Q_0^2} dQ_1^2\, \frac{\alpha_s(Q_1^2)}{2\pi Q_1^2} \int_0^1 \frac{dz_1}{z_1}\, \gamma_{g\to gg}(z_1)\, g_0\left(\frac{x}{z_1}\right) \frac{z_1 Q_1^2}{|x|P} \int_0^t dt_1\, f\left(\frac{z_1 Q_1^2}{|x|P}\, t_1\right) , \tag{44}$$

where $z_1 = x_1/x$. We observe that the effect of the time integral is to limit the range of Q_1^2 values which contribute to $g^{(1)}$. For large values of Q_1^2, the time integral yields unity (all branchings have occurred), whereas for Q_1^2 small, the time integral tends to zero. In order to make progress, we simplify the expression by replacing the integrated decay rate by a step function:

$$g^{(1)}(x,t) \;=\; T(t) \int_{\mu_0^2}^{Q_0^2} dQ_1^2\, \frac{\alpha_s(Q_1^2)}{2\pi Q_1^2} \int_0^1 \frac{dz_1}{z_1}\, \gamma_{g\to gg}(z_1)\, g_0\left(\frac{x}{z_1}\right)\, \theta\left(\frac{z_1 Q_1^2}{|x|P}t \,-\, 1\right) . \tag{45}$$

This is certainly a crude approximation, but it correctly embodies the uncertainty relation between time and virtuality, that is, $t \ge \tau_1 = \gamma_1/Q_1$, where the Lorentz factor is $\gamma_1 = |x|P/Q_1$. Now we differentiate and perform the integration over the virtuality Q_1^2:

$$t\,\frac{\partial}{\partial t}\, g^{(1)}(x,t) = t\,\frac{\partial T(t)}{\partial t}\, \frac{g^{(1)}(x,t)}{T(t)} + T(t) \int_{\mu_0^2}^{Q_0^2} dQ_1^2 \frac{\alpha_s(Q_1^2)}{2\pi} \int_0^1 dz_1\, \gamma_{g\to gg}(z_1)\, g_0\left(\frac{x}{z_1}\right) \frac{t}{|x|P}\, \delta\left(\frac{z_1 Q_1^2}{|x|P}t - 1\right)$$
$$= \left(t\,\frac{\partial}{\partial t}\ln T(t)\right) g^{(1)}(x,t) + \int_0^1 \frac{dz_1}{z_1}\, \frac{\alpha_s\left(\frac{|x|P}{z_1 t}\right)}{2\pi}\, \theta\left(\frac{|x|P}{z_1 t} - \mu_0^2\right) \gamma_{g\to gg}(z_1)\; g^{(0)}\left(\frac{x}{z_1}, t\right) , \tag{46}$$

where in the last step we used Eq. (43) to absorb the Sudakov factor in front of the second term. The time scale of the branching is set by $t = |x|P/(z_1 Q_1^2)$, where $x = x_1 = z_1 x_0$ and x_0 denotes the initial value corresponding to Q_0^2 at t_0. We see that the first order (single-branching) contribution $g^{(1)}$ is determined by the zeroth order (no-branching) contribution $g^{(0)}$.

(iii) The same arguments hold when we calculate the diagram containing n branchings in the LLA. One obtains $g^{(n)}$ as an integral over the branching of $g^{(n-1)}$, with the result

$$t\,\frac{\partial}{\partial t}\, g^{(n)}(x,t) = \left(t\,\frac{\partial}{\partial t}\ln T(t)\right) g^{(n)}(x,t) + \int_0^1 \frac{dz}{z}\, \frac{\alpha_s\left(\frac{|x|P}{zt}\right)}{2\pi}\, \theta\left(\frac{|x|P}{zt} - \mu_0^2\right) \times \gamma_{g\to gg}(z) g^{(n-1)}\left(\frac{x}{z}, t\right) . \tag{47}$$

Evidently, the time scale of the whole branching chain is given by $t = \sum_{j=1}^n t_j = \sum_{j=1}^n |x_j|P/Q_j^2 \simeq x_0 P/(z_1 z_2 \cdots z_n Q_n^2)$ with $x = x_n = z_1 z_2 \cdots z_n x_0$ and initial x_0 at the beginning of the branching chain at Q_0^2 and t_0. Thus the nth branching that follows $n-1$ strongly ordered preceding branchings sets the time scale because $Q_n^2 \ll Q_{n-1}^2 \ll \cdots \ll Q_0^2$ [cf. Eq. (9)]. It is important to realize that here $g^{(n)}$ is completely determined by the cumulative effect of $n-1$ preceding branchings contained in $g^{(n-1)}$. This preserves the "locality property" inherent to the probabilistic parton cascade picture of the LLA, in which a branching only depends on the immediate ancestor.

(iv) According to Eq. (42), the *total* gluon distribution $g(x,t)$ at time t can now be obtained by summing all contributions $g^{(n)}(x,t)$, yielding the differential equation

$$t\,\frac{\partial}{\partial t}\, g(x,t) = \left(t\,\frac{\partial}{\partial t}\ln T(t)\right) g(x,t) + \int_0^1 \frac{dz}{z}\, \frac{\alpha_s\left(\frac{|x|P}{zt}\right)}{2\pi} \theta\left(\frac{|x|P}{zt} - \mu_0^2\right) \times \gamma_{g\to gg}(z) g\left(\frac{x}{z}, t\right) . \tag{48}$$

This equation has the same form as the APL equation for the gluon distribution, except that Q^2 is everywhere replaced by the variable $\frac{|x|P}{zt}$ and, since that variable depends on the ratio $z = x_1/x$, the coupling $\alpha_s(Q^2)$ cannot be taken outside of the z integral.

(v) Next we need to calculate the unitary restoring Sudakov factor $T(t)$. It describes the loss of probability for having no gluon branchings at all up to time t. Thus we have to accumulate the probability for having no branching, one branching, two branchings, etc., and then determine $T(t)$ from the condition that the total probability remains unity at all times:

$$1 \;=\; P(t) \;=\; \sum_{n=0}^{\infty} w^{(n)}(t) \;. \tag{49}$$

Here $w^{(n)}(t)$ denotes the probability for having n branchings up to time t. The probability for no branching at all is evidently

$$w^{(0)}(t) \;=\; \frac{\int_0^1 dx\, g^{(0)}(x,t)}{\int_0^1 dx\, g_0(x)} \;=\; T(t) \;, \tag{50}$$

where g_0 in the denominator is the initial gluon distribution at $t_0 = 0$ and Q_0^2. The probability for one single branching is

$$w^{(1)}(t) = \frac{1}{2}\,\frac{\int_0^1 dx\, g^{(1)}(x,t)}{\int_0^1 dx\, g_0(x)} , \qquad (51)$$

where the factor 1/2 in front arises because $g^{(1)}(x,t)$ counts every branching twice, since it counts the number of gluons (two per branching). Using (47), we find, for the integral in the numerator,

$$\begin{aligned}\int_0^1 dx\, g^{(1)}(x,t) &= T(t)\int_{\mu_0^2}^{Q_0^2} dQ^2\,\frac{\alpha_s(Q^2)}{2\pi Q^2}\int_0^1 \frac{dz}{z}\,\gamma_{g\to gg}(z)\int_0^1 dx\, g_0\left(\frac{x}{z}\right)\,\theta\left(\frac{zQ^2}{|x|P}t-1\right)\\ &= T(t)\int_{\mu_0^2}^{Q_0^2} dQ^2\,\frac{\alpha_s(Q^2)}{2\pi Q^2}\int_0^1 dz\,\gamma_{g\to gg}(z)\int_0^1 dy\, g_0(y)\,\theta\left(\frac{Q^2}{yP}t-1\right) , \end{aligned} \qquad (52)$$

where we have set $y = x/z$. Now we again differentiate with respect to t and carry out the Q^2 integration:

$$\begin{aligned} t\frac{\partial}{\partial t}\left(\int_0^1 dx\, g^{(1)}(x,t)\right) &= \left(t\frac{\partial}{\partial t}\ln T(t)\right)\int_0^1 dx\, g^{(1)}(x,t)\\ &+ \int_0^1 dz\gamma_{g\to gg}(z)\int_0^1 dy\,\frac{\alpha_s\left(\frac{yP}{t}\right)}{2\pi}\,\theta\left(\frac{yP}{t}-\mu_0^2\right)\, g_0(y)\, T(t) . \end{aligned} \qquad (53)$$

Hence, using (43) and (50) and renaming the integration variable, we find

$$t\frac{\partial}{\partial t}w^{(1)}(t) = \left(t\frac{\partial}{\partial t}\ln T(t)\right) w^{(1)}(t) + \frac{1}{2}\int_0^1 dz\,\gamma_{g\to gg}(z)\,\frac{\int_0^1 dx\,\frac{\alpha_s\left(\frac{|x|P}{t}\right)}{2\pi}\,\theta\left(\frac{|x|P}{t}-\mu_0^2\right)\, g^{(0)}(x,t)}{\int_0^1 dx\, g^{(0)}(x,t)\, w^{(0)}(t)} . \qquad (54)$$

We can generalize this result easily to the case of n branchings:

$$t\frac{\partial}{\partial t}w^{(n)}(t) = \left(t\frac{\partial}{\partial t}\ln T(t)\right) w^{(n)}(t) + \frac{1}{2}\int_0^1 dz\,\gamma_{g\to gg}(z)\,\frac{\int_0^1 dx\,\frac{\alpha_s\left(\frac{|x|P}{t}\right)}{2\pi}\,\theta\left(\frac{|x|P}{t}-\mu_0^2\right)\, g^{(n-1)}(x,t)}{\int_0^1 dx g^{(n-1)}(x,t)}\, w^{(n-1)}(t), \qquad (55)$$

where the generalization of (51) is

$$w^{(n)}(t) = \frac{1}{2^n}\,\frac{\int_0^1 dx\, g^{(n)}(x,t)}{\int_0^1 dx\, g_0(x)} . \qquad (56)$$

This equation can be summed over all n, assuming that the *average* running coupling

$$\overline{\alpha_s}\left(t;\mu_0^2\right) \equiv \frac{\int_0^1 dx\,\alpha_s\left(\frac{|x|P}{t}\right)\,\theta\left(\frac{|x|P}{t}-\mu_0^2\right)\, g^{(n-1)}(x,t)}{\int_0^1 dx\, g^{(n-1)}(x,t)} \qquad (57)$$

does not depend on the number of branchings. Since $\alpha_s(Q^2)$ is a slowly varying function of Q^2, this should be a reasonable approximation. Then we have, with Eq. (49),

$$\begin{aligned} 0 = t\frac{\partial}{\partial t}P(t) &= \left(t\frac{\partial}{\partial t}\ln T(t)\right)P(t)\\ &+\frac{1}{2}\int_0^1 dz\gamma_{g\to gg}(z)\frac{\overline{\alpha_s}\left(t;\mu_0^2\right)}{2\pi}P(t) \end{aligned} \qquad (58)$$

or

$$t\frac{\partial}{\partial t}\ln T(t) = -\frac{1}{2}\int_0^1 dz\,\gamma_{g\to gg}(z)\,\frac{\overline{\alpha_s}\left(t;\mu_0^2\right)}{2\pi} . \qquad (59)$$

(vi) This result is now reinserted into the equation (48) for $g(x,t)$, using the x averaged coupling (57) also in the second term on the right hand side, which is necessary to ensure momentum conservation [43]. This yields the final formula

$$\begin{aligned} t\frac{\partial}{\partial t}g(x,t) &= -\frac{1}{2}\int_0^1 dz\gamma_{g\to gg}(z)\frac{\overline{\alpha_s}^g(t)}{2\pi}g(x,t)\\ &+\int_0^1\frac{dz}{z}\gamma_{g\to gg}(z)\frac{\overline{\alpha_s}^g(t)}{2\pi}g\left(\frac{x}{z},t\right), \end{aligned} \qquad (60)$$

where now, instead of (57), the x averaged running coupling strength of gluons is given by

$$\overline{\alpha_s}^g(t) \equiv \frac{\int_0^1 dx\, \alpha_s\left(\frac{|x|P}{t}\right)\, \theta\left(\frac{|x|P}{t} - \mu_0^2\right)\, g(x,t)}{\int_0^1 dx\, g(x,t)} . \tag{61}$$

We emphasize three important properties of the evolution equation (60). First, the probabilistic method used here to derive the evolution of the gluon distribution automatically yields both the contributions of loss and gain of gluons that modify $g(x,t)$ [the first, and second terms, respectively, in (60)]. Second, we note that for asymptotic times $t \to \infty$, the evolution equation recovers the time independent APL equation for the gluon number density. In this limit all the partons have evolved down to virtuality μ_0^2, so that the total gluon number density depends only on the scale μ_0^2. Third, the apparent singularity at $t = 0$ in the argument of α_s in (60) and (61) causes no problem because $t = 0$ never occurs since the earliest point of time is associated with the very first branching of the initial gluons with virtuality Q_0^2, which can occur only after the finite time $t = |x|P/Q_0^2 > 0$.

(vii) For cascading quarks and antiquarks, allowing only for the process $q \to qg$ (the full equations will be derived in Sec. III), we have a similar equation, except for the lack of the factor 1/2 in the loss term:

$$t\frac{\partial}{\partial t}q(x,t) = -\int_0^1 dz\gamma_{q\to qg}(z)\frac{\overline{\alpha_s}^q(t)}{2\pi}q(x,t) + \int_0^1 \frac{dz}{z}\gamma_{q\to qg}(z)\frac{\overline{\alpha_s}^q(t)}{2\pi}q\left(\frac{x}{z},t\right) , \tag{62}$$

where in this case the average coupling is determined by the quark density

$$\overline{\alpha_s}^q(t) \equiv \frac{\int_0^1 dx\, \alpha_s\left(\frac{|x|P}{t}\right)\, \theta\left(\frac{|x|P}{t} - \mu_0^2\right)\, q(x,t)}{\int_0^1 dx\, q(x,t)} . \tag{63}$$

Note that not only is the branching function $\gamma_{q\to qg}$ different from the gluon branching case, but also $\overline{\alpha_s}$ depends implicitly on the quark number density instead of the gluon distribution. The conservation of quark number is derived by integrating the equation over x:

$$t\frac{\partial}{\partial t}\int_0^1 dxq(x,t) = -\int_0^1 dz\gamma_{q\to qg}(z)\frac{\overline{\alpha_s}(t)}{2\pi}\int_0^1 dxq(x,t) + \int_0^1\frac{dz}{z}\int_0^1 dx\gamma_{q\to qg}(z)\frac{\overline{\alpha_s}(t)}{2\pi}q\left(\frac{x}{z},t\right) . \tag{64}$$

Introducing the new integration variable $y = x/z$ in the second term, one immediately finds that it equals the first term, but with positive sign, so that

$$t\frac{\partial}{\partial t}\int_0^1 dx\, q(x,t) = 0 . \tag{65}$$

D. Scattering and fusion processes

As pointed out in Sec. II A, scattering and fusion processes rejuvenate the fragmentation cascade. Let us first consider elastic scatterings between two partons. If a parton (gluon) in an evolving cascade scatters on its way with some other parton from the nuclear medium involving a transverse momentum exchange $p_\perp^2$, then its maximal virtuality is reset to $Q^2 = p_\perp^2$. Since we have related the virtuality Q^2 to the evolution of laboratory time t via the relation

$$t = \sum_{j=1}^{n} \frac{|x_j|P}{Q_j^2} \simeq \frac{|x|P}{Q^2} , \tag{66}$$

with

$$x = x_n = z_1 z_2 \cdots z_n x_0, \qquad Q^2 = Q_n^2 \ll Q_{n-1}^2 \ll \cdots \ll Q_0^2, \tag{67}$$

we therefore need to "reset the clock" for a parton after each scattering process, if the momentum transfer $p_\perp^2$ is larger than the present virtuality of the parton. This is illustrated in Fig. 6. To keep track of these repeated rejuvenations, we introduce a new independent variable

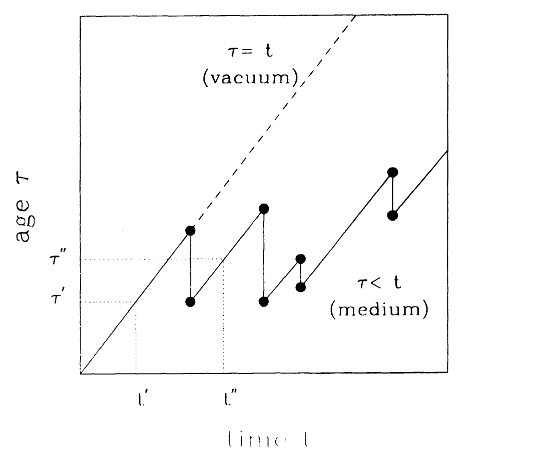

FIG. 6. Connection between the "age" τ of a cascade parton and the time t. In vacuum where the particle evolves (in the LLA) solely by successive branchings with decreasing virtualities Q^2, the age of a parton is measured in time via $t = |x|P/Q^2$. In medium, however, scattering and fusion processes modify the age by rejuvenating the parton in each interaction with the medium. That is, a parton ages slower because it is shifted back to larger virtuality, corresponding to the transverse momentum squared exchanged in the scattering, or to the invariant mass of the compound state in fusions, respectively.

τ to denote the *age of a parton*. In analogy to (66) we define

$$\tau \;\equiv\; \tau(x,t) \;=\; \left(\frac{|x|P}{Q^2}\right)_t \,, \tag{68}$$

which introduces an additional time scale that reflects the external influence of the medium on the time evolution of the parton cascade. Since the scatterings and fusion processes in a medium occur stochastically, the connection between laboratory time t and the age τ cannot be deterministic, but must rather have the nature of a probability distribution $\mathcal{A}(\tau,t,x)$. Here the distribution A denotes that a parton with longitudinal momentum fraction x has age τ at the laboratory time t. As will become clear below, the age distribution will differ for different values of x. It is, however, much easier to derive an evolution equation for the quantity

$$g(x,\tau,t) \;=\; \mathcal{A}(\tau,t,x)\; g(x,t) \tag{69}$$

rather than for the age distribution itself. The function $g(x,\tau,t)$ describes the number of gluons that have longitudinal momentum xP and virtuality $Q^2 = |x|P/\tau$. Of course, if $g(x,\tau,t)$ is known, the age distribution is easily recovered as

$$\mathcal{A}(\tau,t,x) \;=\; \frac{g(x,\tau,t)}{\int_0^\infty d\tau\, g(x,\tau,t)} \;. \tag{70}$$

The age distribution evolves in parallel with laboratory time t in between scatterings and is set back to the age $\tau(p_\perp^2) = |x|P/p_\perp^2$, corresponding to the maximal virtuality $p_\perp^2$ acquired in the collision. The resetting of the clock occurs only if $\tau(p_\perp^2)$ is younger than the present age of the parton, because otherwise the present scattering process cannot be separated from the previous interaction that kicked the parton off mass shell. For simplicity, let us here consider only gluon partons in the cascade and in the medium. The time development of $g(x,\tau,t)$ is governed by four different contributions: (i) a "free-streaming" term describing that τ evolves parallel with t, if the parton does not interact, i.e., the parton "ages naturally"; (ii) a term describing that a branching parton transmits its virtuality to its daughter partons, i.e., these acquire a new age $z\tau$ according to the branching fraction z, so that the variable $Q^2 = |x|P/\tau$ remains unchanged; (iii) a term describing the rejuvenation of partons in two-body scattering processes; and (iv) a term describing the change in the virtuality of a parton by fusion with a parton from the nuclear medium. Hence

$$\frac{\partial}{\partial t}\, g(x,\tau,t) = \left(\frac{\partial g}{\partial t}\right)_{\rm free} + \left(\frac{\partial g}{\partial t}\right)_{\rm branch} + \left(\frac{\partial g}{\partial t}\right)_{\rm scatt} + \left(\frac{\partial g}{\partial t}\right)_{\rm fus} . \tag{71}$$

(i) The free-streaming term is easily obtained from the condition that τ and t evolve in this case parallel, requiring that

$$\begin{aligned} g(x,\tau,t) &= g(x,\tau+d\tau,t+dt) \\ &= g(x,\tau,t) \;+\; \frac{\partial g}{\partial \tau}\,dt \;+\; \frac{\partial g}{\partial t}\,dt \end{aligned} \tag{72}$$

in the absence of other interactions. Hence

$$\left(\frac{\partial g}{\partial t}\right)_{\rm free} \;=\; -\,\frac{\partial}{\partial \tau}\, g(x,\tau,t) \;. \tag{73}$$

(ii) The branching term is obtained simply from Eq. (60) after division by t, if one notes that the variable t in the expression $\alpha_s\left(\frac{|x|P}{zt}\right)$ is related to the virtuality Q^2 of the parent parton and should therefore be replaced by the age variable $\tau = \frac{|x|P}{zQ^2}$. This immediately yields

$$\begin{aligned} \left(\frac{\partial g}{\partial t}\right)_{\rm branch} =& -\frac{1}{2}\int_0^1 dz\; \frac{\overline{\alpha_s}(\tau)}{2\pi\tau}\; \gamma_{g\to gg}(z)\; g(x,\tau,t) \\ &+ \int_0^1 \frac{dz}{z}\; \frac{\overline{\alpha_s}(\tau)}{2\pi\tau}\; \gamma_{g\to gg}(z)\; g\left(\frac{x}{z},\tau,t\right) . \end{aligned} \tag{74}$$

(iii) The scattering term is somewhat more complicated. We begin by noting that the momentum fractions of two partons before (x_1,x_2) and after (x_1',x_2') scattering are related by

$$x_{1;2}' \;=\; \frac{x_1+x_2}{2} \;\pm\; \sqrt{\frac{(x_1-x_2)^2}{4} \;-\; \frac{p_\perp^2}{P^2}} \tag{75}$$

or

$$\begin{aligned} x_1 &= x_1' \;+\; \frac{p_\perp^2}{(x_1'-x_2)P^2}, \\ x_2 &= x_2' \;+\; \frac{p_\perp^2}{(x_2'-x_1)P^2}. \end{aligned} \tag{76}$$

Since only gluons with opposite directions of propagation are allowed to scatter in the parton picture (cf. discussion in Sec. II A), the total scattering rate of cascading gluons is given by

$$\begin{aligned} w = &\int_0^1 dx_1 \int_0^\infty d\tau_1 \int_{-1}^0 dx_2 \\ &\times \int dp_\perp^2\; g(x_1,\tau_1,t)\; \hat{g}(|x_2|)\rho_N\; \frac{d\hat{\sigma}_{gg\to gg}}{dp_\perp^2}, \end{aligned} \tag{77}$$

where ρ_N is the nucleon density in the medium. The loss term due to the scattering of a cascading gluon is therefore

$$\left(\frac{\partial g}{\partial t}\right)_{\rm scatt}^{\rm (loss)} \;=\; -\int_{-1}^0 dx_2 \int dp_\perp^2\; g(x,\tau,t)\; \hat{g}(|x_2|) \times \frac{d\hat{\sigma}_{gg\to gg}}{dp_\perp^2}\;\rho_N . \tag{78}$$

In the gain term we have to distinguish between "soft" and "hard" collisions, as compared with the virtuality of the incoming cascading gluon $Q_1^2 = x_1P/\tau_1$. If $Q_1^2 > p_\perp^2$, the scattering parton will keep its virtuality because the collision cannot be resolved from the previous interaction that originally kicked the parton mass off shell. If $Q_1^2 < p_\perp^2$, the scattered parton will acquire maximum virtuality $p_\perp^2$, corresponding to an age $\tau = |x|P/p_\perp^2$. The gluon scattered out of the nuclear medium, however, always acquires maximum virtuality $p_\perp^2$ because it was spacelike before the interaction. Therefore,

$$\left(\frac{\partial g}{\partial t}\right)^{(\text{gain})}_{\text{scatt}} = \int_0^1 dx_1 \int_0^\infty d\tau_1 \int_{-1}^0 dx_2 \int dp_\perp^2 \; g(x_1,\tau_1,t)\; \hat{g}(|x_2|)\; \frac{d\hat{\sigma}_{gg\to gg}}{dp_\perp^2}\; \rho_N\; \delta(x_1'-x) \times \left\{ \delta\left(\tau - \frac{|x|P}{p_\perp^2}\right) \left[1 + \theta\left(p_\perp^2 - \frac{x_1 P}{\tau_1}\right)\right] + \delta(\tau-\tau_1)\,\theta\left(\frac{x_1 P}{\tau_1} - p_\perp^2\right)\right\} . \quad (79)$$

For the first term in the curly brackets we can perform the $p_\perp^2$ integration, for the second term, the τ_1 integration; the x_1 integration collapses, yielding

$$x_1 = x + \frac{p_\perp^2}{(x-x_2)P^2} , \quad (80)$$

according to Eq. (76). This leaves us with

$$\left(\frac{\partial g}{\partial t}\right)^{(\text{gain})}_{\text{scatt}} = \int_{-1}^0 dx_2\; \hat{g}(|x_2|)\; \frac{|x|P}{\tau^2}\; \frac{d\hat{\sigma}_{gg\to gg}}{dp_\perp^2}\bigg|_{p_\perp^2 = \frac{|x|P}{\tau}} \rho_N \int_0^\infty d\tau_1\; g(x_1,\tau_1,t) \left[1+\theta\left(\frac{\tau_1}{\tau} - \frac{x_1}{|x|}\right)\right] + \int_{-1}^0 dx_2 \int dp_\perp^2\; g(x_1,\tau,t)\; \hat{g}(|x_2|)\; \frac{d\hat{\sigma}_{gg\to gg}}{dp_\perp^2}\; \rho_N\; \theta\left(\frac{x_1 P}{\tau} - p_\perp^2\right) . \quad (81)$$

Note that since x_1 itself depends on $p_\perp^2$ (for fixed x and x_2), the step function in the last term really imposes the integration limit

$$\theta\left(\frac{x_1 P}{\tau} - p_\perp^2\right) = \theta\left(\frac{x(x-x_2)P^2}{(x-x_2)P\tau - 1} - p_\perp^2\right) . \quad (82)$$

(Of course, the integration range is also limited by the condition $x_1 > 0$.)

(iv) Finally we turn to the fusion processes, where the invariant mass M^2 of the produced off-shell parton replaces the momentum transfer $p_\perp^2$ as virtuality scale. Since only those gluons that have reversed their direction of propagation, i.e., with $x < 0$, can fuse with medium gluons (cf. discussion in Sec. II A), the total gluon fusion rate is given by

$$\overline{w} = \int_{-1}^0 dx_1 \int_0^\infty d\tau_1 \int_{-1}^0 dx_2 g(x_1,\tau_1,t)\; \hat{g}(|x_2|) \times \rho_N\; \Gamma_{gg\to g}(x_1,x_2,x_1+x_2)\; \frac{4\pi\alpha_s(M^2)}{M^2}, \quad (83)$$

where the fusion probability $\Gamma_{ab\to c}$ is related to the branching functions $\gamma_{a\to bc}$ by [39]

$$\Gamma_{ab\to c}(x_1,x_2,x_3) = c_{ab\to c}\frac{x_1 x_2}{x_3^2}\; \gamma_{a\to bc}\left(\frac{x_1}{x_3}\right) = c_{ab\to c}\; \frac{x_1 x_2}{x_3^2}\; \gamma_{a\to cb}\left(\frac{x_2}{x_3}\right), \quad (84)$$

where $x_3 = x_1 + x_2$, and the factors in front of $\gamma_{a\to bc}$ arise from the difference of phase-space and flux factors for fusions compared to branchings. The color factors $c_{ab\to c}$ are $c_{gg\to g} = 1/8$, and $c_{qg\to q} = 1/8$, and $c_{q\bar{q}\to g} = 8/9$. Hence we have

$$\Gamma_{gg\to g}(x_1,x_2,x_1+x_2) = \frac{x_1 x_2}{8(x_1+x_2)^2}\gamma_{g\to gg}\left(\frac{x_1}{x_1+x_2}\right), \quad (85)$$

and the invariant mass, neglecting the virtuality of the medium parton [see Eq. (18)], is

$$M^2 = \frac{|x_1|\,P}{\tau_1}\left(1+\frac{x_2}{x_1}\right) = \frac{|x_1+x_2|\;P}{\tau_1} . \quad (86)$$

Here $x_1 < 0$ and τ_1 denote the momentum fraction and virtuality of the shower parton fusing with a parton from the nuclear medium, which has momentum fraction $x_2 < 0$. Since the virtuality of the cascading gluon before fusion was $Q_1^2 = |x_1|P/\tau_1$, we see that the *same* age τ_1 describes its virtuality correctly before *and* after fusion. In other words, the variable τ remains unchanged by fusion processes. The loss and gain term from fusion processes are, therefore,

$$\left(\frac{\partial g}{\partial t}\right)_{fus} = -\int_{-1}^0 dx_2\; g(x,\tau,t)\; \hat{g}(|x_2|)\; \rho_N\; \Gamma_{gg\to g}(x,x_2,x+x_2)\; \frac{4\pi\alpha_s(M^2)}{M^2}\bigg|_{M^2=|x+x_2|P/\tau} + \int_{-|x|}^0 dx_2\; g(x-x_2,\tau,t)\; \hat{g}(|x_2|)\; \rho_N\; \Gamma_{gg\to g}(x-x_2,x_2,x)\; \frac{4\pi\alpha_s(M^2)}{M^2}\bigg|_{M^2=|x|P/\tau} \quad (87)$$

Note that if we would not explicitly distinguish between cascading and nuclear partons, the second term would be supplemented by a factor 1/2 due to the indistinguishability of the two incoming gluons.

We remark that in Eqs. (78), (82), and (87) the absolute value of x_2 in the argument in the nuclear gluon distribution

$\hat{g}(|x_2|)$ accounts for the fact that the measured nucleon structure functions are defined for positive arguments only. However, one has to keep in mind that the nuclear partons are moving in the negative z direction (cf. Sec. II A).

We finally combine all our results to write down the full evolution equation for the off-shell gluon distribution:

$$\begin{aligned}
\frac{\partial}{\partial t} g(x,\tau,t) \;=\; & -\frac{\partial}{\partial \tau} g(x,\tau,t) - \frac{1}{2}\int_0^1 dz\; \gamma_{g\to gg}(z)\, \frac{\overline{\alpha_s}(\tau)}{2\pi\tau}\, g(x,\tau,t) + \int_0^1 \frac{dz}{z}\, \gamma_{g\to gg}(z)\, \frac{\overline{\alpha_s}(\tau)}{2\pi\tau}\, g\left(\frac{x}{z},\tau,t\right) \\
& - \int_{-1}^{0} dx_2\, \hat{g}(|x_2|) \int dp_\perp^2\, \frac{d\hat{\sigma}_{gg\to gg}}{dp_\perp^2}\, \rho_N\, g(x,\tau,t) \\
& + \frac{|x|P}{\tau^2} \int_{-1}^{0} dx_2\, \hat{g}(|x_2|)\, \left.\frac{d\hat{\sigma}_{gg\to gg}}{dp_\perp^2}\right|_{p_\perp^2=|x|P/\tau} \\
& \qquad \times \rho_N \int_0^\infty d\tau_1\, g(x_1,\tau_1,t) \left[1 + \theta\left(\frac{\tau_1}{\tau} - \frac{x_1}{|x|}\right)\right] \\
& + \int_{-1}^{0} dx_2\, \hat{g}(|x_2|) \int^{x_1 P/\tau} dp_\perp^2\, \frac{d\hat{\sigma}_{gg\to gg}}{dp_\perp^2}\, \rho_N\, g(x_1,\tau,t) \\
& - \int_{-1}^{0} dx_2\, \hat{g}(|x_2|)\, g(x,\tau,t)\, \rho_N\, \Gamma_{gg\to g}(x,x_2,x+x_2)\, \left.\frac{4\pi\alpha_s(M^2)}{M^2}\right|_{M^2=|x+x_2|P/\tau} \\
& + \int_{-|x|}^{0} dx_2\, \hat{g}(|x_2|)\, g(x-x_2,\tau,t)\, \rho_N\, \Gamma_{gg\to g}(x-x_2,x_2,x)\, \left.\frac{4\pi\alpha_s(M^2)}{M^2}\right|_{M^2=|x|P/\tau} . \qquad (88)
\end{aligned}$$

Note that the modified equation no longer is a differential equation in the variable $(\ln t)$, but rather in t directly, because the presence of interactions with the medium defines a characteristic time scale $(\sigma\rho)^{-1}$, the mean free time between scatterings, which breaks the scale invariance of the fragmentation cascade.

E. Transverse momentum spread

In order to investigate the development of lateral spread perpendicular to the parton shower axis, one has to study the transverse momentum dependence of the gluon distribution. Recall that the distributrion $g(x,\tau,t)$ is the zeroth moment of the full gluon phase-space distribution $g(x,k_\perp^2,\tau,t)$ [cf. Eq. (34)]:

$$g(x,\tau,t) \;\equiv\; \int dk_\perp^2\; g(x,k_\perp^2,\tau,t) \; . \qquad (89)$$

Instead of keeping the transverse momentum $k_\perp$ as an independent variable in the gluon distribution function, here we will only follow its average growth due to branching, scattering, and fusion processes. In order to do this, we introduce the mean squared transverse momentum distribution $\pi_g(x,\tau,t)$ of gluons as the first moment in $k_\perp^2$, i.e.,

$$\pi_g(x,\tau,t) \;\equiv\; \int dk_\perp^2\; k_\perp^2\; g(x,k_\perp^2,\tau,t) \; . \qquad (90)$$

The evolution equation for $\pi_g(x,\tau,t)$ is easily derived in analogy to the equation (88) for the gluon distribution function $g(x,\tau,t)$. Each loss and gain term is to be weighted with the transverse momentum squared $k_\perp^2$ as it changes by accumulating a certain $p_\perp^2$ generated in the branching, scattering, or fusion processes. Since the loss terms in (88) generally also represent the loss of partons out of a range between $k_\perp^2$ and $k_\perp^2 + dk_\perp^2$, one has simply to replace $g(x,\tau,t)$ by $\pi_g(x,\tau,t)$ in these terms. The gain terms, on the other hand, receive different contributions associated with the different kinematics of branching, scattering, and fusion.

(i) For a branching process $k_{j-1} \to k_j + k_j'$, using (11) and (12) and assuming $Q_j^2, Q_j'^2 \ll Q_{j-1}^2$, one finds

$$\begin{aligned}
k_{\perp j}^2 &\approx z_j^2\, k_{\perp j-1}^2 \;+\; z_j(1-z_j)\, Q_{j-1}^2 \; , \\
k_{\perp j}'^2 &\approx (1-z_j)^2\, k_{\perp j-1}^2 \;+\; z_j(1-z_j)\, Q_{j-1}^2 \; . \qquad (91)
\end{aligned}$$

(ii) For a scattering process between a shower parton with k_{j-1} and a nuclear parton, involving a squared transverse momentum exchange of $p_{\perp j}^2$, with the two corresponding outgoing partons carrying k_j and k_j', respectively, we have

$$\mathbf{k}_{\perp j} \;=\; \mathbf{k}_{\perp j-1} \;+\; \mathbf{p}_{\perp j} \;\; , \qquad \mathbf{k}'_{\perp j} \;=\; -\,\mathbf{p}_{\perp j} \; , \qquad (92)$$

since the nuclear parton initially has no transverse momentum, and

$$p_{\perp j}^2 \;=\; k_{\perp j}^2 \;-\; k_{\perp j-1}^2 \;=\; k_{\perp j}'^2 \; . \qquad (93)$$

(iii) For a fusion process between a cascade parton with k_{j-1} and a parton from the nuclear medium yielding the compound state with k_j, the condition is $\mathbf{k}_{\perp j} = \mathbf{k}_{\perp j-1}$, because as in the scattering case, the nuclear parton initially carries only longitudinal but no transverse momentum; hence

$$k_{\perp j}^2 \;=\; k_{\perp j-1}^2 \; . \qquad (94)$$

Using these kinematic conditions on the change of transverse momentum in the interactions, we find the evolution equation for the first moment in $k_\perp^2$ of the off-shell gluon distribution:

$$\begin{aligned}\frac{\partial}{\partial t}\pi_g(x,\tau,t) \;=\; & -\frac{\partial}{\partial \tau}\pi_g(x,\tau,t) - \frac{1}{2}\int_0^1 dz\,\gamma_{g\to gg}(z)\,\frac{\overline{\alpha_s}(\tau)}{2\pi\tau}\,\pi_g(x,\tau,t)\\
&+\int_0^1 \frac{dz}{z}\,\gamma_{g\to gg}(z)\frac{\overline{\alpha_s}(\tau)}{2\pi\tau}\left[z(1-z)\,\frac{|x|P}{z\tau}\,g\left(\frac{x}{z},\tau,t\right) + z^2\,\pi_g\left(\frac{x}{z},\tau,t\right)\right]\\
&-\int_{-1}^0 dx_2\,\hat{g}(|x_2|)\int dp_\perp^2\,\frac{d\hat{\sigma}_{gg\to gg}}{dp_\perp^2}\,\rho_N\,\pi_g(x,\tau,t)\\
&+\frac{|x|P}{\tau^2}\int_{-1}^0 dx_2\,\hat{g}(|x_2|)\,\left.\frac{d\hat{\sigma}_{gg\to gg}}{dp_\perp^2}\right|_{p_\perp^2=\frac{|x|P}{\tau}}\\
&\quad\times\rho_N\int_0^\infty d\tau_1\,\left\{p_\perp^2\,g(x_1,\tau_1,t)\,\left[1+\theta\left(\frac{\tau_1}{\tau}-\frac{x_1}{|x|}\right)\right]\right.\\
&\qquad\left.+\,\pi_g(x_1,\tau_1,t)\,\theta\left(\frac{\tau_1}{\tau}-\frac{x_1}{|x|}\right)\right\}\\
&+\int_{-1}^0 dx_2\,\hat{g}(|x_2|)\int^{x_1P/\tau} dp_\perp^2\,\frac{d\hat{\sigma}_{gg\to gg}}{dp_\perp^2}\,\rho_N\,\left[p_\perp^2\,g(x_1,\tau,t) + \pi_g(x_1,\tau,t)\right]\\
&-\int_{-1}^0 dx_2\,\hat{g}(|x_2|)\,\pi_g(x,\tau,t)\,\rho_N\,\Gamma_{gg\to g}(x,x_2,x+x_2)\,\left.\frac{4\pi\alpha_s(M^2)}{M^2}\right|_{M^2=|x+x_2|P/\tau}\\
&+\int_{-|x|}^0 dx_2\,\hat{g}(|x_2|)\,\pi_g(x-x_2,\tau,t)\,\rho_N\,\Gamma_{gg\to g}(x-x_2,x_2,x)\,\left.\frac{4\pi\alpha_s(M^2)}{M^2}\right|_{M^2=|x|P/\tau}\quad . \end{aligned}\tag{95}$$

III. THE COUPLED EVOLUTION EQUATIONS FOR QUARKS, ANTIQUARKS, AND GLUONS

In this section we will extend the previous derivation of the gluon evolution to the coupled system of gluons g, quarks q_i, and antiquarks $\bar{q}_i$ with flavors $i = 1,\ldots,f$. The only essential difference to the purely gluonic case is that now the parton cascade evolves via a number of different branching, fusion, and scattering subprocesses which couple the gluons with the quarks and antiquarks. Let us denote the rates for the various interactions of the cascading partons due to $1\to 2$ branchings, $2\to 1$ fusions, and $2\to 2$ scatterings by $(a = q_i, \bar{q}_i, g)$

$$R_a^{(m\to m')}(x,\tau,t) \;\equiv\; \left(\frac{\partial}{\partial t}\,x\,a(x,\tau,t)\right)_{\text{processes}\,m\to m'}\quad , \tag{96}$$

i.e., as the change of the x weighted parton densities, integrated over $k_\perp^2$,

$$x\,a(x,\tau,t) \;=\; \int dk_\perp^2\,x\,a(x,k_\perp^2,\tau,t)\quad . \tag{97}$$

To evaluate the interaction rates $R_a^{(m\to m')}(x,\tau,t)$, we use the well known lowest order perturbative QCD expressions for the branching amplitudes [30,33], fusion amplitudes [39,44], and the parton-parton cross sections [45–47], respectively, and implement those in the formalism described in the preceding section. The corresponding Feynman diagrams are depicted in Figs. 7–9.

For the following we introduce the *parton momentum densities* Q_i ($\bar{Q}_i$) and G:

$$\begin{aligned} Q_i(x,\tau,t) &= x\int dk_\perp^2\,q_i(x,k_\perp^2,\tau,t), G(x,\tau,t),\\ &= x\int dk_\perp^2\,g(x,k_\perp^2,\tau,t), \end{aligned}\tag{98}$$

i.e., the parton number densities q_i ($\bar{q}_i$) and g weighted with the longitudinal momentum fraction x, where $i = 1,\ldots,f$ denotes the quark flavors. Furthermore, for brevity we define the following functions that represent the effective coupling in branching and fusion processes in terms of the running QCD coupling strength $\alpha_s(Q^2)$, Eq. (39):

$$\overline{\alpha_s}(\tau) \equiv \frac{\int_0^1 dx\,\alpha_s\left(\frac{|x|P}{\tau}\right)\,\theta\left(\frac{|x|P}{\tau}-\mu_0^2\right)\,\left(g(x,\tau,t) + \sum_j[q_j(x,\tau,t)+\bar{q}_j(x,\tau,t)]\right)}{\int_0^1 dx\,\left(g(x,\tau,t) + \sum_j[q_j(x,\tau,t)+\bar{q}_j(x,\tau,t)]\right)},$$

$$\xi(\tau) \equiv \frac{\overline{\alpha_s}(\tau)}{2\pi\tau}\;,\quad \zeta(x,\tau) \;\equiv\; \left.\frac{4\pi\alpha_s(M^2)}{M^2}\right|_{M^2=|x|P/\tau}\quad . \tag{99}$$

Finally we recall that $\tau = |x|P/Q^2$ denotes the lifetime of a parton with longitudinal momentum fraction x and virtuality Q^2, which according to (68) sets the typical time scale for producing a parton at x and Q^2 in a parton cascade as it evolves through the nuclear medium, and which reduces to Eq. (66) in vacuum. Furthermore, the variable $z = x/x'$ is the fraction of x values of daughter to mother partons in branchings and fusions, M^2 is the invariant mass squared of two fusing partons, $p_\perp^2$ refers to the relative transverse momentum squared exchanged in parton-parton scatterings, and ρ_N is the nucleon density characterizing the nuclear medium.

A. Branching processes

The net change of the quark number densities due to the branching processes shown in Fig. 7(a) is obtained by adding up the gain term due to quarks of momentum fraction $x_1 > x$ having radiated a gluon of momentum fraction $x_1 - x$, the loss term, identifying all those quarks that had momentum fractions x before radiating a gluon of momentum fraction less than x, and the additional gain term that arises from gluons with momentum fraction $x_1 > x$ decaying in a $q_i\bar{q}_i$ pair with momentum fractions x and $x_1 - x$, respectively. Correspondingly, the net change of the antiquark number distributions due to branchings is given by replacing q_i by $\bar{q}_i$.

The result for the branching rates of quarks (and analogous for antiquarks) is

$$R_{q_i}^{(1\to 2)}(x,\tau,t) \;=\; -\,\hat{A}\,Q_i \;+\; \hat{B}\,G \;, \tag{100}$$

where

$$\begin{aligned} -\,\hat{A}\,Q_i &= -\int_0^1 dz\,\left[Q_i(x,\tau,t) \;-\; Q_i\left(\frac{x}{z},\tau,t\right)\right] \\ &\quad\times\xi(\tau)\;\gamma_{q\to qg}(z)\;, \\ \hat{B}\,G &= \int_0^1 dz\;G\left(\frac{x}{z},\tau,t\right)\;\xi(\tau)\;\gamma_{g\to q\bar{q}}(z)\;. \end{aligned} \tag{101}$$

The branching functions $\gamma_{a\to bc}(z)$ are given by (38).

The change of the gluon distributions is similarly obtained by adding the contributions of Fig. 7(b), namely, the gain of gluons with momentum fraction x due to gluon emission by gluons with momentum fraction $x_1 > x$; the loss term due to gluon emission by gluons with momentum fraction x; the loss term due to gluon decay into $q_i\bar{q}_i$, summed over all quark flavors i; and the gain term due to radiation of gluons with momentum fraction x by quarks and antiquarks with momentum fractions $x_1 > x$. We then obtain, for the branching rates of gluons,

$$\begin{aligned} R_g^{(1\to 2)}(x,\tau,t) &= -\,\hat{C}\,G \;-\; \hat{D}\,G \\ &\quad+\sum_{j=1}^{f}\left(\,\hat{E}\,Q_j \;+\; \hat{E}\,\bar{Q}_j\right), \end{aligned} \tag{102}$$

where

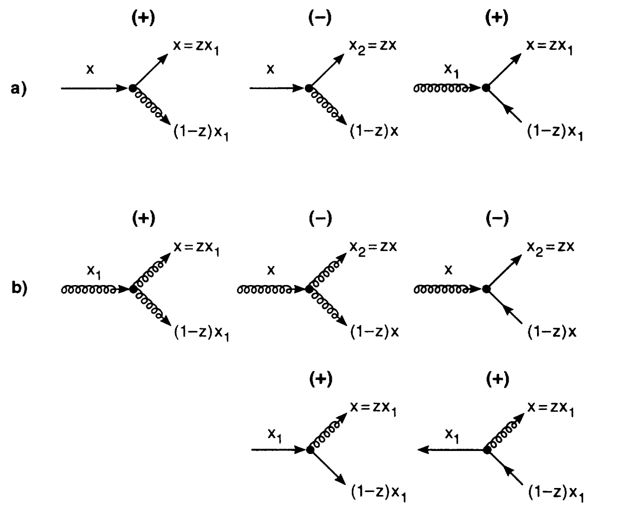

FIG. 7. Feynman diagrams associated with the gain and loss of (a) quarks (antiquarks) and (b) gluons due to elementary branching processes. The parton with momentum fraction x is the "observed" particle.

$$\begin{aligned} -\,\hat{C}\,G &= -\int_0^1 dz\,\left[\frac{1}{2}\,G(x,\tau,t) \;-\; G\left(\frac{x}{z},\tau,t\right)\right] \\ &\quad\times\xi(\tau)\;\gamma_{g\to gg}(z)\;, \\ -\hat{D}\,G &= -\;f\;G(x,\tau,t)\;\int_0^1 dz\;\xi(\tau)\;\gamma_{g\to q\bar{q}}(z)\;, \\ \hat{E}\,Q_j &= \int_0^1 dz\;Q_j\left(\frac{x}{z},\tau,t\right)\;\xi(\tau)\;\gamma_{q\to qg}(z)\;, \\ \hat{E}\,\bar{Q}_j &= \int_0^1 dz\;\bar{Q}_j\left(\frac{x}{z},\tau,t\right)\;\xi(\tau)\;\gamma_{q\to qg}(z)\;. \end{aligned} \tag{103}$$

B. Fusion processes

The parton-parton fusion processes manifestly alter the parton number densities in a similar manner as the branching processes, provided the quark and gluon densities of the nuclear medium (labeled with a caret) are sufficiently dense and the probability for a parton to absorb one of the nuclear partons becomes significant. Recall that we do not consider here fusions among the shower partons or among the nuclear partons themselves. To obtain the fusion rates, we use the expressions for the fusion probabilities $\Gamma_{ab\to c}$ given in (84) in terms of the branching functions $\gamma_{a\to bc}$.

The net change of the quark number distributions due to fusions with partons in the nuclear medium (labeled by a caret) is balanced by the gain and loss of quarks with momentum fraction x due to the processes shown in Fig. 8(a). These are the gain of quarks through fusions of a quark with $x_1 < x$ and a gluon with $x_2 < x$ such that $x_1 + x_2 = x$, the loss of quarks with longitudinal momentum fraction x due to fusions with gluons of x_2, and the loss of quarks with fraction x due to $q_i\bar{q}_i$ annihilation into gluons. The corresponding change in the antiquark number distributions is obtained by interchanging q_i and $\bar{q}_i$.

The result for the fusion rates of quarks or antiquarks is

$$R_{q_i}^{(2\to 1)}(x,\tau,t) \;=\; -\,\hat{A}'[\hat{G}]\,Q_i \;-\; \hat{B}'[\hat{\bar{Q}}_i]\,Q_i \;, \qquad (104)$$

where, in contrast to $\hat{A}$ and $\hat{B}$ in (101), the integral operators $\hat{A}'$ and $\hat{B}'$ are functionals of the densities of quarks $\hat{Q}$ ($\hat{\bar{Q}}$) and gluons $\hat{G}$ of the nuclear medium in which the cascading particles evolve. The integral operators A' and B' are obtained as

$$-\,\hat{A}'[\hat{G}]\,Q_i = -\,\frac{1}{8}\,\rho_N \int_0^1 dz\left[Q_i(x,\tau,t)\hat{G}\left(\frac{x(1-z)}{z}\right)\zeta\left(\frac{x}{z},\tau\right) - Q_i\left(xz,\tau,t\right)\,\hat{G}\left[x(1-z)\right]\zeta\left(x,\tau\right)\right]\gamma_{q\to qg}(z),$$

$$-\,\hat{B}'[\hat{\bar{Q}}_i]\,Q_i = -\,\frac{8}{9}\,\rho_N \int_0^1 dz\; Q_i(x,\tau,t)\hat{\bar{Q}}_i\left(\frac{x(1-z)}{z}\right)\gamma_{g\to q\bar{q}}(z)\,\zeta\left(\frac{x}{z},\tau\right) \;. \qquad (105)$$

The gluon number distributions receive modifications from the fusion processes depicted in Fig. 8(b). There is the gain of gluons with momentum fraction x from fusions of two gluons with $x_1 < x$ and $x_2 < x$ such that $x_1 + x_2 = x$, the loss of gluons with x due to fusions with other gluons from the medium, the gain of gluons with fraction x due to $q_i\bar{q}_i$ annihilation, and the loss of gluons with fraction x due to absorption by quarks or antiquarks. We obtain the following result for the fusion rates of gluons:

$$R_g^{(2\to 1)}(x,\tau,t) = -\,\hat{C}'[\hat{G}]\,G \;-\; \sum_{j=1}^{f}\Big(\hat{D}'[\hat{Q}_j]\,G + \hat{D}'[\hat{\bar{Q}}_j]\,G\Big) + \sum_{j=1}^{f}\Big(\hat{E}'[\hat{\bar{Q}}_j]\,Q_j + \hat{E}'[\hat{Q}_j]\,\bar{Q}_j\Big)\,, \qquad (106)$$

where

$$-\,\hat{C}'[\hat{G}]\,G = -\,\frac{1}{8}\,\rho_N \int_0^1 dz\left[G(x,\tau,t)\hat{G}\left(\frac{x(1-z)}{z}\right)\zeta\left(\frac{x}{z},\tau\right) - G\left(xz,\tau,t\right)\,\hat{G}\left[x(1-z)\right]\zeta\left(x,\tau\right)\right]\gamma_{g\to gg}(z)\;,$$

$$-\,\hat{D}'[\hat{Q}_j]\,G = -\,\frac{1}{8}\,\rho_N\, G(x,\tau,t)\int_0^1 dz\;\hat{Q}_j\left(\frac{x(1-z)}{z}\right)\gamma_{q\to gq}(z)\,\zeta\left(\frac{x}{z},\tau\right)\;,$$

$$-\,\hat{D}'[\hat{\bar{Q}}_j]\,G = -\,\frac{1}{8}\,\rho_N\, G(x,\tau,t)\int_0^1 dz\;\hat{\bar{Q}}_j\left(\frac{x(1-z)}{z}\right)\gamma_{q\to gq}(z)\,\zeta\left(\frac{x}{z},\tau\right)\;,$$

$$\hat{E}'[\hat{\bar{Q}}_j]\,Q_j = \frac{4}{9}\,\rho_N \int_0^1 dz\; Q_j\left(xz\right)\hat{\bar{Q}}_j\left[x(1-z)\right]\,\gamma_{g\to q\bar{q}}(z)\,\zeta\left(x,\tau\right)\;,$$

$$\hat{E}'[\hat{Q}_j]\,\bar{Q}_j = \frac{4}{9}\,\rho_N \int_0^1 dz\; \bar{Q}_j\left(xz\right)\hat{Q}_j\left[x(1-z)\right]\,\gamma_{g\to q\bar{q}}(z)\,\zeta\left(x,\tau\right)\;. \qquad (107)$$

C. Scattering processes

Finally, the collision rates for elastic scatterings of the cascading partons with the partons in the nuclear background medium with nuclear density ρ_N receive various contributions which are diagrammatically shown in Fig. 9. Again, we only account for interactions of the parton cascade with the medium, i.e., those parton collisions that involve a shower parton and a nuclear parton, the latter of which after the scattering becomes a timelike excitation and is added to the cascade. To compress the expressions for the scattering rates, we introduce the function

$$\begin{aligned}\hat{S}_{ab}\{\hat{A}\}\,B \equiv &-\int_{-1}^{0}\frac{dx_2}{x_2}\,\hat{A}(|x_2|)\int dp_\perp^2\,\frac{d\hat{\sigma}_{ab\to ab}}{dp_\perp^2}\,\rho_N\,\frac{B(x,\tau,t)}{x} + \frac{|x|P}{\tau^2}\int_{-1}^{0}\frac{dx_2}{x_2}\,\hat{A}(|x_2|)\,\left.\frac{d\hat{\sigma}_{ab\to ab}}{dp_\perp^2}\right|_{p_\perp^2=\frac{|x|P}{\tau}}\\ &\times\,\rho_N\int_0^{\infty}d\tau_1\,\frac{B(x_1,\tau_1,t)}{x_1}\left[1+\theta\left(\frac{\tau_1}{\tau}-\frac{x_1}{|x|}\right)\right]\\ &+\int_{-1}^{0}\frac{dx_2}{x_2}\,\hat{A}(|x_2|)\int^{x_1P/\tau}dp_\perp^2\,\frac{d\hat{\sigma}_{ab\to ab}}{dp_\perp^2}\,\rho_N\,\frac{B(x_1,\tau,t)}{x_1}\;,\end{aligned} \qquad (108)$$

where $A,B = G, Q_j, \bar{Q}_j$ and the caret labels as before the nuclear parton distributions $\hat{A}$, whereas the distributions B without a caret refer to the cascading partons. The value of x_1 at which the function $B(x_1,\tau,t)$ is to be evaluated is, according to Eq. (80), $x_1 = x + p_\perp^2/[(x-x_2)P^2]$. The values of x_2 are negative, according to our choice of frame, Sec. II A. Since the nuclear parton densities, when represented in terms of the measured nucleon structure

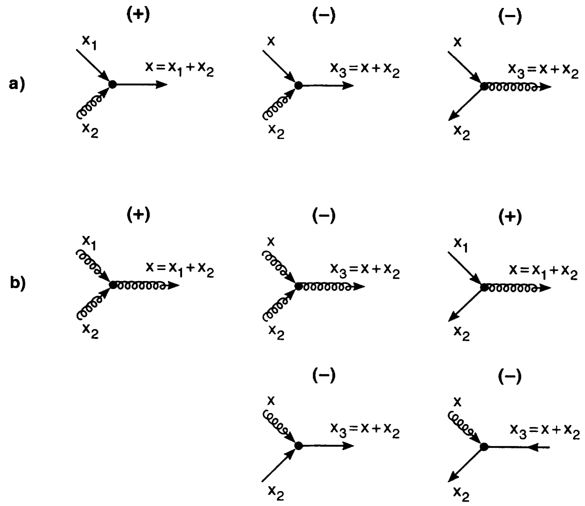

FIG. 8. Feynman diagrams contributing to the gain and loss of (a) quarks (antiquarks) and (b) gluons by fusions with nuclear partons. The parton of interest is the one with momentum fraction x and the nuclear parton from the nucleus structure function has momentum fraction x_2.

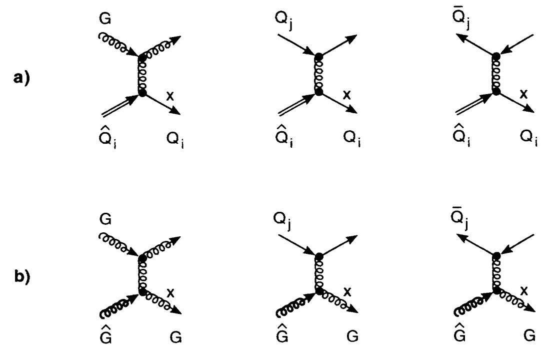

FIG. 9. Diagrams of elementary scattering processes that increase of the number of (a) quarks (antiquarks) and (b) gluons with momentum fraction x due to the liberation of a virtual nuclear parton (labeled by a caret) out of the wave function of the nucleus by an interaction with a cascade parton.

functions, are defined for positive values only, we take here the absolute value, but keep in mind that the partons of the nucleus move in the negative z direction.

For the quarks (and similar antiquarks) we have the processes $q_i g \to q_i g$, $q_i q_j \to q_i q_j$, and $q_i \bar{q}_j \to q_i \bar{q}_j$. Hence

$$R_{q_i}^{(2\to 2)}(x,\tau,t) = \hat{S}_{q_i g}[\hat{Q}_i]\, G + \sum_{j=1}^{f} \left(\hat{S}_{q_i q_j}[\hat{Q}_i]\, Q_j + \hat{S}_{q_i \bar{q}_j}[\hat{Q}_i]\, \bar{Q}_j \right) . \tag{109}$$

On the other hand, for gluons the contributing processes are $gg \to gg$, $gq_j \to gq_j$, and $g\bar{q}_j \to g\bar{q}_j$. Consequently,

$$R_{g}^{(2\to 2)}(x,\tau,t) = \hat{S}_{gg}[\hat{G}]\, G + \sum_{j=1}^{f} \left(\hat{S}_{g q_j}[\hat{G}]\, Q_j + \hat{S}_{g \bar{q}_j}[\hat{G}]\, \bar{Q}_j \right) . \tag{110}$$

The parton-parton cross sections that enter the expressions $\hat{S}_{ab}[\hat{A}]\, B$, Eq. (108), i.e., $d\hat{\sigma}_{ab\to cd}/dp_\perp^2$, for massless partons are related to the squared scattering amplitudes $|\overline{\mathcal{M}}_{ab\to cd}|^2$, averaged over initial spin and color states and summed over the final states, by

$$\frac{d\hat{\sigma}_{ab\to cd}(\hat{s},p_\perp^2)}{dp_\perp^2} = \mathcal{D}_{ab}\,\mathcal{D}_{cd}\, \frac{\pi \alpha_s^2(p_\perp^2)}{\hat{s}^2}\, |\overline{\mathcal{M}}_{ab\to cd}|^2 . \tag{111}$$

Here the variables $\hat{s}, \hat{t}, \hat{u}$ are the kinematic invariants of the parton-parton scattering with $\hat{s} + \hat{t} + \hat{u} = 0$, and $p_\perp^2 = \hat{t}\hat{u}/\hat{s}$ for massless particles. The degeneracy factors $\mathcal{D}_{ab} = (1+\delta_{ab})^{-1}$ account for the identical particle effect in the initial state if a and b are truly indistinguishable, and correspondingly $\mathcal{D}_{cd}$ is the statistical factor for the final state. However, since we explicitly distinguish between nuclear and shower partons, $\mathcal{D}_{ab} = 1$ always. The squared matrix elements for the diagrams in Fig. 9 are standard literature and can be found in, e.g., [45]. For massive quarks the corresponding scattering matrix-elements can be found in Refs. [46,47].

D. The evolution equations for the parton shower functions

Adding all the interaction rates $R^{(1\to 2)}$, $R^{(2\to 1)}$, and $R^{(2\to 2)}$ to the free-streaming term (72), we obtain the following set of evolution equations for the parton densities Q_i of quarks, $\bar{Q}_i$ of antiquarks, and G of gluons, respectively:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\right) Q_i(x,\tau,t) = -\left(\hat{A} + \hat{A}'[\hat{G}] + \hat{B}'[\hat{\bar{Q}}_i]\right) Q_i + \left(\hat{B} + \hat{S}_{qg}[\hat{Q}_i]\right) G + \sum_{j=1}^{f} \hat{S}_{qq}[\hat{Q}_i]\; Q_j + \sum_{j=1}^{f} \hat{S}_{q\bar{q}}[\hat{Q}_i]\; \bar{Q}_j , \tag{112}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right) \bar{Q}_i(x,\tau,t) = -\left(\hat{A} + \hat{A}'[\hat{G}] + \hat{B}'[\hat{Q}_i]\right) \bar{Q}_i + \left(\hat{B} + \hat{S}_{qg}[\hat{\bar{Q}}_i]\right) G$$
$$+ \sum_{j=1}^{f} \hat{S}_{q\bar{q}}[\hat{\bar{Q}}_i]\; Q_j + \sum_{j=1}^{f} \hat{S}_{\bar{q}\bar{q}}[\hat{\bar{Q}}_i]\; \bar{Q}_j \;, \tag{113}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right) G(x,\tau,t) = -\left[\hat{C} + \hat{D} + \hat{C}'[\hat{G}] + \sum_{j=1}^{f}\left\{\hat{D}'[\hat{Q}_j] + \hat{D}'[\hat{\bar{Q}}_j]\right\} - \hat{S}_{gg}[\hat{G}]\right] G$$
$$+ \sum_{j=1}^{f}\left(\hat{E} + \hat{E}'[\hat{Q}_j] + \hat{S}_{gq}[\hat{G}]\right) Q_j + \sum_{j=1}^{f}\left(\hat{E} + \hat{E}'[\hat{Q}_j] + \hat{S}_{g\bar{q}}[\hat{G}]\right) \bar{Q}_j . \tag{114}$$

Note that this set of equations actually describes the time evolution of the parton (longitudinal) momentum distributions $Q = xq$, $\bar{Q} = x\bar{q}$, and $G = xg$ rather than of the parton number densities q, $\bar{q}$, and g. Similarly, one can write down, in the full evolution equations, for the first moments in $k_\perp^2$,

$$\Pi_{q_i}(x,\tau,t) = \int dk_\perp^2\; k_\perp^2\; x\, q_i(x,k_\perp^2,\tau,t) \;,$$
$$\Pi_g(x,\tau,t) = \int dk_\perp^2\; k_\perp^2\; x\, g(x,k_\perp^2,\tau,t), \tag{115}$$

in a straightforward generalization of Eq. (95).

Finally we stress that the evolution Eqs. (112)–(114) can immediately be generalized to treat the cascading partons and the partons of the nuclear medium on the same footing, by dropping our bookkeeping distinction among those two particle sources. This would then describe a dynamically coupled system in which the parton cascade evolution feeds back on the nuclear parton distribution. The only differences to Eqs. (112)–(114) are that the nuclear parton densities also become time dependent, i.e., $\hat{a}(x) \to \hat{a}(x,t)$, where $a = q_i, \bar{q}_i, g$, and in the gain term of $gg \to g$ fusion, as well as in the gg scattering rates, an additional factor of 1/2 would be needed because the two interacting gluons, a cascade parton and a nuclear parton, cannot be distinguished anymore. However, the response of the nuclear density to the penetrating parton shower is naturally delayed and only locally effective, so that the parton cascade, to a good approximation, can be viewed as being unaffected by this time variation of the nuclear medium. Of course this approximation does not apply anymore when there are multiple cascades evolving simultaneously close to each other in phase space, as, e.g., in a nucleus-nucleus collision. In such a case the full space-time history of both the cascading partons as well as the nuclear partons certainly needs to be taken into account.

IV. SUMMARY

To summarize the essence of this work, let us list what we believe are the most important points.

(i) We have derived integro-differential equations for the evolution of a parton cascade in an infinite, homogeneous nuclear medium, describing the parton distributions in terms of the Bjorken variable x and virtuality Q^2 (or "age" τ).

(ii) The Lorentz invariant evolution equations have the character of transport equations in momentum space, familiar from nonequilibrium kinetic theory; however, they include effects of off-shell propagation in addition to collision terms.

(iii) In the absence of a medium the equations reduce to the Altarelli-Parisi-Lipatov equations for the Q^2 evolution of the parton number densities in vacuum.

(iv) Possible immediate applications of the evolution equations are, for example, to the fragmentation of partons in heavy nuclei and to hard QCD probes of a quark-gluon plasma.

(v) The equations can be easily generalized to provide a description of parton transport in ultrarelativistic heavy ion collisions by treating the shower partons and the nuclear partons on the same footing.

(vi) Since our derivation was partially based on heuristic arguments, it would be desirable to obtain a formal justification from fundamental principles of quantum field theory by Green's function methods.

We hope to address these issues in future publications [48].

ACKNOWLEDGMENTS

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[43] This point was brought to our awareness by X.-N. Wang. The requirement for momentum conservation means that $\partial/\partial t \int_0^1 dx\, xg(x,t) = 0$. In order to satisfy this conservation law it is inevitable that both terms on the right hand side of Eq. (48) are proportional to the same coupling $\overline{\alpha_s}^g(t)$ that is independent of x and can therefore be pulled out of the x integration. Then one ends up with an integral of the form $\overline{\alpha_s}^g(t) \int_0^1 dx\, xg(x,t) \int_0^1 dz \gamma_{g\to gg}(z) \left[z - 1/2 \right]$, which is zero, because the z integral vanishes, as can be easily found by inserting the explicit expression for $\gamma_{g\to gg}(z)$.
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