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Inhomogeneous model with a cosmological constant

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(Received 8 November 1993)

We extend an earlier work of Tosa to include the effects of inhomogeneity in the cosmological evolution in Kaluza-Klein spacetime with a cosmological constant. It is observed that the presence of inhomogeneity drastically changes the evolutionary scenario such that our five-dimensional spacetime admits inflation in three-space and dimensional reduction of the extra space. This result is at variance with an earlier conclusion of Tosa for homogeneous models that only a negative cosmological constant with more than one extra spatial dimension can provide inflation of three-space and a small size of the extra one. The model also seems to suggest an alternative mechanism pointing to a smooth transition from a primordial, multidimensional inhomogeneous phase to a four-dimensional homogeneous one.

PACS number(s): 98.80.Hw, 04.50.+h

Kaluza-Klein (KK) theories have shown how gravity and electromagnetism can be unified from Einstein's field equations generalized to five dimensions. The idea has been later extended to include other types of interaction in $(4+d)$-dimensional models where the isometries of the spontaneously compactified d spacelike dimensions account for the gauge symmetries of the effective four-dimensional (4D) theory [1,2]. It is generally known that in KK theories with spontaneous dimensional reduction a very large cosmological constant in four dimensions almost always arises, whereas from observational data $\Lambda \sim 10^{-56}\text{cm}^{-2}$. However, for noncompact internal space one can envision dynamical mechanisms leading to a vanishing 4D cosmological constant [3]. However, it is interesting to point out that geometrically such solutions do not correspond to the topology of direct product of 4D and extra dimensional spaces.

While studying the implications of the different compactification processes it may be pointed out that higher dimensional homogeneous models have been adequately addressed in the literature [4]. But only scant attention has been paid so far to inhomogeneous models and also to the issues concerning it in KK spacetime. But following the Cosmic Background Explorer (COBE) Differential Microwave Radiometer (DMR) experimental findings [5] which point to the unambiguous evidence for the existence of inhomogeneities in the early Universe, there has been a resurgence of interest in models other than homogeneous. On the other hand if higher dimensional spacetime is to be seriously considered as an alternative approach to some real physics, it is only in the realm of the early phase of cosmological evolution that inclusion of an extra space assumes a particular significance. However, our visible Universe is manifestly flat, isotropic and homogeneous, although any plausible explanation of these observed properties continues to be elusive. Explicit computations [6] showed that the so-called "chaotic" models [7] cannot satisfactorily explain these observed phenomena. Invoking a higher dimensional phase one may circumvent the difficulty by arguing that the primordial inhomogeneity may be accounted for through the introduction of extra spatial dimensions which depend both on space and time. With the above considerations in mind the appropriate metric is taken in the form

$$ds^2 = B^2 dt^2 - R^2(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2) - A^2 dy^2, \quad (1)$$

where $B = B(r,t)$, $R = R(t)$, $A = A(r,t)$, and y is a KK parameter taken in the form of a circle.

Two more comments regarding the choice of the metric may be in order. Since A depends both on r and t we are not dealing with any simple product space. The size of the internal space is different at different points of the 4D world in this case.

Second, interest in a 5D spacetime stems from the remarkable fact that both $d=10$ and $d=11$ supergravities yield a solution where a 5D spacetime results following spontaneous dimensional reduction [8].

For the matter field we take an inhomogeneous, purely classical dust distribution $T_{ij} = \rho_i U_j$, where $\rho = \rho(r,t)$ and a comoving system is used. Since in the case of dust the particles follow geodesic world lines one can always set $B = 1$ without any loss of generality.

The Einstein field equations for the metric (1) with $B = 1$ are given by [9]

$$G_{01} = \frac{2\dot{A}'}{A} - \frac{2\dot{R}A'}{RA} = 0 \, , \quad (2)$$

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$$G_1{}^1=\frac{2A'}{AR^2r}-\left[\frac{2\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}+\frac{\ddot{A}}{A}+\frac{2\dot{R}\dot{A}}{RA}\right]=-\Lambda\ , \tag{3}$$

$$G_2{}^2=G_3{}^3=\frac{1}{R^2}\left[\frac{A''}{A}+\frac{A'}{Ar}\right]-\left[2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}+\frac{\ddot{A}}{A}+2\frac{\dot{R}\dot{A}}{RA}\right]=-\Lambda\ , \tag{4}$$

$$G_4{}^4=-3\left[\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}\right]=-\Lambda\ , \tag{5}$$

$$G_0{}^0=\frac{1}{R^2}\left[\frac{A''}{A}+\frac{2A'}{Ar}\right]-3\left[\frac{\dot{R}^2}{R^2}+\frac{\dot{R}\dot{A}}{RA}\right]=-\rho-\Lambda\ , \tag{6}$$

Where an overdot and prime denote differentiation with respect to time and the radial coordinate.

For economy of space, we skip the details of the intermediate steps and write down the final results as ($\Lambda>0$)

$$R^2=a_1e^{pt}+a_2e^{-pt}\quad\left[p^2=\frac{2\Lambda}{3}\right]\ , \tag{7a}$$

$$A=\frac{(b-Cr^2)(a_1e^{pt}+a_2e^{-pt})+p(a_1e^{pt}-a_2e^{-pt})+4C/p^2}{(a_1e^{pt}+a_2e^{-pt})^{1/2}}\ . \tag{7b}$$

One cannot directly get the solutions corresponding to $\Lambda=0$ by simply setting $\Lambda=0$ in Eqs. (7a) and (7b). However, in the limiting case of $\Lambda\rightarrow 0$ the above equations reduce to the ones found earlier by Chatterjee *et al.* [9] for $\Lambda=0$. It may be of interest to see how Eqs. (7a) and (7b) look when $\rho=0$. While Eq. (7a) remains unchanged, we find from Eq. (6) that in this case $b=C=0$. Moreover, if we assume that three-space starts off at $t=0$,

$$\begin{aligned}&a_1=-a_2=R_0{}^2\ \text{(say)}\ ,\\ &A=\frac{pR_0(e^{pt}+e^{-pt})}{(e^{pt}-e^{-pt})^{1/2}}\end{aligned} \tag{7c}$$

which in the limit $\Lambda\rightarrow 0$ yields $A\sim t^{-1/2}$ and $R\sim t^{1/2}$. This is, however, the well known solution obtained earlier by Chodos and Detweiler [10] for a 5D vacuum spacetime with zero cosmological constant. However, when $\Lambda<0$, we get expressions for R and A with "p" replaced by "iq" in Eqs. (7a) and (7b).

When $C=0$, our spacetime becomes homogeneous and so "C" is a measure of inhomogeneity of our cosmology. If we set $a_2=0$ we get a de Sitter like inflation of three-space. In the present work, however, we are primarily concerned with an earlier observation of Tosa [11] in the context of homogeneous spacetime that the expansion of three-space and contraction of the extra ones are possible only when $\Lambda<0$ and the extra space contains more than one spatial dimension. We shall presently see that the presence of inhomogeneity characterized by the nonvanishing value of the constant "C" radically changes the above scenario. Even a 5D inhomogeneous spacetime with any negative, positive, or zero cosmological constant can provide expansion of three-space along with a dimensional reduction of the extra scale.

Following Tosa we demand that the scale factor starts off from $t=0$ in all three cases such that

$$R^2=\begin{cases}R_0{}^2\sinh pt\quad\text{for }\Lambda>0\ , & (8a)\\ R_0{}^2t\quad\text{for }\Lambda=0\ , & (8b)\\ R_0{}^2\sin qt\quad\text{for }\Lambda<0\ , & (8c)\end{cases}$$

Correspondingly we get the expressions for the extra space scale as

$$A=\begin{cases}\dfrac{(b-Cr^2)\sinh pt+a_1\cosh pt+4C/p^2}{(\sinh pt)^{1/2}}\quad\text{for }\Lambda>0\ , & (9a)\\ (b-Cr^2)t^{1/2}+a_1t^{-1/2}-2Ct^{3/2}\quad\text{for }\Lambda=0\ , & (9b)\\ \dfrac{(b-Cr^2)\sin qt+a_1\cos qt+4C/g^2}{(\sin qt)^{1/2}}\quad\text{for }\Lambda<0\ . & (9c)\end{cases}$$

The corresponding mass densities for the above cases are

$$\rho=\begin{cases}\frac{3}{2}\dfrac{p^2(b-Cr^2)}{(b-Cr^2)\sinh^2pt+\dfrac{a_1}{2}\sinh 2pt+\dfrac{4C}{p^2}\sinh pt}\ ,\\ \text{for }\Lambda>0\ , & (10a)\\ \frac{3}{2}\dfrac{(b-Cr^2)}{(b-Cr^2)t^2+a_1t-2Ct^3}\quad\text{for }\Lambda=0\ , & (10b)\\ \frac{3}{2}\dfrac{q^2(b-Cr^2)}{(b-Cr^2)\sin^2qt+\dfrac{a_1}{2}\sin 2qt+\dfrac{4C}{2}\sin qt}\quad\text{for }\Lambda<0\ . & (10c)\end{cases}$$

Our cosmology has evidently a point of symmetry $r=0$, which is at variance with the cosmological principle that envisages a continuous group of symmetry imposed on the points of the Riemannian manifold. So the cosmological principle is clearly given up in our case. Further for $C>0$, the mass density vanishes at the radial coordinate $r=r_b=\sqrt{b/C}$ and this may be set as a natural choice for our coordinate boundary. As the inhomogeneity parameter "C" tends to zero the coordinate boundary recedes more and more from the point of symmetry and the cosmology mimics increasingly a 5D homogeneous model. It may be tempting to suggest that the solution represents a bounded distribution of matter and look for relevant boundary conditions by matching it with a vacuum exterior. As mentioned earlier [see Eqs. (7a) and (7b)] both b and C should vanish separately to ensure that $\rho=0$ for all values of time. In this case one finds that even though $R(t)$ remains unaltered g_{44} (the metric tensor corresponding to the extra dimension) in empty space is obtained only by putting $b=C=0$. So by simple arguments it is possible to conclude that the interior and exterior metric component g_{44} cannot match at $r=r_b$ for all values of time t. So our solutions do not represent a sphere with a definite boundary obeying the conditions of fit with the exterior vacuum. However, when $C<0$, the model expands indefinitely and nowhere in the interior does the matter density ρ vanish. It is encouraging to note that our model is spatially regular as is evident from the behavior of the Kretschmann scalar $R_{ijkl}R^{ijkl}$ with variations of r including the origin.

In what follows we shall discuss, very briefly, the dynamics of the inhomogeneous models presented above in the three distinct cases for a positive, zero, and negative cosmological constant Λ and compare it with the analogous 5D homogeneous models of Tosa referred to earlier.

When $\Lambda>0$, $R\rightarrow 0$ and $A\rightarrow\infty$ as $t\rightarrow 0$. Thus the two scales do not start at the same instant, which is apparently an undesirable feature of the model. During evolution the extra space shrinks to zero during a finite time interval, with suitably chosen values of the arbitrary parameter a and C, one of them being at least negative.

When $\Lambda=0$, $R\rightarrow 0$ and $A\rightarrow\infty$ at the big bang for positive a, while for $a=0$ both the scales start off from zero. On the other hand dimensional reduction takes place in a finite value of times when the constant is positive. The ordinary scale R continues to inflate.

For $\Lambda<0$, the two scale factors once again do not start growing at the same instant; rather, when R starts from zero, the extra dimension begins from infinity. For $C<0$, the dimensional reduction takes place for the extra scale at a finite time t_0 when, however, R also tends to assume a finite value.

It is now possible to discuss the general behavior of the mass density ρ for various cases from the expressions (10a)–(10c). One finds that for all values of Λ ($>=<0$) the matter density approaches an infinitely large magnitude as $t\rightarrow 0$, which shows that a singularity exists at an initial instant. On the other hand in the course of expansion the density ρ decreases but finally assumes an extremely large value when dimensional reduction takes place ($A\rightarrow 0$); that is, the extra volume tends to zero. This happens with the appropriate choices of constants at a finite time in each case. If these constants are chosen such that there is no reduction of the extra dimension, the density ρ straight away vanishes at $t\rightarrow\infty$ in the first two cases, i.e., $\Lambda\geq 0$.

One must also note that in all the cases where the extra volume shrinks to zero the 5D volume R^3A as well vanishes at the same instant.

The most important conclusion of our present work is that the inhomogeneous 5D spacetime may account for inflation of three-space and a simultaneous contraction of the extra scale irrespective of the signature of the cosmological constant. It contrasts sharply with the observations of Tosa. Two desirable features of a multidimensional model are that the two scales should start expanding from the same instant and second as the three-space grows the extra space should shrink. The scenario presented by Tosa, however, does not exhibit both these features at the same instant. It happens, in general, only when the extra space is a compact hypersphere but not in the noncompact torus. In the latter it is observed that only a negative cosmological constant with more than one extra dimension can provide inflation of three-space and the reduction of the extra ones. Even then the big bang is nonsynchronous for the two scales. It is further observed by Tosa that for a 5D spacetime with a nonvanishing matter density, under no circumstances can we observe either a simultaneous bang for the two scales or compactification of the fifth dimension. However, the presence of inhomogeneity drastically alters the above scenario. In our inhomogeneous model ($C\neq 0$), when we get contraction of the fifth dimension and particularly in the special case of $a=0$, the two scales start off simultaneously as mentioned earlier.

Before ending up we would like to stress one very interesting characteristic of the dynamical behavior of our model. As noted earlier the COBE studies point to an unambiguous relic of the primordial inhomogeneity. Various mechanisms have been suggested for a smooth passover from the primordial inhomogeneous phase to the current homogeneous one; the scenario presented in our work seems to provide an alternative resolution to this problem in the following line. It is generally believed that during the compactification transition some dynamical mechanisms would stabilize the extra space at the Planckian length. Although a fully realistic model is still to be explored quantum gravity effects may be a possible candidate for such a mechanism. After the extra space stabilizes the cosmology enters the four-dimensional era without having any reference to the extra dimensions. When the above ansatz is extended to our inhomogeneous model its implication is more profound. Not only do we enter a four-dimensional era following dimensional reduction, it also envisages a smooth transition from multidimensional, inhomogeneous phase to a four-dimensional homogeneous one. So one can avoid choosing very special initial conditions for this purpose. Although we have not been able so far to relate the above mechanism to any specific model, the idea is challenging enough to warrant further investigations in this direction. As a future exercise one should consider the radiation-dominated universe to see if the presence of stress terms changes the above conclusions.

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