SO(10) unification in noncommutative geometry

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We construct an SO(10) grand unified theory in the formulation of noncommutative geometry. The geometry of space-time is that of a product of a continuous four-dimensional manifold times a discrete set of points. The properties of the fermionic sector fix almost uniquely the Higgs structure. The simplest model corresponds to the case where the discrete set consists of three points and the Higgs fields are $\mathbf{16}_s\times\overline{\mathbf{16}}_s$ and $\mathbf{16}_s\times\mathbf{16}_s$. The requirement that the scalar potential for all the Higgs fields not vanish imposes strong restrictions on the vacuum expectation values of the Higgs fields. We show that it is possible to remove these constraints by extending the number of discrete points to six and adding a singlet fermion and a $\mathbf{16}_s$ Higgs field. Both models are studied in detail.

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I. INTRODUCTION

Grand unified theories provide rather attractive mechanisms to unify the weak, electromagnetic, and strong interactions of particle physics and bring order to the choices of quark and lepton representations. At present, the simplest grand unified theories are formulated as spontaneously broken SU(5) and SO(10) gauge theories [1,2]. The SO(10) theories have the attractive feature that all known fermions, plus one right-handed neutrino, in every generation are included in one representation of SO(10). Unfortunately, this feature does not make the theories more predictive than, for example, the SU(5) theories, because there are many choices of patterns for spontaneously breaking SO(10) down to $\mathrm{SU}(3)\times \mathrm{U}(1)_{\mathrm{em}}$ related to different, complicated choices of Higgs fields in various representations of the gauge group [3]. What is called for, in the construction of grand unified theories, is a principle dictating the choice of the Higgs sector. During the past few years, considerable effort has been directed toward understanding this problem by deriving grand unified theories as low-energy limits of the heterotic string. Although this might fundamentally be the right strategy, it has proven to be a rather difficult one, due to the circumstance that it forces one to search for phenomenologically viable candidates among a very large number of string vacua.

It is the purpose of this paper to describe an alternative, probably less profound, but more direct strategy, based on the discovery due to Connes [4,5] and Connes and Lott [6,7] that methods from noncommutative geometry can be applied to (among many other things) model building in particle physics. These authors have shown that the Dirac operators on the one-particle Hilbert space of quarks and leptons are the germ for a geometrical construction of the standard $\mathrm{SU}(3)_c\times \mathrm{SU}(2)_W\times \mathrm{U}(1)_{\mathrm{em}}$ model. In their construction, the Higgs field appears as the component of a generalized gauge field on a generalized Euclidean space-time. Actually, the space-time underlying the Connes-Lott construction is the product of a four-dimensional Riemannian manifold by a discrete two-point set. The philosophy is that, just as electrodynamics gave rise to a new model of space-time, Minkowski space, the standard model of particle physics, or extensions thereof, may give rise to a modification of Minkowski space. Connes and Lott propose a minimal such modification.

One might hope that the geometrical underpinning of the Connes-Lott construction of the standard model could lead to some predictions going beyond those of the original formulation. At first, it seemed that the Connes-Lott construction gives rise to some additional constraints on coupling constants and mass matrices at the tree level (see, e.g., [8]). However, it was subsequently shown by Connes and Lott [7] that the most general construction of the standard model from noncommutative geometry has exactly the same number of free parameters at the tree level as the original construction.[1] Only for some special choices of the gauge field action (which, at first, appeared to be geometrically natural) do additional relations between masses and coupling constants emerge at the tree level. However, it has been shown in [10] that these relations are not stable under the renormalization flow and hence do not survive at the quantum level. Thus, unless new special symmetries are discovered that protect certain relations, one should start from the most

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[1]For other (partly prior) variants of the program initiated in [4–7], see the papers quoted in [9].

general Lagrangian provided by the Connes-Lott construction which is equivalent to the original Lagrangian of the standard model. Perhaps, this may be taken as bad news. However, it turns out that the Connes-Lott theory couples to gravity in a way that is somewhat different from that of the standard formulation. Since, in the Connes-Lott picture, "space-time" consists of two copies of a Riemannian four-manifold, gravitational interactions involve an additional scalar field that determines the *distance* between the two copies of conventional space-time [11]. This distance is related to the weak scale. The new scalar field, which must be considered a dynamical field, appears in the Coleman-Weinberg effective potential of the standard model [12]. This leads to certain inequalities between top-quark and Higgs boson masses, which, however, are not very interesting quantitatively, in the absence of an understanding of the problem of the cosmological constant.

Thus, to date, the main accomplishment of the constructions of the Lagrangian of the standard model by methods of noncommutative geometry described in [5–9] is that it provides a geometrical interpretation of the Higgs field which is unified with the electroweak gauge field. One of the *central observations* made in this paper is that when methods similar to those used in Refs. [5,6] are applied to construct Lagrangians of grand unified theories, certain fairly powerful constraints on the choices of the Higgs field representations emerge. Since the Higgs fields are unified geometrically with the gauge fields, this fact may not sound too surprising. However, it should be contrasted with the situation in the conventional approach, where the choice of the Higgs fields is highly ambiguous.

In a recent paper [13], it has been shown that, by a simple modification of the construction of Connes and Lott, it is possible to obtain unified models such as the SU(5) and the left-right $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ theories. Other models such as the flipped $SU(5)\times U(1)$ model are also within reach of these constructions. The interesting case of SO(10) theories was not treated, because it was not clear how to arrive at a satisfactory theory, in view of the fact that a realistic SO(10) model requires complicated Higgs representations. Meanwhile, it has turned out that the solution is fairly simple, and the construction of a realistic SO(10) model will be the main concern of this paper. All the tools that will be used here are explained in Ref. [13], and a self-contained summary can be found in Sec. 2 of the second item in Ref. [13]. (The results contained there will be freely used in this paper.)

The plan of this paper is as follows. In Sec. II we construct the Dirac operator underlying an SO(10) gauge theory and show that the simplest model involves three copies of conventional space-time. In Sec. III the symmetry-breaking chain is described in detail, and the vacuum expectation values (VEV's) of the Higgs fields are given. In Sec. IV the Higgs potential is analyzed, and it is shown that a potential survives after eliminating certain auxiliary fields *only* if the VEV's of the Higgs fields satisfy certain unphysical constraints. In Sec. V we show that it is possible to *relax* these constraints, provided that one starts from six copies of conventional space-time and certain symmetries are imposed.

II. SO(10) FRAMEWORK

The starting point in Connes' construction [4–8] is the specification of a fermionic sector and of a Dirac operator on the space of spinors, i.e., a model of supersymmetric quantum mechanics. In the SO(10) model [2], the fermions neatly fit in the $\mathbf{16}_s$ spinor representation, repeated three times. A single fermionic family is described by a field $\psi_{\alpha\hat{\alpha}}$, where α is an SO(1,3) Lorentz spinor index with 4 components and $\hat{\alpha}$ is an SO(10) spinor index with 32 components. It satisfies both space-time and SO(10) chirality conditions:

$$(\gamma_5)_\alpha^\beta \psi_{\beta\hat{\alpha}} = \psi_{\alpha\hat{\alpha}} \,, \qquad (\Gamma_{11})_{\hat{\alpha}}^{\hat{\beta}} \psi_{\alpha\hat{\beta}} = \psi_{\alpha\hat{\alpha}} \,, \tag{2.1}$$

where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, $\Gamma_{11} = -i\Gamma_0\Gamma_1\cdots\Gamma_9$, and, for later convenience, we have denoted Γ_{10} by Γ_0. This reduces the independent spinor components to 2 for the spacetime indices and to 16 for the SO(10) indices. The general fermionic action is given by

$$\bar{\psi}^p_{\alpha\hat{\alpha}}(\partial\!\!\!/ + A^{IJ}\Gamma_{IJ})^{\beta\hat{\beta}}_{\alpha\hat{\alpha}}\psi^p_{\beta\hat{\beta}} + \psi^{Tp}_{\alpha\hat{\alpha}} C^{\alpha\beta} H^{pq}_{\hat{\alpha}\hat{\beta}} \psi^q_{\beta\hat{\beta}} \,, \tag{2.2}$$

where C is the charge conjugation matrix, $p,q = 1,2,3$ are family indices, and H is some appropriate combination of Higgs fields breaking the subgroup $SU(2)\times U(1)$ of SO(10) at low energies. An exception of a Higgs field that breaks the symmetry at high energies and yet couples to fermions is the one that gives a Majorana mass to the right-handed neutrinos [11]. The other Higgs fields needed to break the SO(10) symmetry at high energies should not couple to the fermions so as not to give the quarks and leptons super heavy masses.

From the form of Eq. (2.2), we deduce that the gauge and Higgs fields are valued in a subalgebra of the Clifford algebra of SO(10), obtained by projection with the chirality operators acting on the right and left. The number of copies of conventional space-time needed to construct a model in noncommutative geometry is free and is only determined by the requirement of obtaining a realistic model. Since we know that, in a noncommutative construction, the Higgs fields are obtained by introducing more than one copy of Minkowski space, we need to choose a discrete space containing at least three points. On two of the copies, the associated spinors are taken to be identical, and the Higgs fields will not couple to the fermions, as these have the same chirality. On the third copy, the fermions are taken to be the conjugate spinors, as can be deduced from the second term of Eq. (2.2). Thus, between copies 1 and 2, we must impose a permutation symmetry, while between copies 1 and 3 we must require some form of conjugation symmetry. If we insist that the fermionic sector exhibit a Z_2 symmetry, then four copies of Minkowski space are necessary, with the third and fourth copies identified, too. This option will be pursued in the last section. Since both SO(1,3) and SO(10) have conjugation matrices, we take the conjugate spinor to be given by

$$\psi^c \equiv BC\bar{\psi}^T , \tag{2.3}$$

where B is the SO(10) conjugation matrix satisfying $B^{-1}\Gamma_I B = -\Gamma_I^T$. Thus the spinor used in a Connes-Lott type construction of an SO(10) model can be chosen as

$$\Psi = \begin{bmatrix} \psi \\ \psi \\ \psi^c \end{bmatrix} . \tag{2.4}$$

The chirality conditions on the spinor Ψ are given by

$$\begin{aligned} &\gamma_5 \otimes \mathrm{diag}(1,1,-1)\Psi = \Psi , \\ &\gamma_5 \otimes \Gamma_{11} \otimes \mathrm{diag}(1,1,-1)\Psi = \Psi . \end{aligned} \tag{2.5}$$

Before proceeding in our construction, it is useful to address the problem of neutrino masses. The right-handed neutrino must acquire a large mass. This is usually done by coupling the fermions to a **126** or to a $\mathbf{16}_s$ Higgs field with appropriate vacuum expectation values (VEV's) giving a mass to the right-handed neutrino but not to the remaining fermions. The **126** appears already with the Higgs fields that give masses to the fermions. The $\mathbf{16}_s$ can only be obtained by extending the fermionic space by a singlet spinor. This implies that the number of copies of Minkowski space must be increased by 1 or 2, depending on whether the Z_2 symmetry is required or not. In this case, two of the neutral fermions will become superheavy, while the third would remain massless. The fundamental spinor in our construction is then chosen to be

$$\begin{bmatrix} \psi \\ \psi \\ \psi^c \\ \psi^c \\ \lambda \\ \lambda^c \end{bmatrix} , \tag{2.6}$$

where the number of copies associated with the conjugate spinors is doubled. We shall first consider a spinor space corresponding to Eq. (2.4) and treat the more complicated case corresponding to Eq. (2.6) in the last section.

Next, we recall a few basic notions from noncommutative geometry that will be used in our construction of SO(10) models. A smooth manifold M can be studied by analyzing the commutative algebra $C^\infty(M)$ of smooth functions on M. In fact, M can be reconstructed from the structure of $C^\infty(M)$. The basic idea in noncommutative geometry is to define a notion of noncommutative space in terms of a noncommutative (non-Abelian) algebra $\mathcal{A}$. The mathematical structure becomes manageable if $\mathcal{A}$ is assumed to be an *involutive* algebra. This means that there is an antilinear involution $*$ taking $a \in \mathcal{A}$ to $a^* \in \mathcal{A}$ [a^* is the adjoint of a with $(a \cdot b)^* = b^* \cdot a^*$]. It simplifies the theory if one assumes that $\mathcal{A}$ contains an identity element 1. In this case one says that $\mathcal{A}$ is a unital, involutive algebra. It defines a notion of a *compact,* noncommutative space.

Given a unital, involutive algebra $\mathcal{A}$, one can define an algebra

$$\Omega^*(\mathcal{A}) = \bigoplus_{n=0}^{\infty} \Omega^n(\mathcal{A})$$

as the "universal, differential algebra" over $\mathcal{A}$ as follows: One sets $\Omega^0(\mathcal{A}) = \mathcal{A}$ and defines $\Omega^n(\mathcal{A})$ to be the linear space given by

$$\Omega^n(\mathcal{A}) = \left\{ \sum_i a_0^i da_1^i \cdots da_n^i : a_j^i \in \mathcal{A}, \ \forall i,j \right\} , \quad n = 1,2,\ldots .$$

Here da denotes an equivalence class of $a \in \mathcal{A}$, modulo the relations

$$d(a \cdot b) = (da) \cdot b + a \cdot db, \quad d1 = 0, \quad d^2 = 0 . \tag{2.7}$$

An element of $\Omega^n(\mathcal{A})$ is called a *form of degree n.* Let $\alpha \in \Omega^n(\mathcal{A})$ and $\beta \in \Omega^m(\mathcal{A})$. Because of the relations (2.7), one can define the product $\alpha \cdot \beta$ of α with β, and one verifies that $\alpha \cdot \beta \in \Omega^{n+m}(\mathcal{A})$; i.e., $\alpha \cdot \beta$ is a form of degree $n+m$. With this definition of a product of forms, $\Omega^*(\mathcal{A})$ becomes an algebra. Defining

$$(da)^* = -d(a^*) ,$$

one immediately deduces from the definition of $\Omega^n(\mathcal{A})$ and from (2.7) that, for $\alpha \in \Omega^n(\mathcal{A})$, α^* is defined and is again an element of $\Omega^n(\mathcal{A})$.

One-forms play a special role as components of connections on a "line bundle" whose space of sections is given by the algebra $\mathcal{A}$. *A* one-form $\rho \in \Omega^1(\mathcal{A})$ can be expressed as

$$\rho = \sum_i a^i db^i ,$$

a^i, b^i in $\mathcal{A}$, and, since $d1 = 0$, we may impose the condition that

$$\sum_i a^i b^i = 1 ,$$

without loss of generality.

So far, the theory is too general to be useful. In order to analyze a noncommutative space corresponding to a unital, involutive algebra $\mathcal{A}$ more concretely, we introduce the notion of a (Dirac) K cycle for $\mathcal{A}$. Let h be a separable Hilbert space, and let D be a self-adjoint operator on h. We say that (h,D) is a (Dirac) K-cycle for $\mathcal{A}$ if there exists an involutive representation π of $\mathcal{A}$ on h, i.e., a representation (or antirepresentation) of $\mathcal{A}$ satisfying $\pi(a^*) = \pi(a)^*$, with the properties that (i) $\pi(a)$ and $[D, \pi(a)]$ are bounded operators on h, for all $a \in \mathcal{A}$, and (ii) $(D^2+1)^{-1}$ is a compact operator on h. A K-cycle (h,D) for $\mathcal{A}$ is said to be (d,∞) summable iff the trace of $(D^2+1)^{-p/2}$ exists and is finite, for all $p > d$. A K-cycle (h,D) for $\mathcal{A}$ is said to be *even* iff there exists a unitary involution Γ on h, i.e., a bounded operator on h with $\Gamma^* = \Gamma^{-1} = \Gamma$, such that $[\Gamma, \pi(a)] = 0$ for all $a \in \mathcal{A}$, and $\{\Gamma, D\} = \Gamma D + D\Gamma = 0$. Otherwise, (h,D) is called *odd.*

Given a K-cycle (h,D) for $\mathcal{A}$, we define a representation π of $\Omega^*(\mathcal{A})$ on h by setting

$$\pi\left[\sum_i a_0^i da_1^i, \ldots, da_n^i\right] = \sum_i \pi(a_0^i)[D,\pi(a_1^i)]\cdots[D,\pi(a_n^i)] ,$$

for any element $\sum_i a_0^i da_1^i \cdots da_n^i \in \Omega^n(\mathcal{A})$, $n=0,1,2,\ldots$. We also define the spaces of *auxiliary fields*

$$\text{Aux} = \text{Ker}\pi + d\ \text{Ker}\pi ,$$

where

$$\text{Ker}\pi = \bigoplus_{n=0}^{\infty}\left\{\sum_i a_0^i da_1^i \cdots da_n^i : \pi\left[\sum_i a_0^i da_1^i \cdots da_n^i\right] = 0\right\}$$

and

$$d\ \text{Ker}\pi = \bigoplus_{n=0}^{\infty}\left\{\sum_i da_0^i da_1^i \cdots da_n^i : \pi\left[\sum_i a_0^i da_1^i \cdots da_n^i\right] = 0\right\} .$$

It follows from relations (2.7) that Aux is a two-sided ideal in $\Omega^*(\mathcal{A})$, and hence $\Omega_D^*(\mathcal{A})=\Omega^*(\mathcal{A})$ modulo Aux is a universal differential algebra. If $\sum_i a_0^i da_1^i \cdots da_n^i \in \Omega^n(\mathcal{A})$, then

$$\left\{\sum_i \pi(a_0^i)[D,\pi(a_1^i)]\cdots[D,\pi(a_n^i)]+\pi(\beta):\ \beta\in\text{Aux}\right\}$$

represents an n-form

$$\alpha = \sum_i a_0^i da_1^i \cdots da_n^i \mod \text{Aux} ,$$

in $\Omega_D^n(\mathcal{A})$ as an equivalence class of bounded operators on the Hilbert space h. For more details on these somewhat abstract mathematical notions, the reader is referred to [4,7,8,11].

Next, we consider special examples of noncommutative spaces and Dirac K-cycles on which the construction of SO(10) models is based. Let X be a compact, four-dimensional, smooth, Riemannian spin manifold, $\mathcal{A}_1$ the Abelian algebra of smooth functions on X. Since X is compact, $\mathcal{A}_1$ is a unital, involutive algebra. Let h_1 be the Hilbert space of square-integrable Dirac spinors on X, where the volume element used in the definition of integration is given by $\sqrt{g}\, d^4x$ and $g_{\mu\nu}$ is a Riemannian reference metric on X. Let D_1 be the covariant Dirac operator corresponding to $g_{\mu\nu}$ and to a fixed choice of a spin structure on X. Then D_1 is a self-adjoint operator on h_1 and (h_1,D_1) is a K-cycle for $\mathcal{A}_1$. In fact, it is an even K-cycle for $\mathcal{A}_1$, with the involution Γ given by γ^5.

Next, we define an algebra $\mathcal{A}_2$ by setting

$$\mathcal{A}_2 = P_+ \text{Cliff}[\text{SO}(10)]P_+ , \tag{2.8}$$

where $P_\pm = \frac{1}{2}(1\pm\Gamma_{11})$. Clearly, $\mathcal{A}_2$ is a unital, involutive algebra of matrices acting on the complex vector space $P_+\mathbb{C}^{32}$.

Our construction of SO(10) models is based on the noncommutative space described by the noncommutative, unital, involutive algebra

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$$

consisting of $\mathcal{A}_2$-valued, smooth functions on the "Euclidean space-time manifold" X.

Let π_0 denote the representation of $\mathcal{A}$ on the Hilbert space $h_1\otimes\tilde{h}_2$ of square-integrable spinors for SO(1,3)×SO(10), where $\tilde{h}_2=\mathbb{C}^{32}$ is the 32-dimensional vector space on which $\mathcal{A}_2$ acts. Let $\bar{\pi}_0$ denote the antirepresentation defined by

$$\bar{\pi}_0(a) = B\overline{\pi_0(a)}B^{-1} . \tag{2.9}$$

We define $\pi(a)$ by setting

$$\pi(a) = \pi_0(a)\oplus\pi_0(a)\oplus\bar{\pi}_0(a) . \tag{2.10}$$

Then $\pi(a)$ is a bounded operator on the Hilbert space

$$\tilde{h} = h_1\otimes(\tilde{h}_2^{(1)}\oplus\tilde{h}_2^{(2)}\oplus\tilde{h}_2^{(3)}) ,$$

where $\tilde{h}_2^{(i)}\simeq\tilde{h}_2$ are copies of $\mathbb{C}^{32}$, for $i=1,2,3$. Let $\hat{h}$ denote the subpsace of $\tilde{h}$, which is the image of the orthogonal projection onto elements of $\tilde{h}$ of the form

$$\begin{bmatrix} P_+\psi \\ P_+\psi \\ P_-\psi^c \end{bmatrix} ,$$

where $\psi\in h_1\otimes\tilde{h}_2$. Clearly, $T\hat{h}$ is invariant under $\pi(\mathcal{A})$ [as defined in (2.10)]. Finally, we set

$$h = \hat{h}\otimes\mathbb{C}^3 ,$$

where the factor $\mathbb{C}^3$ accounts for the fact that there are three families of quarks and leptons. On h we define a "Dirac operator" D by setting

$$D = D_1\otimes 1\otimes 1 + \gamma_5\otimes D_2 ,$$

where D_2 is a matrix on $(\tilde{h}_2^{(1)}\oplus\tilde{h}_2^{(2)}\oplus\tilde{h}_2^{(2)})\otimes\mathbb{C}^3$ with the property that $\gamma_5\otimes D_2$ leaves h invariant. Then one verifies without difficulties that (h,D) is a Dirac K-cycle for $\mathcal{A}$. Concretely, we choose the Dirac operator D to be given by

$$D = \begin{bmatrix} \partial\!\!\!/\otimes 1\otimes 1 & \gamma_5\otimes M_{12}\otimes K_{12} & \gamma_5\otimes M_{13}\otimes K_{13} \\ \gamma_5\otimes M_{21}\otimes K_{21} & \partial\!\!\!/\otimes 1\otimes 1 & \gamma_5\otimes M_{23} \\ \gamma_5\otimes M_{31}\otimes K_{31} & \gamma_5\otimes M_{32}\otimes K_{32} & \partial\!\!\!/\otimes 1\otimes 1 \end{bmatrix} , \tag{2.11}$$

where the K_{mn} are 3×3 family-mixing matrices commuting with $\pi(\mathcal{A})$. We impose the symmetries $M_{12}=M_{21}=\mathcal{M}_0$, $M_{13}=M_{23}=\mathcal{N}_0$, $M_{31}=M_{32}=\mathcal{N}_0^*$, with $\mathcal{M}_0=\mathcal{M}_0^*$. Similar conditions are imposed on the matrices K_{mn}. For D to leave the subspace h invariant, $\mathcal{M}_0$ and $\mathcal{N}_0$ must have the form

$$\mathcal{M}_0=P_+(m_0+im_0^{IJ}\Gamma_{IJ}+m_0^{IJKL}\Gamma_{IJKL})P_+ \ ,$$
$$\mathcal{N}_0=P_+(n_0^I\Gamma_I+n_0^{IJK}\Gamma_{IJK}+n_0^{IJKLM}\Gamma_{IJKLM})P_- \ , \quad (2.12)$$

where

$$\Gamma_{I_1I_2\cdots I_n}=\frac{1}{n!}\Gamma_{[I_1}\Gamma_{I_2}\cdots\Gamma_{I_n]}$$

are antisymmetrized products of the γ matrices.

Next, we define an involutive "representation" $\pi:\Omega^*(\mathcal{A})\to B(h)$ of $\Omega^*(\mathcal{A})$ by bounded operators on h [$B(h)$ is the algebra of bounded operators on h]: We set

$$\pi_0(a_0da_1da_2\cdots da_n)=\pi(a_0)[D,\pi(a_1)][D,\pi(a_2)]\cdots[D,\pi(a_n)] \ . \quad (2.13)$$

The image of a one-form ρ is

$$\pi(\rho)=\sum_i a^i[D,b^i], \quad \sum_i a^ib^i=1 \ . \quad (2.14)$$

From now on, we shall write a^i and b^i instead of $\pi(a^i)$ and $\pi(b^i)$, respectively. Every one-form ρ determines a covariant differentiation ∇ on h: We set

$$\nabla=D+\pi(\rho) \ . \quad (2.15)$$

Remark. One can think of h as a Hilbert space of sections of a vector bundle for the algebra $\mathcal{A}$. Then ∇ defines a notion of covariant derivative of a section and $\pi(\rho)$ corresponds to the components of the connection ∇.

The curvature of ∇ is then given by

$$\theta=\pi(d\rho)+\pi(\rho^2) \ , \quad (2.16)$$

where

$$\pi(d\rho)=\sum_i [D,\pi(a^i)][D,\pi(b^i)] \ .$$

It is straightforward to compute $\pi(\rho)$, and one gets [10]

$$\pi(\rho)=\begin{bmatrix} A & \gamma_5\mathcal{M}K_{12} & \gamma_5\mathcal{N}K_{13} \\ \gamma_5\mathcal{M}K_{12} & A & \gamma_5\mathcal{N}K_{23} \\ \gamma_5\mathcal{N}^*K_{31} & \gamma_5\mathcal{N}^*K_{32} & B\bar{A}B^{-1}\end{bmatrix} , \quad (2.17)$$

where the fields A, $\mathcal{M}$, and $\mathcal{N}$ are given in terms of the a^i and b^i by

$$A=P_+\left[\sum_i a^i\partial\!\!\!/ b^i\right]P_+ \ ,$$
$$\mathcal{M}+\mathcal{M}_0=P_+\left[\sum_i a^i\mathcal{M}_0b^i\right]P_+ \ , \quad (2.18)$$
$$\mathcal{N}+\mathcal{N}_0=P_+\left[\sum_i a^i\mathcal{N}_0B\bar{b}^iB^{-1}\right]P_- \ .$$

We can expand these fields in terms of the SO(10) Clifford algebra as follows:

$$A=P_+(ia+a^{IJ}\Gamma_{IJ}+ia^{IJKL}\Gamma_{IJKL})P_+ \ ,$$
$$\mathcal{M}=P_+(m+im^{IJ}\Gamma_{IJ}+m^{IJKL}\Gamma_{IJKL})P_+ \ , \quad (2.19)$$
$$\mathcal{N}=P_+(n^I\Gamma_I+n^{IJK}\Gamma_{IJK}+n^{IJKLM}\Gamma_{IJKLM})P_- \ .$$

The self-adjointness condition on $\pi(\rho)$ implies, after using the hermiticity of the Γ_I matrices, that all the fields appearing in the expansion of $A,\mathcal{M}$ are real because both are self-adjoint, while those in $\mathcal{N}$ are complex. The tracelessness condition is on $\mathrm{tr}[\Gamma_1\pi(\rho)]$ where Γ_1 is the grading operator given in the first equation of (2.5). This restricts $a=0$, and then this corresponds to the gauge theory of SU(16). In this case one must also add mirror fermions to cancel the anomaly, which will not be considered here. We shall require instead that the gauge fields acting on the first and third copies have identical components in the Clifford algebra basis. Since

$$B\bar{A}B^{-1}=P_-(-ia+a^{IJ}\Gamma_{IJ}-ia^{IJKL}\Gamma_{IJKL})P_- \ , \quad (2.20)$$

this implies that

$$a_\mu=0 \ ,$$
$$a_\mu^{IJKL}=0 \ . \quad (2.21)$$

The above requirement can be understood as the physical condition that the fermions in the first and third copies will have identical coupling to the gauge fields. Then the fermionic action will be given by

$$(\Psi,\mathcal{P}(d+\rho)\mathcal{P}\Psi)=\int d^4x\,\Psi^*(x)\mathcal{P}[D+\pi(\rho)]\mathcal{P}\Psi(x) \ , \quad (2.22)$$

where

$$\mathcal{P}=\mathrm{diag}(P_+,P_+,P_-) \ .$$

To transform this expression from Euclidean space to Minkowski space in order to impose the space-time chirality condition, we have to perform the following substitutions: $\gamma^4\to i\gamma^0$, $\gamma_5\to -i\gamma_5$, $\psi^*\to\bar{\psi}$, $\psi^{c*}\to-\bar{\psi}^c$. Because of space-time chirality, the field $\mathcal{M}$ decouples from the fermions. Then this is the field that must acquire a vacuum expectation value breaking SO(10) at very large energies. The field $\mathcal{N}$ does couple to fermions and must acquire expectation values that give the small fermionic masses, except for possible large values of the components that give a mass to the right-handed neutrino.

Now we are ready to write the fermionic action in terms of the component fields

$$I_f=\int d^4x\{2\bar{\psi}_+[i(\partial\!\!\!/+A)\psi_++\gamma_5(\mathcal{N}+\mathcal{N}_0)\psi_+^cK_{13}]+\bar{\psi}_+^c[i(\partial\!\!\!/+A)\psi_+^c+\gamma_5(\mathcal{N}^*+\mathcal{N}_0^*)\psi_+K_{13}^*]\} \ , \quad (2.23)$$

where $\psi_+=P_+\psi$ and by SO(10) chirality is equal to ψ. From here on and when convenient, we shall denote $\mathcal{M}$ by $P_+\mathcal{M}P_+$ and $\mathcal{N}$ by $P_+\mathcal{N}P_-$. Equation (2.23) can be simplified by using the properties of the charge conjugation matrices B and C:

$$B^{-1}\Gamma_IB=-\Gamma_I^T \ ,$$
$$C^{-1}\gamma_\mu C=-\gamma_\mu^T \ . \quad (2.24)$$

After rescaling $\psi\to(1/\sqrt{3})\psi$, the action (2.23) simplifies to

$$I_f=\int d^4x\left[\bar{\psi}_+ i(\partial\!\!\!/+A)\psi_+ -\frac{1}{\sqrt{3}}[\psi_+^T B^{-1}C^{-1}(\mathcal{N}^*+\mathcal{N}_0^*)\psi_+ K_{13}^* + \text{H.c.}]\right] . \quad (2.25)$$

Thus we have achieved our goal of constructing a Dirac operator that gives the appropriate interactions of an SO(10) unified gauge theory.

III. SO(10) SYMMETRY BREAKING

The symmetry-breaking pattern that breaks the gauge group SO(10) must be coded into the Dirac operator D. The Higgs fields at our disposal are $\mathcal{M}$ and $\mathcal{N}$. In terms of SO(10) representations these are **1**, **45**, and **210** in $\mathcal{M}$ and complex **10**, **120**, and **126** in $\mathcal{N}$. To be explicit we shall work in a specific Γ-matrix representation first introduced by Georgi and Nanopolous [2]. The 32×32 Γ matrices are represented in terms of tensor products of five sets of Pauli matrices $\sigma_i,\tau_i,\eta_i,\rho_i,\kappa_i$ where $i=1,2,3$. To these matrices we assign the following matrices on the tensor product space:

$$\begin{aligned}
&\sigma_i\rightarrow 1_2\otimes 1_2\otimes 1_2\otimes 1_2\otimes\sigma_i ,\\
&\tau_i\rightarrow 1_2\otimes 1_2\otimes 1_2\otimes\tau_i\otimes 1_2 ,\\
&\eta_i\rightarrow 1_2\otimes 1_2\otimes\eta_i\otimes 1_2\otimes 1_2 ,\\
&\rho_i\rightarrow 1_2\otimes\rho_i\otimes 1_2\otimes 1_2\otimes 1_2 ,\\
&\kappa_i\rightarrow\kappa_i\otimes 1_2\otimes 1_2\otimes 1_2\otimes 1_2 .
\end{aligned} \quad (3.1)$$

The Γ matrices are then given by

$$\begin{aligned}
&\Gamma_i=\kappa_1\rho_3\eta_i ,\\
&\Gamma_{i+3}=\kappa_1\rho_1\sigma_i ,\\
&\Gamma_{i+6}=\kappa_1\rho_2\tau_i ,\\
&\Gamma_0=\kappa_2 ,\\
&\Gamma_{11}=\kappa_3 ,
\end{aligned} \quad (3.2)$$

where $i=1,2,3$, and when it is obvious, we shall omit the tensor product symbols. In this basis an SO(10) chiral spinor will take the form

$$\psi_+=\begin{bmatrix}\chi_+\\ 0\end{bmatrix} , \quad (3.3)$$

where χ is a $\mathbf{16}_s$ in the space $V_\rho\otimes V_\eta\otimes V_\tau\otimes V_\sigma$, with $V_\rho\equiv\cdots\equiv V_\sigma\equiv C^2$. The SO(10) conjugation matrix is defined by $B\equiv-\Gamma_1\Gamma_3\Gamma_4\Gamma_6\Gamma_8$, which, in the basis of Eq. (3.2) becomes

$$B=\kappa_1\rho_2\eta_2\tau_2\sigma_2\equiv\kappa_1 b , \quad (3.4)$$

where the matrix $b=\rho_2\eta_2\tau_2\sigma_2$ is the conjugation matrix in the space of the 16 component spinors. The action of B on a chiral spinor is then

$$B\psi_+=\begin{bmatrix}0\\ b\chi_+\end{bmatrix} . \quad (3.5)$$

The advantage of this system of matrices is that both spinors χ_+ and $bC\bar{\chi}_+{}^T$ have the same form, except that the first one is left handed and the second one is right handed. To correctly associate the components of χ_+ with quarks and leptons, we consider the action of the charge operator [3] on χ_+:

$$\begin{aligned}
Q&=\frac{i}{6}(\Gamma_{45}+\Gamma_{69}+\Gamma_{78})-\frac{i}{2}\Gamma_{12}\\
&=-\tfrac{1}{6}(\sigma_3+\tau_3+\rho_3\tau_3\sigma_3)+\tfrac{1}{2}\eta_3 ,
\end{aligned} \quad (3.6)$$

which gives

$$Q\chi_+=\text{diag}(0,\tfrac{2}{3},\tfrac{2}{3},\tfrac{2}{3},-1,-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},1, -\tfrac{2}{3},-\tfrac{2}{3},-\tfrac{2}{3},0)\chi_+ . \quad (3.7)$$

Thus the components of the left-handed spinor χ_+ are written as

$$\chi_+=\begin{bmatrix} n_L\\ u_L^1\\ u_L^2\\ u_L^3\\ e_L\\ d_L^1\\ d_L^2\\ d_L^3\\ -(d_R^3)^c\\ (d_R^2)^c\\ (d_R^1)^c\\ -(e_R)^c\\ (u_R^3)^c\\ -(u_R^2)^c\\ -(u_R^1)^c\\ (n_R)^c \end{bmatrix} , \quad (3.8)$$

where the c in this equation stands for the usual charge conjugation, e.g., $d^c=C\bar{d}^T$. The upper and lower components in χ are mirrors, with the signs chosen so that the spinor $bC\bar{\chi}_+{}^T$ has exactly the same form as χ_+, but with the left- and right handed signs L and R interchanged.

We now specify the vacuum expectation values (VEV's) $\mathcal{M}_0$ and $\mathcal{N}_0$. The group SO(10) is broken at high energies by $\mathcal{M}$, which contains the representations **45** and **210**. By taking the VEV of the **210** to be $\mathcal{M}^{0123}=O(M_G)$, the SO(10) symmetry is broken to $SO(4)\times SO(6)$, which is isomorphic to $SU(4)_c\times SU(2)_L\times SU(2)_R$. The $SU(4)_c$ is further broken to $SU(3)_c\times U(1)_c$ by the VEV of the **45**.

Therefore we write [2,3]

$$P_+\mathcal{M}_0P_+=P_+([M_G\Gamma_{0123}-iM_1(\Gamma_{45}+\Gamma_{78}+\Gamma_{69})]P_+ \\ =\tfrac{1}{2}(1+\kappa_3)[-M_G\rho_3+M_1(\sigma_3+\tau_3+\rho_3\tau_3\sigma_3)] . \quad (3.9)$$

Therefore $\mathcal{M}_0$ breaks SO(10) to $SU(3)_c\times U(1)_c \times SU(2)_L\times SU(2)_R$, which is also of rank 5. The rank is reduced by giving a VEV to the components of **126** that couple to the right-handed neutrino. Therefore the VEV of $\mathcal{N}_0$ must contain the term

$$M_2\left[\frac{1}{2^5}\right](\kappa_1+i\kappa_2)(\rho_1+i\rho_2)(\eta_1+i\eta_2) \times(\tau_1+i\tau_2)(\sigma_1+i\sigma_2) . \quad (3.10)$$

In terms of the Γ matrices, Eq. (3.8) has a rather complicated form

$$\tfrac{1}{8}(\{[\Gamma_{13489}+i(1\to 2)]+i(4\to 5)\}-i(8\to 7)) . \quad (3.11)$$

The VEV of $\mathcal{N}_0$ breaks $U(1)_c\times SU(2)_R$ to $U(1)_Y$, and the surviving group would be the familiar $SU(3)_c \times SU(2)_L\times U(1)_Y$. The generators of $SU(2)_L\times SU(2)_R$ are [2]

$$T^i_{L,R}=-\frac{i}{2}(\tfrac{1}{2}\epsilon^{ijk}\Gamma_{jk}\pm\Gamma^{i0}) \\ =\tfrac{1}{2}(1\pm\kappa_3\rho_3)\eta^i , \quad (3.12)$$

while $SU(4)_c$ is generated by

$$-i\Gamma_{i+3,j+3}=\epsilon_{ijk}\sigma^k , \\ -i\Gamma_{i+6,j+6}=\epsilon_{ijk}\tau^k , \quad (3.13) \\ -i\Gamma_{i+3,j+6}=\rho_3\tau_j\sigma_i .$$

It is straightforward to check that the only generators that leave $\mathcal{M}_0$ and the part of $\mathcal{N}_0$ given by (3.10) invariant are those of the standard model. We shall explicitly identify these generators in order to proceed to the next stage of breaking $SU(2)_L\times U(1)_Y$ without any ambiguity. The eight SU(3) generators are given by $(1-\rho_3\tau_3)\sigma_i$, $(1-\rho_3\sigma_3)\tau_i$, $\rho_3(\tau_1\sigma_1+\tau_2\sigma_2)$, and $\rho_3(\tau_2\sigma_1-\tau_1\sigma_2)$. Finally, the $U(1)_Y$ generator is

$$Y=-\tfrac{1}{3}(\sigma_3+\tau_3+\rho_3\tau_3\sigma_3)+\tfrac{1}{2}(1-\kappa_3\rho_3)\eta_3 , \quad (3.14)$$

and its action on the spinor χ_+ is given by

$$Y\chi_+=\text{diag}(-1,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},-1,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\tfrac{2}{3},\tfrac{2}{3},\tfrac{2}{3},2,-\tfrac{4}{3},-\tfrac{4}{3},-\tfrac{4}{3},0)\chi_+ . \quad (3.15)$$

This is related to the charge operator Q by

$$Q=\tfrac{1}{2}Y+T^3_L , \quad (3.16)$$

where the action of the $SU(2)_L$ isospin T^3_L on χ_+ is given by $T^3_L=\tfrac{1}{2}(1+\rho_3)\eta_3$.

For the last stage of symmetry breaking of $SU(2)_L\times U(1)_Y$, we can use the field $\mathcal{N}$, which contains the complex representations **10**, **120**, and **126**. The most general VEV that preserves the group $SU(3)_c\times U(1)_Q$ is

$$P_+\mathcal{N}_0P_-=\tfrac{1}{2}(1+\kappa_3)\{(is\Gamma_0+p\Gamma_3)+[a'\Gamma_{120}-ia\Gamma_{123}+b'(\Gamma_{453}+\Gamma_{783}+\Gamma_{693})-ib(\Gamma_{450}+\Gamma_{690}+\Gamma_{780})] \\ -[ie(\Gamma_{01245}+\Gamma_{01269}+\Gamma_{01278})+f(\Gamma_{31245}+\Gamma_{31269}+\Gamma_{31278})]\}+\text{term in (3.11)} . \quad (3.17)$$

Use of the explicit matrix representation for the Γ matrices simplifies Eq. (3.17) to

$$P_+\mathcal{N}_0P_-\kappa_1=\tfrac{1}{2}(1+\kappa_3)\Bigg[s+p\rho_3\eta_3+a\rho_3+a'\eta_3+(b'+b\rho_3\eta_3+e\eta_3+f\rho_3)(\sigma_3+\tau_3+\rho_3\tau_3\sigma_3) \\ +M_2\left[\frac{1}{2^5}\right](\rho_1+i\rho_2)(\eta_1+i\eta_2)(\tau_1+i\tau_2)(\sigma_1+i\sigma_2)\Bigg] , \quad (3.18)$$

where all terms containing η_3 break $SU(2)_L\times U(1)_Y$. Having specified all the VEV's that break SO(10) down to the low-energy symmetry, it is straightforward, though tedious, to write down the fermionic masses generated through the symmetry breaking. These are

$$I_{f\text{ mass}}=-\frac{1}{\sqrt{3}}\int d^4x(\{[s+p+3(e+f)]K_{(pq)}+[a+a'+3(b+b')]K_{[pq]}\}\bar{N}^p_R N^q_L \\ +\{[s+p-(e+f)]K_{(pq)}+[a+a'-(b+b')]K_{[pq]}\}\bar{u}^p_R u^q_L \\ +\{[s-p-3(e-f)]K_{(pq)}+[a-a'-3(b-b')]K_{[pq]}\}\bar{e}^p_R e^q_L \\ +[(s-p+e-f)K_{(pq)}+(a-a'+b-b')K_{[pq]}]\bar{d}^p_R d^q_L+[M_2K_{(pq)}(N^{pc}_R)^T C^{-1}N^{qc}_R]+\text{H.c.}) , \quad (3.19)$$

where we have denoted the family mixing matrix K_{13} by K. For the neutral fields N_L and N_R, we have a seesaw mechanism, giving the right-handed neutrino a large Majorana mass [14,15], and the neutrino mass matrix takes the simple form (ignoring generation mixing)

$$\begin{array}{c|cc} & N_L & N_R^c \\ \hline N_L & 0 & m \\ N_R^c & m & M_2 \end{array} , \tag{3.20}$$

where m is of the order of the weak scale. This matrix has two eigenstates of masses M_2 and m^2/M_2. The free parameters at this stage are M_G, M_1, M_2, a, a', b, b', e, f, s, and p and the matrix K_{pq}. However, when we examine the scalar potential in the next section, it will become clear that in order for the potential, or some terms in it, not to vanish, the above parameters must be related. Also we note that, since both the symmetric and antisymmetric parts of K_{pq} enter the fermionic mass matrix, it cannot be completely removed. By performing a unitary transformation on $\chi_+^p \rightarrow U_q^p \chi_+^q$ such that $U^* U = 1$, the matrix K_{pq} is transformed to $(U^T K U)_{pq}$. Since K is an arbitrary complex matrix, the matrix U can be used to eliminate 9 out of the 18 real parameters. We shall come back to the fermionic mass terms after having examined the bosonic sector.

IV. BOSONIC ACTION

In the noncommutative formulation of the Yang-Mills action, an essential ingredient is the Dirac operator. The curvature of the one-form ρ is defined by

$$\theta = d\rho + \rho^2 . \tag{4.1}$$

The Yang-Mills action in the noncommutative setting is given by

$$I_b = \tfrac{1}{4} \mathrm{Tr}_\omega (\theta^2 |D|^{-4}) , \tag{4.2}$$

where Tr_ω is the Dixmier trace. It was shown in [11] that one can equivalently use the heat-kernel expression

$$\lim_{\epsilon \to 0} \frac{\mathrm{tr}(\theta^2 e^{-\epsilon |D|^2})}{\mathrm{tr}(e^{-\epsilon |D|^2})} . \tag{4.3}$$

For both definitions it can be shown that the Yang-Mills action is equal to [4,5]

$$I = \tfrac{1}{4} \int d^4x \, \mathrm{Tr}\{ \mathrm{tr}[\pi^2(\theta)] \} . \tag{4.4}$$

To compute $\pi(\theta)$, the expression $\pi(d\rho)$ must be evaluated from the definition of ρ:

$$\pi(d\rho) = \sum_i [D, a^i][D, b^i] , \tag{4.5}$$

and this must be expressed in terms of the fields appearing in $\pi(\rho)$. Since $\pi(d\rho)$ is not necessarily zero when $\pi(\rho)$ is, one must quotient out the space $\mathrm{Ker}(\pi) + d\,\mathrm{Ker}(\pi)$. The space of auxiliary fields can be determined by computing $\pi(d\rho)|_{\rho=0}$. Since the Yang-Mills action is qudratic in the curvature θ, the process of working on the quotient space is equivalent to keeping nondynamical auxiliary fields and eliminating them through their equations of motion. The remaining potential of the Higgs fields would be orthogonal to the space of auxiliary fields. Depending on the VEV's of the Higgs fields, it can happen that the potential for some terms vanishes. This means that the potential becomes flat in certain directions and that some VEV's are not determined at the classical level. This could be cured at the quantum level when radiative corrections to the potential are taken into account.

The Yang-Mills action in Eq. (4.4) has been derived for an N point space in [13]. Here we simply quote the result

$$\begin{aligned} I = \sum_{m=1}^{N} \mathrm{Tr} \Bigg[& \tfrac{1}{2} F_{\mu\nu}^m F^{\mu\nu m} - \Bigg| \sum_{p \neq m} |K_{mp}|^2 |\phi_{mp} + M_{mp}|^2 - (Y_m + X'_{mm}) \Bigg|^2 \\ & + \sum_{p \neq m} |K_{mp}|^2 \left| \partial_\mu (\phi_{mp} + M_{mp}) + A_{\mu m} (\phi_{mp} + M_{mp}) - (\phi_{mp} + M_{mp}) A_{\mu p} \right|^2 \\ & - \sum_{n \neq m} \sum_{p \neq m,n} \left| K_{mp} K_{pn} [(\phi_{mp} + M_{mp})(\phi_{pn} + M_{pn}) - M_{mp} M_{pn}] - X_{mn} \right|^2 \Bigg] , \end{aligned} \tag{4.6}$$

where the A^m are the gauge fields in the $m-m$ entry of $\pi(\rho)$ and ϕ_{mn} are the scalar fields in the $m-n$ entry of $\pi(\rho)$. The X_{mn}, X'_{mn}, and Y_m are fields whose unconstrained elements are auxiliary fields that can be eliminated from the action. Their expressions in terms of the a^i and b^i are

$$X_{mn} = \sum_i a_m^i \sum_{p \neq m,n} K_{mp} K_{pn} (M_{mp} M_{pn} b_n^i - b_m^i M_{mp} M_{pn}) , \quad m \neq n , \tag{4.7}$$

$$X'_{mm} = \sum_i a_m^i \partial^2 b_m^i + (\partial^\mu A_\mu^m + A^{\mu m} A_\mu^m) , \tag{4.8}$$

$$Y_m = \sum_{p \neq m} \sum_i a_m^i |K_{mp}|^2 |M_{mp}|^2 b_m^i . \tag{4.9}$$

In the case at hand, the discrete space has three points. Because of the permutation and complex conjugation symmetry, the a_m^i are related to each other. This in turn relates some of the auxiliary fields to one another. To use Eq. (4.5), we must compute the different terms as functionals of the component fields appearing in $\pi(\rho)$. We first write

$$A=\frac{g}{4}\gamma^{\mu}A_{\mu}^{IJ}\Gamma_{IJ} ,$$

where g is the SO(10) gauge coupling constant. Then the kinetic term for the gauge field A_{μ}^{IJ} as given by the first term in Eq. (4.6), after computing the sum and the trace over Cliff [SO(10)], is equal to

$$-4g^2F_{\mu\nu}^{IJ}F^{\mu\nu IJ} , \tag{4.10}$$

where the field strength is

$$F_{\mu\nu}^{IJ}=\partial_{\mu}A_{\nu}^{IJ}-\partial_{\nu}A_{\mu}^{IJ}+g(A_{\mu}^{IK}A_{\nu}^{KJ}-A_{\nu}^{IK}A_{\mu}^{KJ}) . \tag{4.11}$$

The Higgs kinetic terms have two parts, corresponding to $\mathcal{M}$ and $\mathcal{N}$. Using the decomposition of $\mathcal{M}$ and $\mathcal{N}$ in the Cliff [SO(10)] basis, one gets the result

$$\begin{aligned}64\,\mathrm{Tr}|K_{12}|^2\{&(\partial_{\mu}m)^2+2[D_{\mu}(m+m_0)_{IJ}]^2\\&+4[D_{\mu}(m+m_0)_{IJKL}]^2\}\\&+64|K_{13}|^2[|D_{\mu}(n+n_0)_I|^2+3|D_{\mu}(n+n_0)_{IJK}|^2\\&+5|D_{\mu}(n+n_0)_{IJKLM}|^2] ,\end{aligned} \tag{4.12}$$

where the D appearing in this equation is the covariant derivative with respect to the SO(10) gauge group and the m's and n's are defined in Eq. (2.19). For example, $D_{\mu}n_I=\partial_{\mu}n_I+gA_{\mu}^{IJ}n_J$. The masses of the components of the gauge fields A_{μ}^{IJ} corresponding to the broken generators of SO(10) are provided by the VEV's $\mathcal{M}_0$ and $\mathcal{N}_0$. The most complicated part is the Higgs potential, since this involves new fields, some of which are related, and the nondynamical ones must be eliminated through their equations of motion. It is given by

$$\begin{aligned}V(\mathcal{M},\mathcal{N})=2\left||K_{12}|^2|\mathcal{M}+\mathcal{M}_0|^2+|K_{13}|^2|\mathcal{N}+\mathcal{N}_0|^2-(Y_1+X'_{11})\right|^2+\left||K_{31}|^2|\mathcal{N}+\mathcal{N}_0|^2+|K_{12}|^2|\mathcal{M}+\mathcal{M}_0|^2-(Y_3+X'_{33})\right|^2\\+2\left||K_{13}|^2|\mathcal{N}+\mathcal{N}_0|^2-|\mathcal{N}_0|^2-X_{12}|^2+2|K_{12}K_{23}[(\mathcal{M}+\mathcal{M}_0)(\mathcal{N}+\mathcal{N}_0)-\mathcal{M}_0\mathcal{N}_0]-X_{13}\right|^2 ,\end{aligned} \tag{4.13}$$

where we have used the symmetry that equates some of the K's and X's. We now write the explicit expressions for the X and Y fields. First, we have

$$\begin{aligned}X'_{11}&=\sum_i a^i\partial^2b^i+(\partial^{\mu}A_{\mu}+A^{\mu}A_{\mu}) ,\\X'_{33}&=B\overline{X}'_{11}B^{-1} .\end{aligned} \tag{4.14}$$

Next, we have, for the Y's,

$$\begin{aligned}Y_1&=\sum_i a^i|K_{12}|^2|\mathcal{M}_0|^2b^i+2\sum_i a^i|K_{13}|^2|\mathcal{N}_0|^2b^i ,\\Y_3&=B\overline{Y}_1B^{-1} .\end{aligned} \tag{4.15}$$

Finally, we have for the X_{mn}, $m\neq n$, the expressions

$$\begin{aligned}X_{12}&=|K_{13}|^2\left[\sum_i a^i|\mathcal{N}_0|^2B\overline{b}^iB^{-1}-|\mathcal{N}_0|^2\right] ,\\X_{13}&=K_{12}K_{23}\left[\sum_i a^i\mathcal{M}_0\mathcal{N}_0B\overline{b}^iB^{-1}-\mathcal{M}_0\mathcal{N}_0\right] ,\end{aligned} \tag{4.16}$$

and the other X's related to the above ones by permutation symmetry. It is easy to notice that X'_{11} and X'_{33} are auxiliary fields that do not depend on the K matrices. Therefore eliminating these fields would result in expressions orthogonal to the corresponding K space. Eliminating the remaining auxiliary fields Y_1, Y_3, X_{12}, and X_{13} is much more complicated. If all of these were independent, the potential would vanish, after eliminating them. However, if the VEV's $\mathcal{M}_0$ and $\mathcal{N}_0$ are chosen in a special way, then it is possible for the potential to survive. One must arrange for a relation between the auxiliary fields, so that, after eliminating the independent combinations, the potential that corresponds to the given vacuum will result. A close look at the potential in Eq. (4.13) shows that if all of the X and Y fields are independent, the potential disappears after eliminating them. In this case one would have a flat potential and the true vacuum must be determined from the radiative corrections. The presence of the auxiliary fields ensure that the potential is gauge invariant. By comparing X_{12} and Y_1, one sees that they can be related only if $\sum a^i|\mathcal{M}_0|^2b^i$ is not an independent field. This can happen if

$$M_G=M_1 , \tag{4.17}$$

so that $|\mathcal{M}_0|^2=4M_1^2$, and we get the relation

$$Y_1=|K_{12}|^2|\mathcal{M}_0|^2+|K_{13}|^2|\mathcal{N}_0|^2+X_{12} . \tag{4.18}$$

Next, for the term in the potential depending on X_{13} not to vanish, X_{13} must not be an independent field and must be a function of $\mathcal{N}$. This is possible if $\mathcal{M}_0\mathcal{N}_0$ is proportional to $\mathcal{N}_0$. This condition is extremely restrictive, but fortunately has one solution given by

$$\begin{aligned}&\mathcal{M}_0\mathcal{N}_0=2M_1\mathcal{N}_0 ,\\&a'=b'=0 ,\\&f=-s=\frac{a}{2} ,\\&p=3e=\tfrac{3}{2}b ,\end{aligned} \tag{4.19}$$

and the free parameters in the theory are M_1,M_2,a,b and the matrices K_{12},K_{13}. The equation for X_{13} simplifies to

$$X_{13}=K_{13}(2M_1\mathcal{N}) . \tag{4.20}$$

Then the only independent fields to be eliminated are X_{12} and X'_{11}. The resulting potential is

$$V(\mathcal{M},\mathcal{N})=[\mathrm{Tr}|K_{12}|^4-(\mathrm{Tr}|K_{12}|^2)^2]||\mathcal{M}+\mathcal{M}_0|^2-4M_1^2|^2 +2\,\mathrm{Tr}|K_{12}K_{13}|^2|(\mathcal{M}+\mathcal{M}_0-2M_1)(\mathcal{N}+\mathcal{N}_0)|^2 \;. \tag{4.21}$$

We note that the form of the potential (4.21), being the sum of squares, is very restrictive. The most general potential one constructs in the conventional approach using the same Higgs representations would involve many more terms. By insisting on the use of the Higgs in the combinations $\mathcal{M}$ and $\mathcal{N}$, the number of possible terms would decrease. Only when the further requirement of having a potential which is a sum of squares is imposed will the potentials in the noncommutative construction and in the conventional approach agree. The total bosonic action is the sum of the terms (4.10), (4.12), and (4.21), multiplied by an overall constant. We choose this constant to be $1/16g^2$ to get the canonical kinetic energy for the gauge fields. The kinetic energy for the scalar fields $\mathcal{M}$ and $\mathcal{N}$ is normalized canonically after rescaling

$$\mathcal{M}\to\frac{g}{2\sqrt{2\,\mathrm{Tr}|K'|^2}}\mathcal{M}\;,\qquad \mathcal{N}\to\frac{g}{2\sqrt{\mathrm{Tr}|K|^2}}\mathcal{N}\;, \tag{4.22}$$

where we have denoted K_{13} by K and K_{12} by K'. After rescaling, the bosonic action becomes

$$I_{\mathrm{bosonic}}=\int d^4x\left\{-\frac{1}{4}F^{IJ}_{\mu\nu}F^{\mu\nu IJ}+\frac{1}{32}\mathrm{Tr}\left[\frac{1}{2}[D_\mu(\mathcal{M}+\mathcal{M}_0)]^2+|D_\mu(\mathcal{N}+\mathcal{N}_0)|^2\right] +\frac{g^2}{2^5\times 32}\left[\frac{\mathrm{Tr}|K'|^4}{(\mathrm{Tr}|K'|^2)^2}-1\right]\mathrm{Tr}||\mathcal{M}+\mathcal{M}_0|^2-4M_1^2|^2+\frac{g^2}{2^3\times 32}|(\mathcal{M}+\mathcal{M}_0-2M_1)(\mathcal{N}+\mathcal{N}_0)|^2\right\}\;. \tag{4.23}$$

Finally, the fermionic action becomes

$$I_f=-\frac{g}{\sqrt{3\,\mathrm{Tr}|K|^2}}\int d^4x\{K_{pq}[(a+3b)\bar{N}^p_R N^q_L+(a-3b)\bar{e}^p_R e^q_L] +K_{qp}[(-a+b)\bar{u}^p_R u^q_L-(a+b)\bar{d}^p_R d^q_L]+M_2K_{(pq)}(N^p_R)^TC^{-1}N^q_R+\mathrm{H.c.}\}\;. \tag{4.24}$$

By examining the gauge kinetic term, one finds the usual SO(10) relations among the gauge coupling constants

$$g_2=g_3=g=(\tfrac{5}{3})^{1/2}g_1\;, \tag{4.25}$$

implying that $\sin^2\theta_W$ at the unification scale M_1 is $\frac{3}{8}$. From the $\mathcal{N}$-kinetic term, one sees that the mass of the W gauge boson is

$$m_W^2=\frac{g^2}{4}(a^2+3b^2)\;. \tag{4.26}$$

From the fermionic mass terms, one deduces, using the fact that the top quark mass is much heavier than the other fermionic masses, that

$$m_t=g|b-a|\;. \tag{4.27}$$

Comparing with m_W, we get the relation

$$m_t=2m_W\frac{|1-b/a|}{\sqrt{1+3b^2/a^2}}\;, \tag{4.28}$$

and this gives upper and lower bounds on the top quark mass,

$$\frac{2}{\sqrt{3}}m_W\le m_t\le\frac{4}{\sqrt{3}}m_W=186.13\ \mathrm{GeV}\;, \tag{4.29}$$

which agrees with present experimental limits. However, such relations could not be maintained at the quantum level [10]. Unfortunately, the same matrix K_{qp} appears for the u^p and d^p quarks, implying that the same transformation can be used for u^p and d^p to diagonalize K_{qp}. This in turn implies that this model does not allow for a Cabibbo angle, and this phenomenologically rules out this model. At this point one option would be to relax some of the constraints such as Eq. (4.19), which would make the potential flat in certain directions. This would make the fermionic masses given by (3.19) not suffer from the absence of a Cabibbo angle. Only a study of the radiative corrections and minimizing the effective action can determine whether this possibility is realistic. Alternatively, we can look for modifications in the building structures so that this model becomes acceptable. This result shows that model building in noncommutative geometry is so constrained that most models could be ruled out on phenomenological grounds.

V. REALISTIC SO(10) MODEL

The model presented in the previous sections is minimal in the sense that the number of points in the internal geometry and the Higgs fields cannot be reduced. If one insists on a Z_2 symmetry between the different copies, then the number of points would have to be even, and we have to take two copies where the conjugate spinors are placed, instead of the one copy considered before. It will be seen that this extension cannot have a potential after eliminating the auxiliary fields. Therefore this model has to be further extended by one or two points to get the $\mathbf{16}_s$ Higgs field, and this will ensure that the potential can be arranged to survive. The fermionic space is extended with a singlet spinor. Two of the neutral fermions will become superheavy, while the third one would remain massless. The fermionic space will then be given by

$$\Psi\equiv\begin{bmatrix}\psi\\ \psi\\ \psi^c\\ \psi^c\\ \lambda\\ \lambda^c\end{bmatrix}, \tag{5.1}$$

and $\mathcal{A}_2$ is

$$\mathcal{A}_2\equiv \text{Cliff}[SO(10)]\oplus R\ . \tag{5.2}$$

The Hilbert space is the direct sum of the Hilbert space C^{32} and the Hilbert space C. The involutive map π is now taken to be

$$\pi=\pi_0\oplus\pi_0\oplus\overline{\pi}_0\oplus\overline{\pi}_0\oplus\pi_1\oplus\pi_1\ . \tag{5.3}$$

To every element $f\in\mathcal{A}$, we associate a sixtet $(f_1,\ldots,f_6)$, of matrix-valued functions on X, where $f_1,\ldots,f_4$ are 32×32 matrices, and f_5,f_6 are 1×1 matrices. The decomposition of π corresponds to

$$f\to\text{diag}(f_1,\ldots,f_6)\ , \tag{5.4}$$

with the symmetry

$$\begin{aligned}&f_1=f_2=a\ ,\\ &f_3=f_4=B\overline{a}B^{-1}\ ,\\ &f_5=f_6=f'\ ,\end{aligned} \tag{5.5}$$

i.e., a permutation symmetry $1\leftrightarrow 2$ and $3\leftrightarrow 4$. In this decomposition, the operator D becomes

$$D=\begin{bmatrix}\partial\!\!\!/\otimes 1\otimes 1 & \gamma_5\otimes M_{12}\otimes K_{12} & \cdots & \gamma_5\otimes M_{16}\otimes K_{16}\\ \gamma_5\otimes M_{21}\otimes K_{21} & \partial\!\!\!/\otimes 1\otimes 1 & \cdots & \gamma_5\otimes M_{26}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_5\otimes M_{61}\otimes K_{61} & \gamma_5\otimes M_{62}\otimes K_{62} & \cdots & \partial\!\!\!/\otimes 1\end{bmatrix}, \tag{5.6}$$

where the K_{mn} are 3×3 matrices commuting with the a_i and b_i. Therefore we shall take

$$\begin{aligned}&M_{12}=M_{21}=\mathcal{M}_0\ ,\\ &M_{34}=M_{43}=BM_{12}B^{-1}\ ,\\ &M_{13}=M_{23}=M_{14}=M_{24}=\mathcal{N}_0\ ,\\ &M_{15}=M_{16}=M_{25}=M_{26}=H_0\ ,\\ &M_{35}=M_{45}=M_{36}=M_{46}=BM_{15}\ ,\\ &M_{56}=0\ ,\end{aligned} \tag{5.7}$$

and a similar symmetry for the K_{mn}. We further restrict the elements of the algebra to those of the form

$$\mathcal{P}f\mathcal{P}\ ,$$

where the projection operator is

$$\mathcal{P}=\text{diag}(P_+,P_+,P_-,P_-,1,1)\ . \tag{5.8}$$

For $\pi(\rho)$ one then gets

$$\pi(\rho)=\begin{bmatrix}A & \gamma_5\mathcal{M}K_{12} & \gamma_5\mathcal{N}K_{13} & \gamma_5\mathcal{N}K_{14} & \gamma_5 HK_{15} & \gamma_5 HK_{16}\\ \gamma_5\mathcal{M}K_{12} & A & \gamma_5\mathcal{N}K_{23} & \gamma_5\mathcal{N}K_{24} & \gamma_5 HK_{25} & \gamma_5 HK_{26}\\ \gamma_5\mathcal{N}^*K_{31} & \gamma_5\mathcal{N}^*K_{32} & B\overline{A}B^{-1} & \gamma_5\mathcal{M}'K_{34} & \gamma_5 H'K_{35} & \gamma_5 H'K_{36}\\ \gamma_5\mathcal{N}^*K_{41} & \gamma_5\mathcal{N}^*K_{42} & \gamma_5\mathcal{M}'K_{43} & B\overline{A}B^{-1} & \gamma_5 H'K_{45} & \gamma_5 H'K_{46}\\ \gamma_5 H^*K_{51} & \gamma_5 H^*K_{52} & \gamma_5 H'^*K_{53} & \gamma_5 H'^*K_{54} & 0 & 0\\ \gamma_5 H^*K_{61} & \gamma_5 H^*K_{62} & \gamma_5 H'^*K_{63} & \gamma_5 H'^*K_{64} & 0 & 0\end{bmatrix}, \tag{5.9}$$

where the new functions A, $\mathcal{M}$, $\mathcal{N}$, and H are given in terms of the a^i and b^i by

$$\begin{aligned}&A=P_+\left[\sum_i a^i\partial\!\!\!/ b^i\right]P_+,\quad \mathcal{M}+\mathcal{M}_0=P_+\left[\sum_i a^i\mathcal{M}_0 b^i\right]P_+\ ,\\ &\mathcal{N}+\mathcal{N}_0=P_+\left[\sum_i a^i\mathcal{N}_0 B\overline{b}^i B^{-1}\right]P_-,\quad H+H_0=P_+\left[\sum_i a^i H_0 b'^i\right]\ ,\end{aligned} \tag{5.10}$$

and $\mathcal{M}'=B\bar{\mathcal{M}}B^{-1}$, $H'=B\bar{H}$. We shall make the same physical requirement as in Eqs. (2.20) and (2.21) that reduces the gauge group from U(16) to SO(10). The fermionic action, in terms of the component fields, is given by

$$I_f=\int d^4x\{2\bar{\psi}_+[i(\partial\!\!\!/+A)\psi_+ +2\gamma_5(\mathcal{N}+\mathcal{N}_0)\psi_+^c K_{13}+\gamma_5(H+H_0)\lambda^c K_{15}]$$
$$-2\bar{\psi}_+^c[i(\partial\!\!\!/+A)\psi_+^c+2\gamma_5(\mathcal{N}^*+\mathcal{N}_0^*)\psi_+K_{13}^*\gamma_5 B(\bar{H}+\bar{H}_0)\lambda K_{15}^*]$$
$$+\bar{\lambda}[i\partial\!\!\!/\lambda+2\gamma_5(H+H_0)^T B^{-1}\psi_+^c K_{15}]-\bar{\lambda}^c[i\partial\!\!\!/\lambda^c+2\gamma_5(H^*+H_0^*)\psi_+K_{15}^*]\}\ , \tag{5.11}$$

where $\psi_+=P_+\psi$ and by SO(10) chirality is equal to ψ. This expression can be simplified by using the properties of the charge conjugation matrices B and C, and after rescaling $\psi\to\frac{1}{2}\psi$ and $\lambda\to(1/\sqrt{2})\lambda$, the fermionic action (5.11) simplifies to

$$I_f=\int d^4x\left\{\bar{\psi}_+i(\partial\!\!\!/+A)\psi_+ +\bar{\lambda}i\partial\!\!\!/\lambda - \left[\psi_+^T B^{-1}C^{-1}(\mathcal{N}^*+\mathcal{N}_0^*)\psi_+K_{13}^* + \frac{1}{\sqrt{2}}\lambda^T C^{-1}(H^*+H_0^*)\psi_+K_{15}^*+\frac{1}{\sqrt{2}}\psi_+^T C^{-1}(\bar{H}+\bar{H}_0)\lambda K_{35}+\text{H.c.}\right]\right\}\ . \tag{5.12}$$

The only change in the breaking mechanism is that $U(1)_c\times SU(2)_R$ is broken also by the H_0 whose VEV is given by

$$H_0=M_3\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}\ . \tag{5.13}$$

The fermionic action is modified slightly from Eq. (3.19) to become

$$I_{f\,\text{mass}}=-\int d^4x(\{[s+p+3(e+f)]K_{(pq)}+[a+a'+3(b+b')]K_{[pq]}\}\bar{N}_R^p N_L^q$$
$$+\{[s+p-(e+f)]K_{(pq)}+[a+a'-(b+b')]K_{[pq]}\}\bar{u}_R^p u_L^q$$
$$+\{[s-p-3(e-f)]K_{(pq)}+[a-a'-3(b-b')]K_{[pq]}\}\bar{e}_R^p e_L^q$$
$$+[(s-p+e-f)K_{(pq)}+(a-a'+b-b')K_{[pq]}]\bar{d}_R^p d_L^q$$
$$+[\sqrt{2}M_3K'_{pq}\bar{N}_R^p\lambda_L^q+M_2K_{(pq)}(N_R^{pc})^T C^{-1}N_R^{qc}]+\text{H.c.})\ , \tag{5.14}$$

where we have denoted the family mixing matrices K_{13}, K_{15}, and K_{56} by K, K', and K'', respectively. Since we have three neutral fields N_L, N_R^c, and λ_L and their mass eigenstates are mixed, the mass matrix must be diagonalized. Ignoring the mixing due to the generation matrices, the mass matrix is of the form

$$\begin{array}{c|ccc} & N_L & N_R^c & \lambda_L\\ \hline N_L & 0 & m & 0\\ N_R^c & m & M_2 & M_3\\ \lambda_L & 0 & M_3 & 0\end{array}\ , \tag{5.15}$$

and we shall assume a mass hierarchy $m\ll M_2,M_3$, and $M_2\sim M_3$. Diagonalization of the matrix (5.13) produces two massive fields whose masses are of order M_2, and the third will be a massless left-handed neutrino. The kinetic term for the gauge field A_μ^{IJ} is equal to

$$-4g^2F_{\mu\nu}^{IJ}F^{\mu\nu IJ}\ , \tag{5.16}$$

and the Higgs kinetic terms have three parts corresponding to $\mathcal{M}$, $\mathcal{N}$, and H. They are given by

$$2|K_{12}|^2\text{Tr}\{[D_\mu(\mathcal{M}+\mathcal{M}_0)]^2\}+8|K_{13}|^2\text{Tr}[|D_\mu(\mathcal{N}+\mathcal{N}_0)|^2]+12|K_{15}|^2|D_\mu(H+H_0)|^2\ , \tag{5.17}$$

where the D appearing in this equation is the covariant derivative with respect to the SO(10) gauge group. The mass terms of the gauge fields corresponding to the broken generators of SO(10) are provided by the VEV's $\mathcal{M}_0$, $\mathcal{N}_0$, and H_0. The Higgs potential is very complicated in this case. It is given by

$$2\left||K_{12}|^2|\mathcal{M}+\mathcal{M}_0|^2+2|K_{13}|^2|\mathcal{N}+\mathcal{N}_0|^2+2|K_{15}|^2|H+H_0|^2-(Y_1+X'_{11})\right|^2$$
$$+2\left|2|K_{31}|^2|\mathcal{N}+\mathcal{N}_0|^2+|K_{12}|^2|\mathcal{M}+\mathcal{M}_0|^2+2|K_{13}|^2|H+H_0|^2-(Y_3+X'_{33})\right|^2+2\left|4|K_{51}|^2|H+H_0|^2-(Y_5+X'_{55})\right|^2$$
$$+2\left|2|K_{13}|^2(|\mathcal{N}+\mathcal{N}_0|^2-|\mathcal{N}_0|^2)+2|K_{15}|^2(|H+H_0|^2-|H_0|^2)-X_{12}\right|^2$$
$$+8\left|K_{12}K_{23}[(\mathcal{M}+\mathcal{M}_0)(\mathcal{N}+\mathcal{N}_0)-\mathcal{M}_0\mathcal{N}_0]+K_{14}K_{43}[(\mathcal{N}+\mathcal{N}_0)(\overline{\mathcal{M}}+\overline{\mathcal{M}}_0)-\mathcal{N}_0\overline{\mathcal{M}}_0]\right.$$
$$\left.+2K_{15}K_{53}[(H+H_0)B(\overline{H}+\overline{H}_0)-H_0B\overline{H}_0]-X_{13}\right|^2$$
$$+8\left|K_{12}K_{25}(\mathcal{M}+\mathcal{M}_0)(H+H_0)+2K_{13}K_{35}(\mathcal{N}+\mathcal{N}_0)B(\overline{H}+\overline{H}_0)-X_{15}\right|^2$$
$$+2\left|2K_{31}K_{14}(|\mathcal{N}^*+\mathcal{N}_0^*|^2-|\mathcal{N}_0^*|^2)+2|K_{35}|^2[|B(\overline{H}+\overline{H}_0)|^2-|B\overline{H}_0|^2]-X_{34}\right|^2$$
$$+8\left|2K_{31}K_{15}[(\mathcal{N}^*+\mathcal{N}_0^*)(H+H_0)-\mathcal{N}_0^*H_0^*]+2K_{34}K_{45}[|B(\overline{H}+\overline{H}_0)|^2-|B\overline{H}_0|^2]-X_{35}\right|^2$$
$$+2\left|4|K_{51}|^2(|H^*+H_0^*|^2-|H_0^*|^2)-X_{56}\right|^2, \tag{5.18}$$

where we have used the symmetry that equates some of the K's and the X's. The explicit expressions for the X and Y fields are

$$X'_{11}=\sum_i a^i\partial^2 b^i+(\partial^\mu A_\mu+A^\mu A_\mu),\quad X'_{33}=B\overline{X}'_{11}B^{-1},\quad X'_{55}=\sum_i a'^i\partial^2 b'^i . \tag{5.19}$$

Next, we have

$$\begin{aligned}Y_1&=\sum_i a^i|K_{12}|^2|\mathcal{M}_0|^2b^i+2\sum_i a^i|K_{13}|^2|\mathcal{N}_0|^2b^i+\sum_i a^i|K_{15}|^2|H_0|^2b^i ,\\ Y_3&=B\overline{Y}_1B^{-1} ,\\ Y_5&=2M_2^2(|K_{51}|^2+|K_{53}|^2) .\end{aligned} \tag{5.20}$$

The expressions for X_{mn}, $m\neq n$, are now given by

$$\begin{aligned}X_{12}&=2|K_{13}|^2\left[\sum_i a^i|\mathcal{N}_0|^2B\overline{b}^iB^{-1}-|\mathcal{N}_0|^2\right]+2|K_{15}|^2\left[\sum_i a^i|H_0|^2b^i-|H_0|^2\right] ,\\ X_{13}&=K_{12}K_{23}\left[\sum_i a^i\mathcal{M}_0\mathcal{N}_0B\overline{b}^iB^{-1}-\mathcal{M}_0\mathcal{N}_0\right]+K_{14}K_{43}\left[\sum_i a^i\mathcal{N}_0B\overline{\mathcal{M}}_0B^{-1}-\mathcal{N}_0B\overline{\mathcal{M}}_0B^{-1}\right]\\ &\quad+2|K_{15}|^2\left[\sum_i a^iH_0\overline{H}_0\overline{b}^iB^{-1}-|H_0|^2B^{-1}\right] ,\\ X_{15}&=|K_{12}K_{25}|\left[\sum_i a^i\mathcal{M}_0H_0b'^i-\mathcal{M}_0H_0\right]+2K_{13}K_{35}\left[\sum_i a^i\mathcal{N}_0B\overline{H}_0b'^i-\mathcal{N}_0B\overline{H}_0\right] ,\\ X_{34}&=B\overline{X}_{12}B^{-1} ,\\ X_{35}&=B\overline{X}_{15} ,\\ X_{56}&=0 ,\end{aligned} \tag{5.21}$$

and the other X's are related to the ones above by permutation symmetry $X_{12}=X_{21}$, $X_{34}=X_{43}$, $X_{13}=X_{14}=X_{23}=X_{24}$, $X_{16}=X_{26}$, and $X_{36}=X_{46}$. We also have similar identities for the K's, and in addition, we have assumed the relations $K_{12}=\overline{K}_{34}$ and $K_{15}=\overline{K}_{35}$. In analogy with the previous model, we must impose the relation

$$M_G=M_1 , \tag{5.22}$$

in order to get a relation between X_{12} and Y_1:

$$Y_1=|K_{12}|^2|\mathcal{M}_0|^2+2|K_{13}|^2|\mathcal{N}_0|^2+X_{12} . \tag{5.23}$$

For the term in the potential involving X_{13} to survive, we should be able to express this field in terms of the other scalar fields. By examining the expression for X_{13}, we notice that a simplification occurs if we require that

$$K_{12}=\overline{K}_{12} , \tag{5.24}$$

because the terms involving $\mathcal{M}_0\mathcal{N}_0^{(1)}$ drop out, where $\mathcal{N}_0^{(1)}$ is the part of $\mathcal{N}_0$ independent of M_2. In this case X_{13} can be made to be zero, provided that we take

$$M_1M_2=-\frac{K_{15}\overline{K}_{15}}{2K_{12}K_{13}}M_3^2 , \tag{5.25}$$

where we have used the relation

$$H_0H_0^* = \frac{M_2}{M_3^2}\mathcal{N}_0^{(2)} , \quad (5.26)$$

where $\mathcal{N}_0^{(2)}$ is the part in $\mathcal{N}_0$ dependent on M_2. If no relation is taken between K_{12} and $\bar{K}_{12}$, then the only way for the potential to survive is to impose a relation on $\mathcal{M}_0\mathcal{N}_0$ identical to the one for the simpler model as well as a relation between M_2 and M_3. This case will not be interesting for us, since the fermionic mass matrices, apart from the neutral fields sector, are identical to those in the previous model and thus would suffer from the same problem of absence of the Cabibbo angle. The auxiliary field X_{15} is not independent and is equal to

$$X_{15} = uH , \quad (5.27)$$

where

$$u = 2K_{13}\bar{K}_{15}[s + p - 3(b + b') + 2(a + a') + M_2] - 2K_{12}K_{25}M_1 . \quad (5.28)$$

After eliminating the auxiliary fields Y_1 and X'_{33}, the potential becomes

$$\begin{aligned} V(\mathcal{M},\mathcal{N},H) = {} & [\mathrm{Tr}|K_{12}|^4 - (\mathrm{Tr}|K_{12}|^2)^2]\mathrm{Tr}(|\mathcal{M}+\mathcal{M}_0|^2 - |\mathcal{M}_0|^2)^2 \\ & + 4\left|K_{13}K_{12}[(\mathcal{M}+\mathcal{M}_0)(\mathcal{N}+\mathcal{N}_0) + (\mathcal{N}+\mathcal{N}_0)B(\bar{\mathcal{M}}+\bar{\mathcal{M}}_0)B^{-1}] + 2K_{15}\bar{K}_{15}[(H+H_0)B(\bar{H}+\bar{H}_0)]\right|^2 \\ & + 8\left|K_{12}K_{15}(\mathcal{M}+\mathcal{M}_0)(H+H_0) + 2K_{13}\bar{K}_{15}(\mathcal{N}+\mathcal{N}_0)B(\bar{H}+\bar{H}_0) - u(H+H_0)\right|^2 \\ & + 16[\mathrm{Tr}|K_{15}|^4 - (\mathrm{Tr}|K_{15}|^2)^2]\left||H^* + H_0^*|^2 - M_3^2|^2 + 16\,\mathrm{Tr}|K_{15}|^4||H^* + H_0^*|^2 - M_3^2\right|^2 . \end{aligned} \quad (5.29)$$

This potential would only coincide with the one constructed in the conventional approach with the requirements of using the Higgs fields in the combinations $\mathcal{M}$ and $\mathcal{N}$ and having the potential to be the sum of squares. Otherwise, there will be many more terms possible in the conventional approach.

Therefore the fermionic mass terms are still given by Eq. (5.14) and do not suffer the problem encountered before. This completes our study of the model and shows that it is possible to obtain a good SO(10) model. A complete phenomenological analysis will be left for the future.

VI. SUMMARY AND CONCLUSION

We have seen that a realistic SO(10) model can be constructed using the noncommutative geometry setting of Connes. The attractiveness of this model stems from the fact that all the fermions fit into one representation, making the spinor space particularly simple. Depending on the number of discrete points extending the continuous geometry, the Higgs structure is predicted uniquely. We found two models: The first one is quite simple and has a very restrictive form for the fermion masses. This turns out to be unrealistic if one insists on requiring the potential to determine all the VEV's at the classical level. Including radiative corrections to the potential may cure this problem. The second example is more complicated, but the Higgs structure is essentially the same as that of the first model, with the difference of an additional $\mathbf{16}_s$ Higgs field. The fermionic masses are not as restricted as those in the first model. We hope to study the spectrum in more detail in the future. A study of the quantum system is not meaningful before having determined those symmetries of the system that are characteristic of the noncommutative geometry setting.

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