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Fractal behaviors in proton-nucleus interactions at 800 GeV

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The multifractal moments with suppressed statistical contributions have been investigated in interactions of 800 GeV protons with emulsion nuclei in one- (η/ϕ) and two-dimensional (η-ϕ) spaces in different multiplicity intervals. The scaled factorial moments have also been determined and the relationship between the fractal indices and the intermittency indices discussed. The dependence of generalized dimensions on order gives clear evidence of a self-similar cascade mechanism in the interactions.

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I. INTRODUCTION

The study of nonstatistical fluctuations in distributions of secondary particles produced in high-energy interactions has become an important tool in understanding the mechanism of multiparticle production [1–4]. Prompted by sharp spikes in single cosmic ray events, Bialas and Peschanski [5] proposed to study event-to-event particle density fluctuations in rapidity bins of decreasing width. They suggested a power-law dependence of the scaled factorial moments on the rapidity bin size, called intermittency in hydrodynamic turbulence [6]. Evidence for intermittency has been reported in a variety of processes (e^+e^-, μp, hadron-hadron, hadron-nucleus, and nucleus-nucleus collisions) [1–4].

The power-law behavior of factorial moments indicates self-similarity and the existence of fractal properties [7] in the multiparticle production process. A fractal or self-similar object satisfies a power-law scaling which reflects the underlying dynamics. Various methods have been suggested for investigating fractal structures in multiparticle production [8–12]. Hwa [10] suggested a set of multifractal moments G_q to investigate the large density fluctuations in terms of the multifractal formalism. If the particle production process exhibits self-similar behavior, power-law dependence of the G_q moments on the phase space bin size is expected [10,11]. However, because of finite multiplicities in events at finite energy, the self-similarity cannot be expected at finer scales of resolution. Hence, the G_q moments are not free of statistical fluctuations. In order to suppress the statistical contribution, Hwa and Pan [12] have suggested a modified form of the G_q, i.e., G_q' and applied it to hadronic collisions [12,13]. They have used the Monte Carlo code ECCO [14] based on the geometrical branching model [15] in one- [12], two-, and three-dimensional (3D) [13] phase spaces. This formalism has also been applied to high-energy nuclear collisions [16] in the geometrical branching model using ECCO and the European Muon Collaboration (EMC) data on multiparticle production in μp and μd collisions [17]. Hwa and Pan [12] have also discussed the fractal interpretation of intermittency and derived a relation between the fractal indices and the intermittency indices, thus revealing a correspondence between the decreasing power of G_q' moments and increasing power of the scaled factorial moments.

We have already observed [18] intermittent behavior in multiparticle production in proton interactions with emulsion nuclei at 800 GeV, which is presently the highest available energy for fixed targets. Multiplicity dependence of intermittency has been observed [18] in one-dimensional pseudorapidity (η) and azimuthal angle (ϕ) and two-dimensional (2D) (η-ϕ) spaces, wherein intermittency strength is found to decrease with increasing multiplicity. In the present work, we investigate the fractal structure of multiplicity fluctuations in proton-nucleus interactions in one and two dimensions using the modified G_q moments introduced by Hwa and Pan [12]. We have also studied the relation between the fractal indices and intermittency indices, as well as their dependences on order, multiplicity, and phase space dimension in order to obtain insight into the dynamical mechanism

TABLE I. Characteristics of the three multiplicity intervals. The errors are statistical.

Multiplicity interval	Number of events	$\langle N_s \rangle$	$\langle N_s \rangle$ in $\Delta\eta$=0.5–5.5
$11 \le N_s \le 20$	996	15.23±0.48	14.26±0.45
$21 \le N_s \le 30$	491	24.88±1.12	23.36±1.05
$N_s \ge 31$	250	41.02±2.59	38.42±2.43

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responsible for hadronization in proton-nucleus interactions.

II. EXPERIMENT

The experimental data analyzed here comes from 40 Ilford G5 emulsion pellicles of dimensions $10 \times 8 \times 0.06$ cm^3 which were exposed to a proton beam of energy 800 GeV at Fermilab. The experimental details have been given earlier [19]. Following the usual emulsion nomenclature [20] secondary particles having ionization $I \leq 1.4 I_0$ (I_0 being the ionization of the primary) were termed as shower tracks and their multiplicity denoted by N_s. Tracks having ionization $I > 1.4 I_0$ were called heavy tracks, and their multiplicity denoted by N_h. The space angle (θ) and the azimuthal angle (ϕ) of the shower tracks with respect to the beam axis were measured by the 3D coordinate method. The uncertainty in angle measurement is 8×10^{-4} radians for space angle and 8×10^{-2} radians for azimuthal angle. The pseudorapidity η [$= -\ln(\tan\theta/2)$] is determined, which is a good approximation to rapidity at very high energy. The accuracy is ~ 0.1 units in pseudorapidity.

We have divided the sample of interactions into three categories with different multiplicity ranges: i.e., (i) $11 \leq N_s \leq 20$, (ii) $21 \leq N_s \leq 30$, and (iii) $N_s \geq 31$. The analysis has been performed in the pseudorapidity range $\Delta\eta = 0.5-5.5$. Table I gives the number of events, $\langle N_s \rangle$ and $\langle N_s \rangle$ in $\Delta\eta$ for the three multiplicity ranges considered.

III. METHOD OF ANALYSIS

For the analysis of multiplicity fluctuations in pseudorapidity space η, the pseudorapidity interval $\Delta\eta$ is divided into M bins of width $\delta\eta = \Delta\eta / M$. The multifractal moment G_q of order q is defined for a single event by [11]

$$G_q = \sum_{m=1}^{M} (K_m / N)^q , \qquad (1)$$

where N is the number of shower particles in the event in the pseudorapidity range $\Delta\eta$ and K_m is the number of shower particles in the mth bin. Here, q is a real number, and the summation is carried over nonempty bins only. If the particle production process exhibits multifractal structure, the power-law behavior [11]

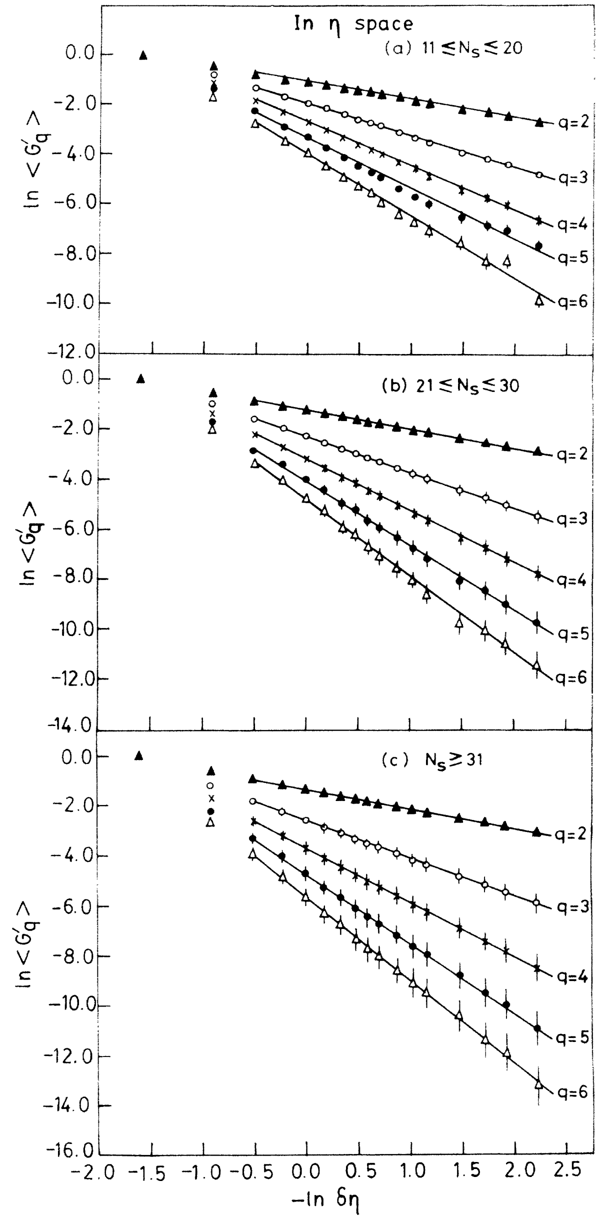

FIG. 1. $\ln\langle G'_q \rangle$ as a function of $-\ln\delta\eta$ for different multiplicity intervals. The errors shown are statistical. Lines represent the linear fits to the data.

TABLE II. Values of the slopes $\tau'_q(\eta)$ in pseudorapidity space for different multiplicity intervals. The numbers in parentheses are standard errors.

Multiplicity interval	$\tau'_2(\eta)$	$\tau'_3(\eta)$	$\tau'_4(\eta)$	$\tau'_5(\eta)$	$\tau'_6(\eta)$
$11 \leq N_s \leq 20$	0.694	1.269	1.740	2.032	2.493
	(0.005)	(0.014)	(0.024)	(0.052)	(0.067)
$21 \leq N_s \leq 30$	0.751	1.425	2.038	2.575	3.044
	(0.004)	(0.009)	(0.016)	(0.031)	(0.058)
$N_s \geq 31$	0.785	1.503	2.166	2.777	3.340
	(0.006)	(0.010)	(0.014)	(0.020)	(0.038)

$$G_q \propto (\delta\eta)^{\tau_q} \tag{2}$$

is expected. This power-law behavior does not occur in the limit $\delta\eta \to 0$, since in that limit K_m for a nonempty bin is 1 and G_q approaches N^{1-q}. Thus at finite energy, where N is not very large, the self-similarity or fractal structure cannot be expected at infinitesimal $\delta\eta$. The slope τ_q in Eq. (2) is determined only for the large $\delta\eta$ region.

Hwa and Pan have suggested [12] a modified form of the G_q moments defined by

$$G'_q = \sum_{m=1}^{M} (K_m/N)^q \theta(K_m - q) \tag{3}$$

for positive integral orders q, where $\theta(K_m - q)$ is the step function that is 1 for $K_m \geq q$ and 0 for $K_m < q$. The definitions (1) and (3) differ only by the θ function. In statistical and geometrical problems where N is very large and $N/M \gg q$, the θ function is inessential, and the two definitions give the same result. But in particle physics, N is limited and so the need of the θ function arises in order to suppress the statistical contribution.

For an ensemble of events, the averaging is done as

$$\langle G'_q \rangle = \frac{1}{N_{ev}} \sum_{N_{ev}} G'_q \tag{4}$$

where N_{ev} is the total number of events in the ensemble. A self-similar pattern of the multiplicity fluctuations leads to the power-law behavior [12]

$$\langle G'_q \rangle \propto (\delta\eta)^{\tau'_q} \tag{5}$$

for all bin widths $\delta\eta$ down to the smallest.

The most complete description of scaling properties of fractal objects is given in terms of an infinite set of generalized (Renyi) dimensions [9,21,22]. The generalized dimensions D'_q for the G'_q moments can be determined from τ'_q by the relation [12]

$$D'_q = \tau'_q/(q-1) . \tag{6}$$

Hwa and Pan have also developed a new approach for a close comparison between the multifractal moments and the scaled factorial moments [12]. The scaled factorial moments that have been extensively studied are [5]

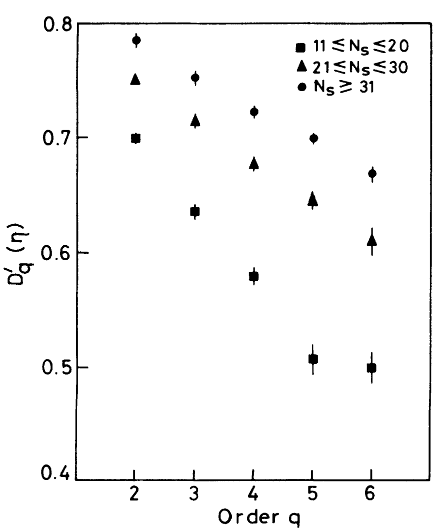

FIG. 2. The generalized dimensions $D'_q(\eta)$ in η space as a function of q for different multiplicity intervals.

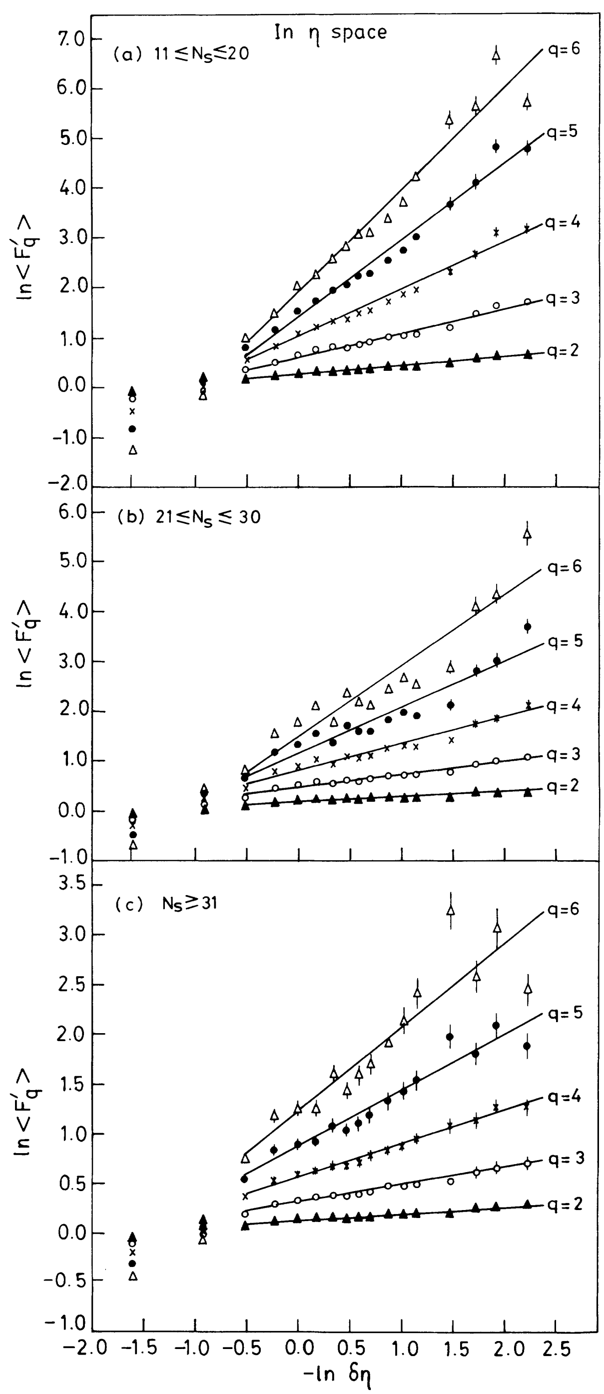

FIG. 3. $\ln\langle F'_q \rangle$ as a function of $-\ln\delta\eta$ for different multiplicity intervals. The errors shown are statistical. Lines represent the linear fits to the data.

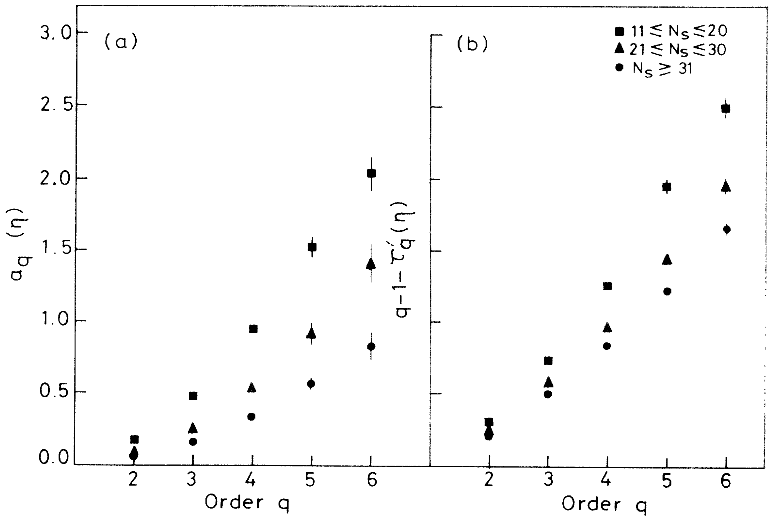

FIG. 4. Comparison of the exponents of F_q' and G_q' moments in pseudorapidity space: (a) $a_q(\eta)$ and (b) $q-1-\tau_q'(\eta)$ as functions of order q. The errors shown are standard.

$$\langle F_q \rangle = \frac{1}{N_{\mathrm{ev}}} \sum_{N_{\mathrm{ev}}} M^{q-1} \sum_{m=1}^{M} \frac{K_m(K_m-1)\cdots(K_m-q+1)}{\langle N \rangle^q} \tag{7}$$

where $\langle N \rangle$ is the average number of shower particles of the events in the ensemble.

For comparison with G_q' moments Hwa and Pan have considered a slightly different form of the scaled factorial moments defined by [12]

$$\langle F_q' \rangle = \frac{1}{N_{\mathrm{ev}}} \sum_{N_{\mathrm{ev}}} M^{q-1} \sum_{m=1}^{M} \frac{K_m(K_m-1)\cdots(K_m-q+1)}{N^q} . \tag{8}$$

The power-law dependence of $\langle F_q' \rangle$ on $\delta\eta$ is predicted to be of the form

$$\langle F_q' \rangle \propto (\delta\eta)^{-a_q} \tag{9}$$

where the slope a_q characterizes the strength of the fluctuations.

A relation between the G_q' moments and F_q' moments has been developed [12], thus providing a fractal interpretation for intermittency. The fractal index τ_q' measures the strength of multifractality, while the intermittency index a_q is a measure of the strength of intermittency. Thus, a correspondence between intermittency and multifractality can be obtained by relating the indices a_q and τ_q'. The two indices have been shown to be approximately related as [12]

$$a_q \approx q-1-\tau_q' . \tag{10}$$

The relationship is not exact because F_q' and G_q' are different moments and approach each other only in the limiting case of infinite N. It is clear from Eq. (10) that the deviation of a_q from zero is equivalent to the deviation of τ_q' from $q-1$. Thus to compare the fractal behavior of $\langle F_q' \rangle$ and $\langle G_q' \rangle$ we should compare a_q and $q-1-\tau_q'$.

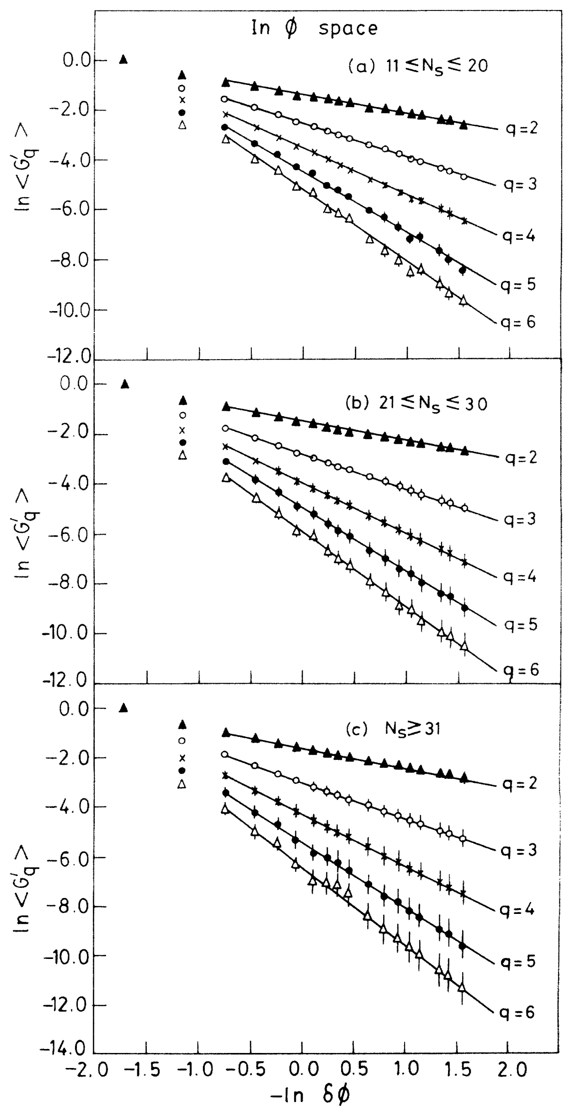

FIG. 5. $\ln\langle G_q' \rangle$ as a function of $-\ln\delta\phi$ for different multiplicity intervals. The errors shown are statistical. Lines represent the linear fits to the data.

TABLE III. Values of the slopes $\tau'_q(\phi)$ in azimuthal space for different multiplicity intervals. The numbers in parentheses are standard errors.

Multiplicity interval	$\tau'_2(\phi)$	$\tau'_3(\phi)$	$\tau'_4(\phi)$	$\tau'_5(\phi)$	$\tau'_6(\phi)$
$11 \le N_s \le 20$	0.718	1.336	1.854	2.455	2.690
	(0.005)	(0.011)	(0.019)	(0.042)	(0.055)
$21 \le N_s \le 30$	0.742	1.404	2.001	2.535	2.986
	(0.003)	(0.008)	(0.014)	(0.020)	(0.032)
$N_s \ge 31$	0.773	1.453	2.066	2.638	3.170
	(0.004)	(0.009)	(0.015)	(0.029)	(0.057)

IV. RESULTS AND DISCUSSION

A. Analysis in one-dimensional pseudorapidity space

The values of $\langle G'_q \rangle$ are calculated using Eqs. (3) and (4) for the three multiplicity intervals: $11 \le N_s \le 20$, $21 \le N_s \le 30$, and $N_s \ge 31$ for q ranging from 2 to 6. Figures 1(a)–1(c) show $\ln\langle G'_q \rangle$ as a function of $-\ln\delta\eta$ for the three multiplicity intervals considered. A linear dependence of $\ln\langle G'_q \rangle$ on $-\ln\delta\eta$ is observed indicating self-similarity in the interactions. The absolute value of $\ln\langle G'_q \rangle$ for a fixed q and $\delta\eta$ is highest for the $N_s \ge 31$ sample and lowest for the $11 \le N_s \le 20$ sample. Hence, a multiplicity dependence of the G'_q moments is clearly observed. The slope values $\tau'_q(\eta)$ in η space are calculated by the least-squares fitting of the data points in the range $-\ln\delta\eta = (-0.511)-(2.219)$ and are listed in Table II. The values of $\tau'_q(\eta)$ for each q are seen to increase with increasing multiplicity, being minimum for $11 \le N_s \le 20$ and maximum for the $N_s \ge 31$ sample.

The values of generalized dimensions $D'_q(\eta)$ in η space are calculated from $\tau'_q(\eta)$ using Eq. (6). Figure 2 shows the variation of $D'_q(\eta)$ with order q for the three multiplicity intervals. The errors shown are standard errors.

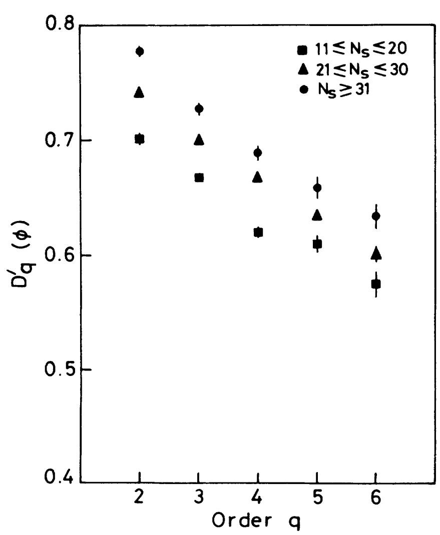

FIG. 6. The generalized dimensions $D'_q(\phi)$ in ϕ space as a function of q for different multiplicity intervals.

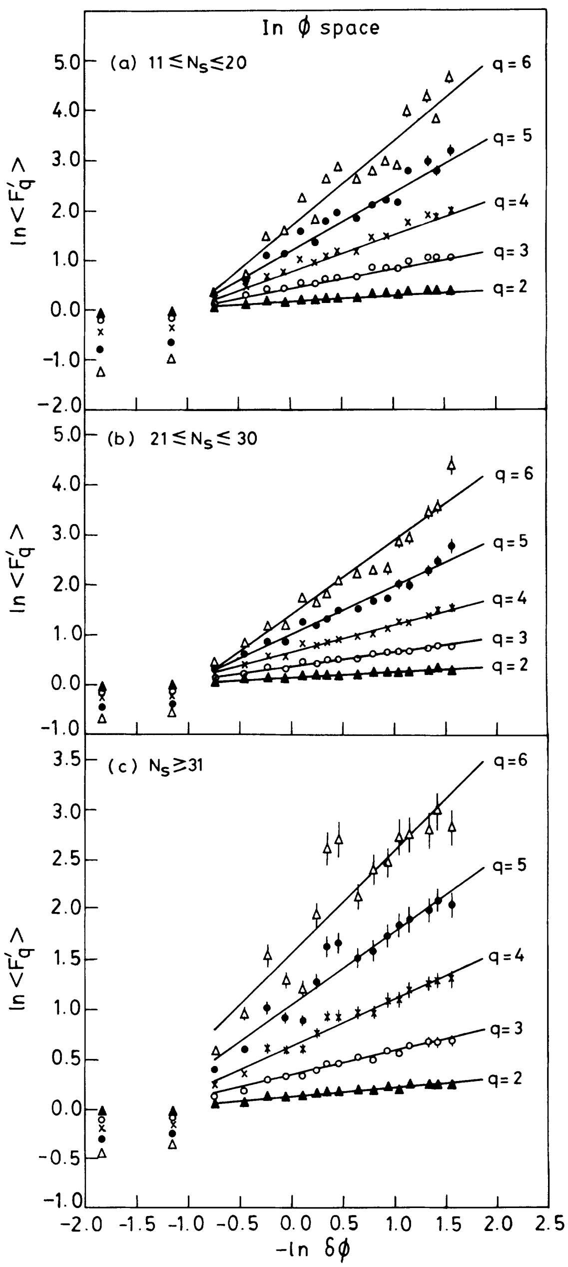

FIG. 7. $\ln\langle F'_q \rangle$ as a function of $-\ln\delta\phi$ for different multiplicity intervals. The errors shown are statistical. Lines represent the linear fits to the data.

Since $D'_q(\eta) < D'_{q'}(\eta)$ for $q > q'$, the structure is clearly multifractal [9,22]. Dependence of $D'_q(\eta)$ on q also indicates a self-similar cascade mechanism in proton-nucleus interactions [23]. The values of $D'_q(\eta)$ for each q are also seen to increase with increasing multiplicity. The dependence of slope values $\tau'_q(\eta)$ and generalized dimensions $D'_q(\eta)$ on multiplicity indicates that the multifractal structure is more pronounced for higher multiplicity events as compared to the lower multiplicity events.

The scaled factorial moments $\langle F'_q \rangle$ in pseudorapidity space are calculated using Eq. (8). Figures 3(a)–3(c) show the variation of $\ln\langle F'_q \rangle$ as a function of $-\ln\delta\eta$. A linear rise of $\ln\langle F'_q \rangle$ with $-\ln\delta\eta$ is observed. The values of $\ln\langle F'_q \rangle$ are highest for the $11 \le N_s \le 20$ sample and lowest for the $N_s \ge 31$ sample for a fixed q and $\delta\eta$. Thus the F'_q moments also exhibit multiplicity dependence. The slope values $a_q(\eta)$ are determined by the least-squares fitting of the data points in the same $\delta\eta$ range as for the G'_q moments to enable a comparison between G'_q and F'_q moments. Figure 4(a) shows the variation of $a_q(\eta)$ with order q. For a fixed q the slope values $a_q(\eta)$ are highest for $11 \le N_s \le 20$ and lowest for the $N_s \ge 31$ sample implying that intermittency effects are stronger in events with low multiplicity. The multiplicity dependence of intermittency has been thoroughly investigated both experimentally [4] and theoretically [9,24]. Because of several sources contributing to particle production in high multiplicity events, the intermittency effects are masked as compared to the lower multiplicity events [9].

The multiplicity dependence of the F'_q moments is complementary to the multiplicity dependence of the G'_q moments as the intermittency indices $a_q(\eta)$ decrease with increasing multiplicity and the fractal indices $\tau'_q(\eta)$ increase with increasing multiplicity. In order to compare the deviation of $a_q(\eta)$ from zero with the deviation of $\tau'_q(\eta)$ from $q-1$, $q-1-\tau'_q(\eta)$ is plotted as a function of q in Fig. 4(b). Comparison of Figs. 4(a) and 4(b) shows that the dependences of $a_q(\eta)$ and $q-1-\tau'_q(\eta)$ on q have similar trends for the three multiplicity intervals, although their values are significantly different. Similar behavior has been observed in EMC data on μp and μd collisions [17]. Hence the approximate relation (10) between $a_q(\eta)$ and $\tau'_q(\eta)$ is valid for our data and reveals that the increasing power of $\langle G'_q \rangle$ corresponds to the decreasing power of $\langle F'_q \rangle$.

B. Analysis in one-dimensional azimuthal space

The analysis described in Sec. IV A for studying fluctuations in pseudorapidity distributions can be applied to study fluctuations in azimuthal angle distributions as well. We consider only the shower tracks with pseudorapidity lying in the range $\Delta\eta = 0.5-5.5$. The entire range of $\phi = 0-2\pi$ is divided into M_ϕ bins of size $\delta\phi = 2\pi/M_\phi$.

The values of $\ln\langle G'_q \rangle$ are calculated using Eqs. (3) and (4) for $q = 2-6$ and shown in Figs. 5(a)–5(c) as a function of $-\ln\delta\phi$ for the three multiplicity intervals. A linear dependence of $\ln\langle G'_q \rangle$ on $-\ln\delta\phi$ is observed indicating the presence of fractal structure in azimuthal angle distributions also. For a certain value of q and $\delta\phi$, the absolute value of $\ln\langle G'_q \rangle$ is highest for the $N_s \ge 31$ sample and lowest for the $11 \le N_s \le 20$ sample, showing that the G'_q moments in ϕ space are also multiplicity dependent.

The slope values $\tau'_q(\phi)$ in ϕ space are calculated by the least-squares fitting of the data points in the range $-\ln\delta\phi = (-0.739)-(1.563)$. Table III shows the values of $\tau'_q(\phi)$ for the three multiplicity intervals. The slope values are found to increase with order q for each multiplicity interval and for a fixed q increase with increasing multiplicity. Hence, the fractal structure is more pronounced at higher multiplicities. The values of generalized dimensions $D'_q(\phi)$ in ϕ space are calculated from $\tau'_q(\phi)$ using Eq. (6) and shown in Fig. 6 as a function of the order q for the three multiplicity intervals. The errors shown are standard errors. The variation of $D'_q(\phi)$ with q indicates that the structure is multifractal. The values of $D'_q(\phi)$ are also seen to increase with increasing multiplicity for each q.

The scaled factorial moments $\langle F'_q \rangle$ in ϕ space are calculated using Eq. (8). Figures 7(a)–7(c) show the variation of $\ln\langle F'_q \rangle$ as a function of $-\ln\delta\phi$ for different multiplicity intervals. A linear rise of $\ln\langle F'_q \rangle$ with $-\ln\delta\phi$ is

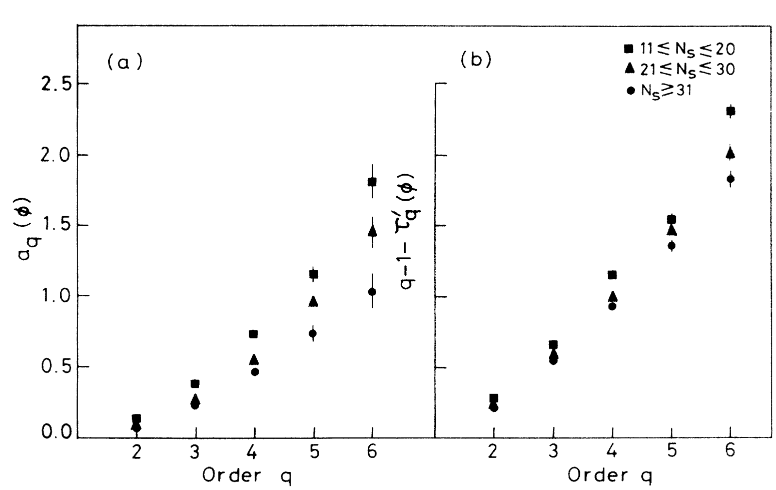

FIG. 8. Comparison of the exponents of F'_q and G'_q moments in azimuthal space: (a) $a_q(\phi)$ and (b) $q-1-\tau'_q(\phi)$ as functions of order q. The errors shown are standard.

observed. The slope values $a_q(\phi)$ are computed by the least-squares fitting of the data points. Figure 8(a) shows the variation of $a_q(\phi)$ with order q. The slope values $a_q(\phi)$ are seen to decrease with increasing multiplicity. Thus in ϕ space also, intermittency effects are stronger in low multiplicity events.

To compare the F'_q and the G'_q moments, values of $q-1-\tau'_q(\phi)$ are plotted as a function of order q in Fig. 8(b). Within errors, the dependence of $a_q(\phi)$ and $q-1-\tau'_q(\phi)$ on order q are similar for the three multiplicity intervals.

C. Analysis in two-dimensional (η-ϕ) space

Ochs and Wosiek [25] proposed that intermittency effects should be stronger if the fluctuations are analyzed simultaneously in two or three variables. This has been observed in several experimental investigations where stronger intermittency is seen in higher dimensional phase spaces as compared to a lower dimensional phase space [1–4]. In our data of proton-nucleus interactions at 800 GeV we have observed stronger intermittency in two-dimensional (η-ϕ) space as compared to one-dimensional (η or ϕ) space [18]. Here, we investigate the fractal structure of fluctuations in two-dimensional (η-ϕ) space using the G'_q moments and relate them to the scaled factorial moments F'_q.

For analysis in two-dimensional (η-ϕ) space, the ranges of η and ϕ are $\Delta\eta=0.5$–5.5 and $\Delta\phi=0$–2π. The $\Delta\eta\times\Delta\phi$ space is divided into $M_{\eta\text{-}\phi}=M_\eta\times M_\phi$ cells of area $\delta\eta\delta\phi=(\Delta\eta/M_\eta)\times(\Delta\phi/M_\phi)$ with $M_\eta=M_\phi$. The G'_q moments are calculated using Eq. (3) with K_m the number of particles in the mth cell and the summation over $M_{\eta\text{-}\phi}=M_\eta\times M_\phi$ cells. The values of $\ln\langle G'_q\rangle$ are calculated using Eq. (4) for $q=2$ to 5. Figures 9(a)–9(c) show a linear dependence of $\ln\langle G'_q\rangle$ on $-\ln(\delta\eta\delta\phi)$ for the three multiplicity intervals, indicating the presence of self-similar fractal structure in two-dimensional (η-ϕ) space as well. The absolute values of $\ln\langle G'_q\rangle$ for a certain q and ($\delta\eta\delta\phi$) value are highest for the $N_s\geq 31$ sample and least for the $11\leq N_s\leq 20$ sample, making explicit the multiplicity dependence of the G'_q moments in two-dimensional (η-ϕ) space.

The slopes $\tau'_q(\eta\text{-}\phi)$ in (η-ϕ) space are obtained by the least-squares fitting of the data points with $-\ln(\delta\eta\delta\phi)=(-1.250)$–$(2.098)$. Table IV shows that

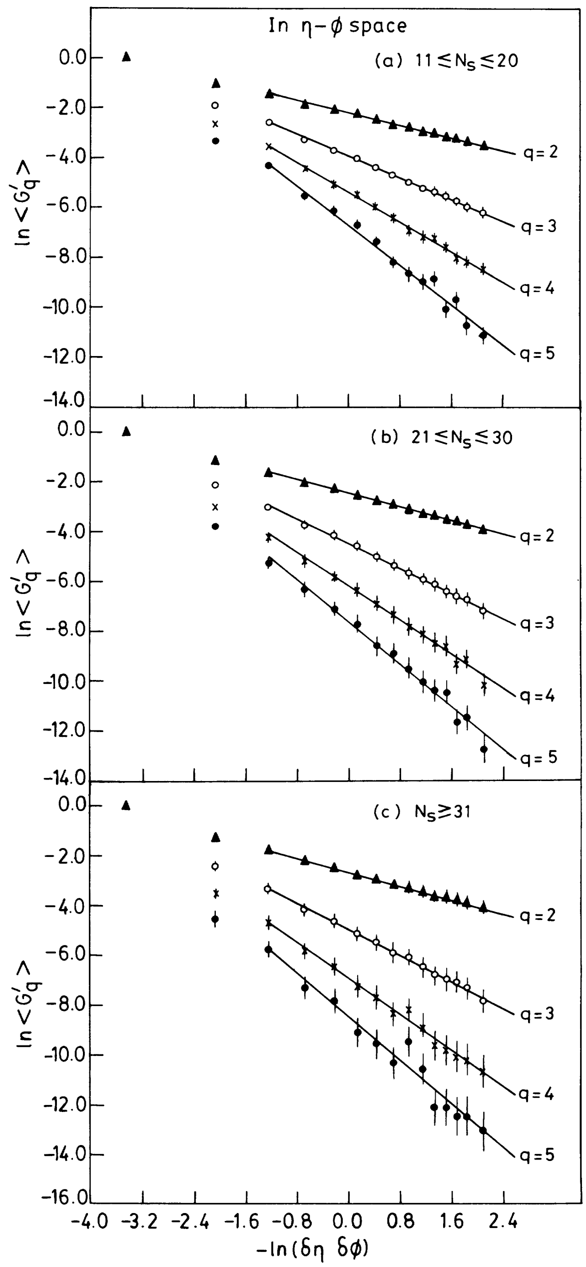

FIG. 9. $\ln\langle G'_q\rangle$ as a function of $-\ln(\delta\eta\,\delta\phi)$ for different multiplicity intervals. The errors shown are statistical. Lines represent the linear fits to the data.

TABLE IV. Values of the slopes $\tau'_q(\eta\text{-}\phi)$ in two-dimensional (η-ϕ) space for different multiplicity intervals. The numbers in parentheses are standard errors.

Multiplicity interval	$\tau'_2(\eta\text{-}\phi)$	$\tau'_3(\eta\text{-}\phi)$	$\tau'_4(\eta\text{-}\phi)$	$\tau'_5(\eta\text{-}\phi)$
$11\leq N_s\leq 20$	0.621	1.096	1.500	2.020
	(0.007)	(0.010)	(0.032)	(0.082)
$21\leq N_s\leq 30$	0.680	1.247	1.709	2.125
	(0.007)	(0.018)	(0.051)	(0.087)
$N_s\geq 31$	0.690	1.307	1.793	2.174
	(0.005)	(0.020)	(0.053)	(0.129)

the slope values $\tau'_q(\eta\text{-}\phi)$ increase with increasing multiplicity. Comparison of Table IV with Tables II and III shows that the slope values $\tau'_q(\eta\text{-}\phi)$ in two-dimensional $(\eta\text{-}\phi)$ space are smaller than the corresponding values of $\tau'_q(\eta)$ or $\tau'_q(\phi)$ in one-dimensional (η or ϕ) space.

The values of generalized dimensions $D'_q(\eta\text{-}\phi)$ in $(\eta\text{-}\phi)$ space are calculated using Eq. (6). Figure 10 shows $D'_q(\eta\text{-}\phi)$ as a function of order q for the three multiplicity intervals. The errors shown are standard errors. The values of $D'_q(\eta\text{-}\phi)$ are seen to decrease with increasing q in each multiplicity interval, suggesting that the structure is multifractal [22]. The $D'_q(\eta\text{-}\phi)$ values are highest for the $N_s \geq 31$ sample and lowest for the $11 \leq N_s \leq 20$ sample. This makes evident that even in two-dimensional $(\eta\text{-}\phi)$ space the fractal structure is more pronounced in high multiplicity events. The values of generalized dimensions in two-dimensional $(\eta\text{-}\phi)$ space are smaller than the corresponding values in one-dimensional (η or ϕ) space.

The scaled factorial moments F'_q in $(\eta\text{-}\phi)$ space are calculated using Eq. (8). Figures 11(a)–11(c) show a linear rise of $\ln\langle F'_q\rangle$ with $-\ln(\delta\eta\delta\phi)$ in each multiplicity interval. The multiplicity dependence of the F'_q moments is also explicit from the figures. The values of $\ln\langle F'_q\rangle$ for a fixed q and $(\delta\eta\delta\phi)$ are seen to decrease with increasing multiplicity. The slopes $a_q(\eta\text{-}\phi)$ are determined by the least-squares fitting of the data points and are shown in Fig. 12(a) as a function of the order q. The slope values $a_q(\eta\text{-}\phi)$ decrease with the increasing multiplicity for each q, thus indicating stronger intermittency in low multiplicity events. The slope values $a_q(\eta\text{-}\phi)$ are larger than the corresponding values of $a_q(\eta)$ or $a_q(\phi)$ implying stronger intermittency in two-dimensional $(\eta\text{-}\phi)$ distributions.

To compare the fractal indices $\tau'_q(\eta\text{-}\phi)$ with the intermittency indices $a_q(\eta\text{-}\phi)$, $q-1-\tau'_q(\eta\text{-}\phi)$ is plotted as a function of q in Fig. 12(b). Figure 12 shows that the dependences of $a_q(\eta\text{-}\phi)$ and $q-1-\tau'_q(\eta\text{-}\phi)$ on q show similar trends. Comparing Fig. 12 with Figs. 4 and 8 of one-dimensional η and ϕ spaces, it is observed that the

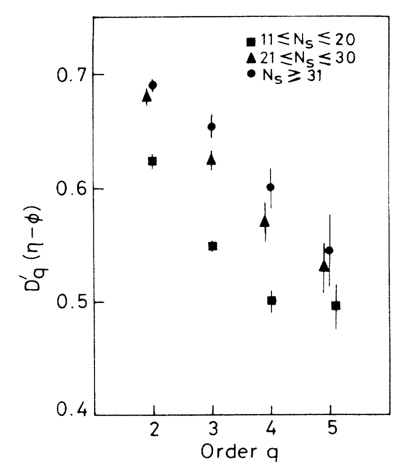

FIG. 10. The generalized dimensions $D'_q(\eta\text{-}\phi)$ in two-dimensional $(\eta-\phi)$ space as a function of q for different multiplicity intervals.

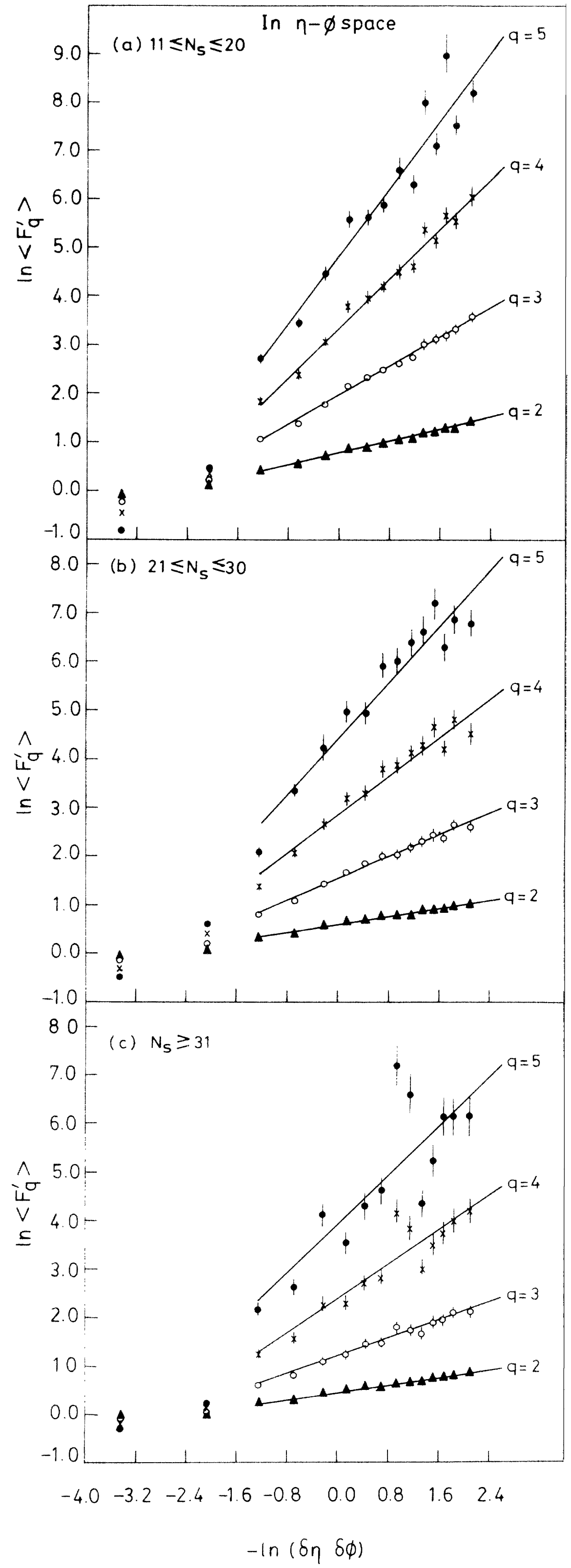

FIG. 11. $\ln\langle F'_q\rangle$ as a function of $-\ln(\delta\eta\delta\phi)$ for different multiplicity intervals. The errors shown are statistical. Lines represent the linear fits to the data.

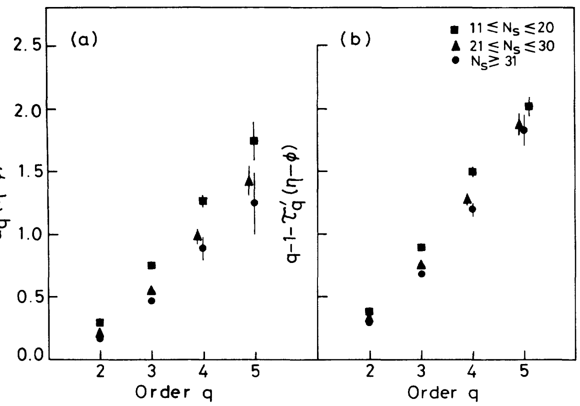

FIG. 12. Comparison of the exponents of F_q' and G_q' in two-dimensional (η-ϕ) space: (a) $a_q(\eta\text{-}\phi)$ and (b) $q-1-\tau_q'(\eta\text{-}\phi)$ as functions of order q. The errors shown are standard.

approximate relation (10) between the intermittency indices and fractal indices is satisfied to a better extent in two-dimensional (η-ϕ) space than in one-dimensional (η or ϕ) spaces.

V. CONCLUSIONS

The multifractal moments G_q' and the scaled factorial moments F_q' exhibit clear power-law behavior which is characteristic of a self-similar system in one-dimensional η, ϕ, and two-dimensional (η-ϕ) spaces. The fractal indices τ_q' and the intermittency indices a_q show different dependences on multiplicity and phase space dimension as follows.

In one-dimensional η, ϕ, and two-dimensional (η-ϕ) spaces, the fractal indices τ_q' increase with increasing multiplicity as $\tau_q'(11 \le N_s \le 20) < \tau_q'(21 \le N_s \le 30) < \tau_q'(N_s \ge 31)$ whereas the intermittency indices a_q decrease with increasing multiplicity as $a_q(11 \le N_s \le 20) > a_q(21 \le N_s \le 30) > a_q(N_s \ge 31)$.

In each of the three multiplicity intervals it is seen that $\tau_q'(\eta)$ and $\tau_q'(\phi) > \tau_q'(\eta\text{-}\phi)$, $a_q(\eta)$ and $a_q(\phi) < a_q(\eta\text{-}\phi)$. Thus the fractal indices τ_q' decrease with increasing phase space dimension, and the intermittency indices a_q increase with increasing phase space dimension.

The intermittency strength increases with decreasing multiplicity and increasing phase space dimension whereas the strength of the multifractal moments G_q' decreases with decrease in multiplicity and increase in phase space dimension.

A correspondence between the increasing strength of the G_q' moments and decreasing strength of the F_q' moments has been obtained through the relation $a_q \sim q-1-\tau_q'$. The intermittency index a_q and deviation of the fractal index τ_q' from $q-1$ show similar dependence on order q for the different multiplicity intervals in η, ϕ, and (η-ϕ) spaces, thus providing a connection between intermittency and multifractality. A remarkable equivalence between a_q and $q-1-\tau_q'$ is observed in two-dimensional (η-ϕ) space.

The dependence of generalized dimensions D_q' on q in η, ϕ, and (η-ϕ) spaces indicates multifractal structure and the presence of self-similar cascading mechanism in the interactions.

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[1] *Intermittency in High Energy Collisions,* Proceedings of the Santa Fe Workshop, Santa Fe, New Mexico, 1990, edited by F. Cooper, R. C. Hwa, and I. Sarcevic (World Scientific, Singapore, 1991).
[2] W. Kittel, in *Proceedings of the Twentieth International Symposium on Multiparticle Dynamics,* Gut Holmecke, Germany, 1990, edited by R. Baier and D. Wegener (World Scientific, Singapore, 1991), p. 401.
[3] *Fluctuations and Fractal Structures,* Proceedings of the Ringberg Workshop on Multiparticle Production, Ringberg Castle, Germany, 1991, edited by R. C. Hwa, W. Ochs, and N. Schmitz (World Scientific, Singapore, 1992).
[4] A. Bialas, in *Quark Matter '90,* Proceedings of the Eighth International Conference on Ultrarelativistic Nucleus-Nucleus Collisions, Menton, France, edited by J. P. Blaizot *et al.* [Nucl. Phys. **A525**, 345c (1991)].
[5] A. Bialas and R. Peschanski, Nucl. Phys. **B273**, 703 (1986); **B308**, 857 (1988).
[6] B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).
[7] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1982); G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).
[8] I. M. Dremin, Mod. Phys. Lett. A **3**, 1333 (1988); P. Carruthers, Int. J. Mod. Phys. A **4**, 5587 (1989); Ph. Brax and R. Peschanski, Nucl. Phys. **B346**, 65 (1990).
[9] P. Lipa and B. Buschbeck, Phys. Lett. B **223**, 465 (1988).
[10] R. C. Hwa, Phys. Rev. D **41**, 1456 (1990).
[11] C. B. Chiu and R. C. Hwa, Phys. Rev. D **43**, 100 (1991).
[12] R. C. Hwa and J. C. Pan, Phys. Rev. D **45**, 1476 (1992).
[13] J. C. Pan and R. C. Hwa, Phys. Rev. D **46**, 4890 (1992).
[14] R. C. Hwa and J. C. Pan, Phys. Rev. D **45**, 106 (1992).
[15] R. C. Hwa, Phys. Rev. D **37**, 1830 (1988); W. R. Chen and R. C. Hwa, *ibid.* **39**, 179 (1989); R. C. Hwa and X. N. Wang, *ibid.* **39**, 2561 (1989).
[16] R. C. Hwa and J. C. Pan, Phys. Rev. D **46**, 2941 (1992).
[17] I. Derado, R. C. Hwa, G. Jancso, and N. Schmitz, Phys. Lett. B **283**, 151 (1992).
[18] R. K. Shivpuri and Vandana Anand, Z. Phys. C **59**, 47 (1993).
[19] R. K. Shivpuri and V. K. Verma, Phys. Rev. D **47**, 123 (1992).
[20] C. Gupt, R. K. Shivpuri, N. S. Verma, and A. P. Sharma,

Phys. Rev. D **26**, 2202 (1982).
[21] A. Renyi, *Probability Theory* (North-Holland, Amsterdam, 1970).
[22] H. G. E. Hentschel and I. Procaccia, Physica **8D**, 435 (1983); P. Grassberger and I. Procaccia, *ibid.* **13D**, 34 (1984).
[23] A. Bialas and R. C. Hwa, Phys. Lett. B **253**, 436 (1991).
[24] J. C. Pan, Phys. Rev. D **45**, 834 (1992); Saul Barshay, Z. Phys. C **47**, 199 (1990); W. Ochs, in *Proceedings of the Twentieth International Symposium on Multiparticle Dynamics* [2].
[25] W. Ochs and J. Wosiek, Phys. Lett. B **214**, 617 (1988); W. Ochs, *ibid.* **247**, 101 (1990).