PHYSICAL REVIEW D VOLUME 50, NUMBER 1 1 JULY 1994

Intermittency and a phase transition in a lattice gas model

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(Received 27 September 1993)

On the basis of a lattice gas model and the convolution formula with a cell construction scheme, we demonstrate that intermittency, i.e., the power law behavior of the moments in rapidity space, is caused by a phase transition between the ordered phase and disordered phase with respect to the particle number distribution. In this model the critical moments are directly connected with an order parameter at the phase transition point. The indices of these moments are simply given by the critical exponent near the critical point. It is pointed out that the critical indices of this phase transition are not constant, but depend on the size of the rapidity interval, if the system is not near enough to the critical point.

PACS number(s): 13.85.Hd

I. INTRODUCTION

Inspired by the stimulating proposal of intermittency in rapidity space with respect to the factorial moments [1], several authors [2–6] have presented their theoretical ideas to examine intermittent behavior [7] from the viewpoint of a phase transition between hadronic matter and a quark-gluon plasma. Most of the authors investigated this behavior supposing the intermittency of the moments of rapidity distributions as a striking signal of short-range fluctuations characterized in cooperative phenomena near the critical point. However, some authors [5] have considered that such behavior is only to show the relevance to short distance correlations.

If the intermittent power law behavior of the moments is some signal of critical phenomena, i.e., not a mere representation of the short distance correlations, this apparent short-range fluctuation in rapidity space is due to yield a fluctuation of a macroscopic quantity of state connected with some order parameter. In the critical state the fluctuation gives the singularity of the macroscopic quantity of state, and so we have a critical power law behavior of an order parameter. Therefore, one should make clear the relation between the characteristic moments and the order parameter. This convenient relation makes us comprehend the similarity between the intermittent power law behavior of the moments and the power law of the order parameter. This is a reliable way to ascertain whether one can surely find evidence of critical behavior in intermittency.

The aim of this paper is to show the universal relation between intermittent behavior and cooperative phenomena with regard to phase transitions. We will present a method to obtain the factorial moments of the particle number distribution from an order parameter in view of the universal correspondence between the Ising model and the lattice gas model [8]. In the same manner to illustrate the universal singularity [9] of the order parameters near the critical points of these models, we relate the reason why these factorial moments show intermittency, i.e., the characteristic power law behavior in rapidity space. It is pointed out that the indices of these moments are simply given by the critical exponent in the Ising model and the lattice gas model near the critical point. We deduce this power law behavior from the convolution formula of probability theory [10] with the cell construction scheme of these lattice systems. In the framework of the convolution theory it is easy to comprehend that self-similarity of the cascading models [1, 4] and the fractal systems [2, 3] is an extremely restricted condition for the singular factorial moments. It may be possible to loosen this condition for intermittency in connection with phase transitions.

In the next section, on the basis of a lattice gas model we will present a method to express the factorial moments of the particle number distribution of lattice gas among configured cells in terms of an order parameter, i.e., the macroscopic fluctuation of the number of distributed particles. This order parameter is equivalent to magnetization in the Ising model. In Sec. III, according to the convolution theory for phase transitions, we demonstrate that the power law behavior of the moments is derived from the singular probability densities for finding particles in the rapidity interval space near the critical point. In the framework of the Ising model and the lattice gas model, the indices of these moments are simply derived from the critical exponent of these models. The Gaussian model, which is given by modifying the Ising model, provides more realistic indices (see Table I). In Sec. IV, we summarize our conclusions.

II. MOMENTS IN THE LATTICE GAS MODEL

It is well known that the lattice gas model for the liquid-gas transition is mathematically equivalent to the

TABLE I. Comparison of the indices ϕ_l between the experimental data [16] and the model calculations. Theoretical values for Gaussian model are given by Eq. (3.12) with $\zeta = 0.199$ and $\sigma = 0.256$.

	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
Data of ϕ_l in the two-dimensional (y, ϕ) distribution	0.153 (5)	0.481 (20)	0.897 (100)	1.32 (20)	
Two-dimensional Ising (lattice gas) model	0.125	0.250	0.375	0.500	0.625
Gaussian model based on the convolution formula	0.150	0.413	0.874	1.68	3.10

Ising model of a ferromagnetic phase transition. According to the lattice gas model [11], the number of particles of lattice gas corresponds to the number of down spins in the Ising model. The z component of the spin s_i at the lattice point i is connected with the gas density ρ_i as

$$s_i = 2\left(\frac{1}{2} - \rho_i\right). \tag{2.1}$$

If the lattice consists of N sites, with the help of Eq. (2.1) the magnetization per lattice site is given by

$$I = \frac{\sum_i s_i}{N} = \frac{N_+ - N_-}{N} = \frac{\Delta N}{N}, \tag{2.2}$$

where N_+ (N_-) is the number of up (down) spins, and $\Delta N = N_+ - N_-$ indicates the fluctuation of the number of particles. Putting $N = V$ and $N_- = \sum_i \rho_i$ [11], we obtain from Eqs. (2.1) and (2.2) the specific volume per particle:

$$v = \frac{V}{N_-} = \frac{1}{\langle\rho\rangle} = \frac{2}{1-I}, \tag{2.3}$$

where $\langle\rho\rangle = \sum_i \rho_i/N$. It turns out from the above-mentioned correspondence between the lattice gas model and the Ising model that magnetization can be represented by $I = \Delta N/V$ [12], and, hence, if the fluctuation of the number of particles becomes large, the order parameter I ought to be nonzero even if the volume of the system V is large enough. This means the system is in ordered phase.

Let us imagine the cell construction of a lattice gas, i.e., particles partitioned among M cells, such as Kadanoff's construction of block spins [13]. In the framework of cell construction the total number of particles (n) distributed in M cells is provided by

$$q_1 + q_2 + \cdots + q_M = n, \tag{2.4}$$

where q_i is the number of particles in the ith cell. This lattice gas model yields the lth moment for a given configuration as follows:

$$f_l(M) = \left[(1/M)\sum_{i=1}^{M} {q_i}^l\right] \Big/ \left[(1/M)\sum_{i=1}^{M} q_i\right]^l = M^{l-1}n^{-l}\sum_{i=1}^{M} {q_i}^l. \tag{2.5}$$

In the case of equidistribution, i.e., $q_1 = q_2 = \cdots = q_M = 1$, we obtain $\sum_{i=1}^{M} {q_i}^l = M$, and thus $f_l = 1$ because of $n = M$. If the system shows an extreme fluctuation, e.g., $q_r = n$ at $i = r$, we have $f_l = M^{l-1}$. These results are essentially the same as the moments for some specific configurations in the Ising model as shown in Ref. [3].

In order to see the relation between this model and the Ising model, we set $M = N$ and $n = N_-$ for the minimum cells, assuming $q_i = \rho_i$. With the help of Eq. (2.3) it is possible to rewrite Eq. (2.5) into

$$f_l(M) = \left(\frac{M}{n}\right)^{l-1} = \left(\frac{N}{N_-}\right)^{l-1} = v^{l-1}. \tag{2.6}$$

Therefore, we get from Eqs. (2.2), (2.3), and (2.6) the moment f_l in terms of an order parameter I in the form

$$\ln f_l(I) = (l-1)\ln\left(\frac{2}{1-I}\right), \tag{2.7}$$

where $I < 1$. This formula is valid also for large cells in which the spins and the gas densities are renormalized [14] to satisfy Eq. (2.1). If the system is extremely ordered like $I \sim 1$, Eq. (2.7) yields

$$\ln f_l \sim (l-1)I', \tag{2.8}$$

where $I' = I + \ln 2$. It turns out from Eq. (2.8) that $\ln f_l$ is proportional to the order parameter I in ordered phase, while $\ln f_l$ becomes constant for $I = 0$ in disordered phase. Thus, we can verify the phase transition of this system by analyzing the behavior of $\ln f_l$.

III. CONVOLUTION FORMULA FOR INTERMITTENCY

As shown in the previous section, in the lattice gas model the lth moment f_l is represented by $f_l(M) \propto M^{l-1}$ in terms of the number of lattice cells M as Eq. (2.6). This moment is quite similar to the intermittent power law behavior of the moment of the particle number distribution in rapidity space.

In the lattice gas model and the Ising model the order parameter I is given by $I \sim \epsilon^\beta$ near the critical point in terms of the temperature T and the critical temperature T_c, where β is the critical exponent and $\epsilon = (T - T_c)/T_c$. Therefore, we have

$$f_l \sim I^{l-1} \sim \epsilon^{(l-1)\beta}. \tag{3.1}$$

The number of subdivisions M, which is inversely proportional to the correlated cell size, is considered to be proportional to the inverse of the correlation length $\xi \sim \epsilon^{-\nu}$, where ν is the critical exponent in this lattice system. Accordingly, we obtain

$$f_l \sim \epsilon^{(l-1)\beta} \sim M^{(l-1)\beta/\nu} \sim M^{\phi_l}, \tag{3.2}$$

and hence $\phi_l = (l-1)\beta/\nu$. The two-dimensional Ising model and the lattice gas model yield $\phi_l = 0.125(l-1)$ because of $\beta = 0.125$ and $\nu = 1$. This result is the same as the speculation [15] based on the two-dimensional Ising model. It is natural to identify M cells of lattice gas with M intervals in rapidity space, in accordance with Satz's illustration [3] for obtaining intermittency from the Ising model, since the Ising model and the lattice gas model are mathematically equivalent [11]. Our result is compared with the experimental data [16] in Table I.

In order to investigate in detail intermittency for the particle number distribution, we must know the behavior of the moments according to the decreasing size of rapidity interval where fluctuations of the particle number become the macroscopic quantity. Therefore, we divide large cells into small ones. This corresponds to separating rapidity intervals Δy_i $(i = 1, 2, \ldots, M)$ into small segments.

Let us consider random separation of the ith rapidity interval Δy_i in m segments such as

$$\Delta y_i = \eta_{i1} + \eta_{i2} + \cdots + \eta_{im}, \tag{3.3}$$

where η_{ij} is the rapidity size of the jth segment in the ith interval. Since these segments are independent random variables, the probability density $p_i(\Delta y_i)$ for finding particles in the ith rapidity interval is given by the convolution of the respective probabilities $p_{ij}(\eta_{ij})$ in each segment as [10]

$$p_i(\Delta y_i) = \sum_m Q(m) \int \cdots \int p_{i1}\left(\Delta y_i - \sum_{j=2}^{m} \eta_{ij}\right) p_{i2}(\eta_{i2}) \cdots p_{im}(\eta_{im}) d\eta_{i2} \cdots d\eta_{im}, \tag{3.4}$$

where $Q(m)$ is the weight function to convolute each probability.

This probability density p_i is equivalent to $q_i/\sum_i q_i$ in the lattice gas model with the cell construction scheme. Accordingly, we have the lth moment for the particle number distribution of Eq. (2.5) in the form

$$f_l(M) = \frac{1}{M} \sum_{i=1}^{M} (Mp_i)^l. \tag{3.5}$$

The normalization condition requires

$$p_1 + p_2 + \cdots + p_M = 1. \tag{3.6}$$

In our lattice gas model the probability density for finding particles in the ith rapidity interval is equivalent to the particle number density in the ith cell which corresponds to the spin variable of the same cell. According to Kadanoff's construction of block spins [13] and the renormalization group method [14], we can deduce the singular critical order parameters with respect to these variables, i.e., magnetization of the ferromagnetic transition (density minus critical density in the case of the liquid-gas transition), by means of iteration of cell construction, assuming scale invariance near the critical point. In the framework of the convolution theory [10], this assumption of scale invariance is essentially the same as the bootstrap condition [17] for a probability density such as $\ln p_i \sim \ln p_{ij}$. Thus, we obtain the scale-independent critical probability density p_c near the critical point. Since this probability density does not depend on the cell size, it should be independent of the rapidity interval η_{ij}.

If we set the asymptotic value of the probability density near the critical point as $p_{ij} = p_{ci}$ in the ith rapidity interval and put

$$\int p_{ij}(\eta_{ij}) d\eta_{ij} = \int p_{ci}(\eta) d\eta = W_{ci}, \tag{3.7}$$

in which we divide Δy_i into m segments of the same rapidity size η, we have, from Eq. (3.4),

$$p_i(\Delta y_i) = \sum_m Q(m) W_{ci}^m. \tag{3.8}$$

The normalization condition (3.6) provides $\sum_i p_i = \sum_i \sum_m Q(m) W_{ci}^m = 1$. The scale invariant bootstrap condition demands $W_{ci} \sim W_c$. W_c is defined like Eq. (3.7) in the full rapidity region. Therefore, the moment (3.5) near the critical point is given by

$$f_l(M) = \frac{1}{M} \sum_{i=1}^{M} (Mp_i)^l = \sum_m Q(m) W_c{}^{ml}, \tag{3.9}$$

where $f_1(M) = 1$. As for the independent separation of rapidity intervals, the Poisson distribution $Q(m) = (\langle m \rangle^m / m!) \exp(-W_c \langle m \rangle)$ yields

$$f_l(M) = \exp\left[\langle m \rangle W_c \left(W_c{}^{l-1} - 1\right)\right] = M^{\phi_l}, \tag{3.10}$$

where $\phi_l = \langle m \rangle W_c \left(W_c{}^{l-1} - 1\right) / \ln M$. Assuming $\left(W_c{}^{l-1} - 1\right) \approx \ln W_c{}^{l-1}$ for $W_c{}^{l-1} \approx 1$ and $\langle m \rangle = \ln M / (W_c \ln \lambda)$, we have $\phi_l = \ln W_c{}^{l-1} / \ln \lambda$. This is the case of a recurrent self-similar separation of rapidity intervals in λ segments based on the fractal structure in the rapidity space [1].

As shown in the beginning of this section, the indices predicted from the two-dimensional Ising model and the lattice gas model are not satisfactory to be compared with the experimental data (see Table I). Since the Gaussian model [14], which can be obtained by modifying the Ising model, provides more realistic critical exponent $\nu = 0.5$, we should take the Gaussian-type probability density $p_c(\eta) = \exp(-\sigma\eta^2)$ instead of the δ function for obtaining the critical indices from the convolution formula. Integration of $p_c(\eta)$ in the full rapidity region yields

$$\int p_c(\eta)d\eta = \int \exp(-\sigma\eta^2)d\eta = \sqrt{\frac{\pi}{4\sigma}} = W_c. \quad (3.11)$$

Equation (3.11) and the mean value $\langle m\rangle = \zeta \ln M/W_c$ lead to

$$\phi_l = \langle m\rangle\, W_c(W_c^{l-1} - 1)/\ln M = \zeta\{(\pi/4\sigma)^{(l-1)/2} - 1\}. \quad (3.12)$$

It is shown in Table I that these indices are consistent with the experimental data [16] especially in the large M region where our convolution formula becomes sufficiently valid.

The self-similar separation model demands $W = \int p(\eta)d\eta$ to be constant in the rapidity space, which provides the constant critical exponent ϕ. However, it turns out from our model that, around the critical point, W depends on the finite lattice size (η) in the strict sense, and so W may depend on the rapidity interval. If the observed system is not near enough to the critical point in rapidity space, we cannot observe the finite lattice size which is small enough to be compared with the lattice constant of this system. Therefore, it is probable that the critical exponent ϕ slightly depends on the observed rapidity interval through W, if the system does not approach near enough to the critical point.

IV. CONCLUSIONS

We have shown that intermittency, i.e., fluctuations of the number of particles in different sizes in rapidity space, surely reflects a phase transition between the ordered phase and disordered phase. We have presented a lattice gas model which easily leads us to comprehend that the fluctuation of the number of particles represents the order parameter like magnetization of spin systems, and the scaled moments are given only by this order parameter. We, thus, have concluded that intermittency with respect to the moments results from cooperative phenomena of the system near the critical point.

Intermittency of the system is characterized by the power law behavior of the scaled moments such as the power law singularity of the order parameters in various phase transitions. The indices of these moments are simply obtained from the critical exponent near the critical point. This behavior of the moments is derived from the critical value of the probability number density in rapidity space on the basis of the convolution formula with the cell construction scheme. We have shown our model calculation in comparison with the experimental data. In contrast with other results obtained from usual self-similar fractal models, we have pointed out that the critical indices should depend on the lattice size, i.e., the size of rapidity interval, if the system is not near enough to the critical point. Recent experimental data [18] show the energy dependence of f_l, and hence may support this consequence.

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