Asymptotically free ϕ^4 theory with even elements of a Grassmann algebra

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Bilinear composites of anticommuting constituents are even elements of a Grassmann algebra, which are nilpotent commuting (NC) variables. We study a ϕ^4 theory where the ϕ field has NC Fourier components and we find that it is asymptotically free.

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In this paper we investigate the perturbative properties of a ϕ^4 theory where the Fourier components of the ϕ field are even elements of a Grassmann algebra, which are nilpotent commuting (NC) variables. We find that for attractive self-coupling such a theory is asymptotically free, as is the ordinary theory [1]. But while the latter has a Euclidean action unbounded from below, with NC variables the partition function is well defined *irrespective of the attractive or repulsive character of the coupling.*

Even elements of a Grassmann algebra are products of odd elements. Since the latter describe fermions, from the physical point of view NC variables describe composites of fermions. We think that such variables can be used for two distinct purposes:

(i) To construct phenomenological models, disregarding the way they can be related to fermionic constituents. One such application would be to use a NC (therefore composite) Higgs field to avoid triviality [2] in the scalar section of the standard model of electroweak interactions.

(ii) To study bosonization in fermionic systems. NC fields have in fact been introduced by a change of variables (in the sense specified below) in the Berezin integral defining the partition function of a model of composite gauge fields [3]. It is conceivable, for instance, that by a change of variables of this kind one could derive an effective action for superconductivity similar to the Ginzburg-Landau one, but with NC Fourier components of the field. The model we are going to discuss might be a prototype of such an effective theory.

The use of NC variables finds in the theory of superconductivity a paradigmatic example, and we will mention later a result relevant to the present work. But they should be useful in the study of all bilinear composites of fermions, of which other well-known examples are density fluctuations in the Tomonaga model of the electron gas, spin waves in ferroantiferromagnetic metals, and nucleon Cooper pairs in the interacting boson model of nuclear physics. Finally we are also considering the possibility of using NC variables in bound state problems. Among these, quark bound states seem to be the most interesting case to investigate, in the hope that because of confinement one could get rid of the constituents odd variables altogether.

In any case when one is interested in correlation functions which do not involve the constituent fields but only those combinations which define the composites, one would like to be able to treat the composites themselves as independent variables. To do this we need a definition of the integral over the composites such as to give, when the composites are expressed in terms of the constituent fields, the same results as the Berezin integral over the latters. When the integration variables are trilinear composites (baryons), which are odd elements of the Grassmann algebra, the integral must still be the Berezin integral. When the integration variables are bilinear composites, which are even elements of the Grassmann algebra, we need a new definition. Notice that by adding to the action irrelevant terms which are quadratic in the trilinear or bilinear composites, we can perform a new type of perturbative expansion.

Let us start by reporting the definition of the integral over even elements. For a single complex NC variable

$$a:\ a^2=0,\quad aa^*=a^*a\ , \tag{1}$$

it is defined according to

$$\int da^*da\,a^*a=1\ , \tag{2}$$

all other integrals vanishing. If

$$a=c_1c_2,\quad a^*=c_2^*c_1^*\ , \tag{3}$$

the c_i's being odd Grassmann variables, the definition (1) gives the same result as the Berezin integral on the c_i's:

$$\int dc_1^*dc_1dc_2^*dc_2c_2^*c_1^*c_1c_2=1\ . \tag{4}$$

Notice that, according to such a definition,

$$\int da^*da\,\exp(a^*a)=1 \tag{5}$$

with a plus sign in the exponent. The generalization to more degrees of freedom,

$$a_h:\ a_h^2=0,\quad a_ha_k=a_ka_h,\quad a_h^*a_k=a_ka_h^*\ , \tag{6}$$

is obvious:

$$\int\prod_h da_h^*da_ha_h^*a_h=1\ , \tag{7}$$

all other integrals vanishing. It is then easy to see that

$$\int[da^*da]\exp\left(\sum_{h,k}a_h^*A_{h,k}a_k\right)=\mathrm{per}(A) \tag{8}$$

where

$$[da^*da]=\prod_h da_h^* da_h \tag{9}$$

and per(A) is the permanent of the matrix A.

Let us now explain in which sense relations such as Eq. (3) can be considered a change of variables. They cannot obviously be inverted, and therefore an action defined in terms of the c^*'s and c's cannot in general be expressed through the a^*'s and a's. But in any nonvanishing Berezin integral all the c^*'s and c's must appear, so that we can always rearrange them into products of the a^*'s and a's. We can therefore perform the change of variables (3) at any nonvanishing order of perturbation theory, or more generally in the evaluation of any nonvanishing quantity.

Before proceeding further we must introduce the notion of order of nilpotency of a NC variable a. This is the smallest integer n^* for which $a^n=0$ for $n>n^*$. We have discussed so far complex NC variables of order 1. Now the sum of two NC variables of order 1 is of order 2: The order of nilpotency is changed by a change of variables. Changing variables, one has to change accordingly the rule of integration. This can be done in simple cases, but it might turn out to be too difficult in general, and in particular for a Fourier transformation. We will therefore confine ourselves to those manipulations which do not alter the order of nilpotency, and we will distinguish whether the field itself or its Fourier components are NC variables of given order. We will refer to these fields as nilpotent in configuration or momentum space, respectively. They need to be treated in a quite different way. The reason is that the propagator for ordinary fields is evaluated by diagonalizing the wave operator, namely by a Fourier transformation which we have excluded. The propagator of a NC field in configuration space, therefore, must be determined by a hopping expansion. It can thus be shown that such a field behaves as an ordinary field in space-time dimensions where the self-avoiding random walk is a free theory [4]. This happens in more than four dimensions, and conjecturally also in four dimensions. For NC fields in momentum space the evaluation of the propagator does not instead present any difficulty because the wave operator is diagonal in momentum space.

The restriction to a given order of nilpotency is not too severe a limitation, as it might appear at first sight. It is in fact compatible with many relevant symmetries. One example is provided by the quoted model of composite gauge fields [3]. Another example is the following [4]. Consider a NC scalar field of order 1 in configuration space:

$$\phi(x):\ [\phi(x)]^2=0,\ \ \phi^*(x)\phi(y)=\phi(y)\phi^*(x)\ . \tag{10}$$

It is easy to see that the integration measure is invariant under the local gauge transformations

$$\phi(x)\rightarrow e^{i\theta(x)}\phi(x)\ . \tag{11}$$

Moreover these transformations do not change the order of nilpotency, so that this field can be coupled to a gauge field [4]. Another symmetry compatible with a given order of nilpotency is Lorenz invariance. This is trivial for nilpotency in configuration space, but not in momentum space. Under the Lorenz transformation

$$x\rightarrow \Lambda x+d \tag{12}$$

the Fourier components $a(p)$ of the ϕ field transform according to

$$a(p)\rightarrow e^{-id\Lambda p}a(\Lambda p)\ . \tag{13}$$

Also these transformations do not alter the order of nilpotency.

Let us now define the model to be studied. We will work in Euclidean space. To take into account nilpotency it is convenient that the arguments of the nilpotent variables be discrete. Therefore we consider our system in a box.

The Fourier transform of a function f is

$$f(x)=\frac{1}{L^2}\sum_p \tilde{f}(p)e^{ipx} \tag{14}$$

where L is the side of the box and the sum is over the discrete momenta:

$$p_\mu(n)=\frac{2\pi}{L}n_\mu,\ \ n_\mu \text{ integer}\ . \tag{15}$$

For a scalar field,

$$\tilde{\phi}(p)=\frac{1}{\omega(p)}[a^*(p)+a(-p)]\ , \tag{16}$$

where

$$\frac{1}{\omega(p)}=(p^2+m^2)^{-1/2}\exp\left[\frac{-p^2}{\Lambda^2}\right] \tag{17}$$

is the regulated propagator. We are going to investigate the case where $a^*(p),a(p)$ are NC variables of order 1. Note that, with the regularization (17), the ϕ field is of infinite order of nilpotency, but it can be made of finite order by, for instance, a lattice regularization.

We assume the free action to be

$$S_0=\int d^4x\,\phi(x)[-\Box+m^2]_\Lambda\phi(x)\ , \tag{18}$$

the wave operator being regulated in such a way as to have the propagator (17). Note that S_0 differs from the action of ordinary scalars by a factor $\frac{1}{2}$.

Free propagators are defined by

$$\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\rangle_0=\frac{1}{Z_0}\int[da^*da]\tilde{\phi}(p_1)\times\tilde{\phi}(p_2)\exp(S_0) \tag{19}$$

where

$$Z_0=\int[da^*da]\exp(S_0)\ . \tag{20}$$

This definition is analogous to that of the propagators of ordinary scalars in terms of holomorphic variables, apart from the plus sign in the exponent. As we will see this sign is necessary, as a consequence of Eq. (5), to get the right propagator. It is worthwhile mentioning that we get a positive sign for the action also in two fermionic models where the NC field is obtained by a change of variables from an ordinary partition function. One of these models concerns a many-fermion system with a pairing interaction [4]. This confirms the interpretation that NC fields describe composites of fermions.

It is convenient to introduce the variables

$$A(p)=a^*(p)+a(-p),\quad B(p)=A(p)A(-p) \tag{21}$$

which simplify the expression of

$$S_0=\sum_p B(p) \tag{22}$$

and

$$\exp(S_0)=[1+B(0)]\prod_p^{*}[1+2B(p)+2B^2(p)]\ . \tag{23}$$

The star means that the product must be taken over all directions and absolute values of $p\neq 0$, but not over its orientations.

In order to evaluate correlation functions we must arrange the products of $\tilde{\phi}(p)$ into products of $B(p)$'s and use the relations

$$\int[da^*da]B(p)\exp(S_0)=Z_0\ , \tag{24}$$

$$\int[da^*da]B^2(p)\exp(S_0)=\tfrac{1}{2}Z_0,\quad p\neq 0\ . \tag{25}$$

In such a way we get, for the free propagator,

$$\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\rangle_0=\delta(p_1+p_2)\frac{1}{\omega^2(p_1)} \tag{26}$$

the expression valid for ordinary scalars. Many-point correlation functions, however, are different from the corresponding ones of ordinary scalars, due to the exclusion principle obeyed by $\tilde{\phi}(p)$.

Let us now introduce the interaction

$$S_I=\frac{1}{4!}g\int d^4x\,\phi^4(x)$$
$$=\frac{1}{4!}g\frac{1}{L^4}\sum_{p_1,p_2,p_3,p_4}\delta(p_1+p_2+p_3+p_4)\prod_{i=1}^{4}\frac{A(p_i)}{\omega(p_i)} \tag{27}$$

and study the correlation functions

$$\langle\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)\rangle=\frac{1}{Z}\int[da^*da]\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)\times\exp(S_0+S_I)\ . \tag{28}$$

In the above equation,

$$Z=\int[da^*da]\exp(S_0+S_I)\ . \tag{29}$$

We will perform the standard loop expansion, assuming that, as usual, the disconnected contributions cancel out. This cancellation is such a general property that it would be surprising if it were not true in the present case, the more so since it has been proved for a NC field in configuration space [4].

The two-point connected correlation function to one loop is

$$\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\rangle_1=\frac{1}{Z_0}\frac{1}{4!}g\int[da^*da]\frac{A(p_1)}{\omega(p_1)}\frac{A(p_2)}{\omega(p_2)}\times\frac{1}{L^4}\sum_{q_1,q_2,q_3,q_4}\delta(q_1+q_2+q_3+q_4)\prod_{i=1}^{4}\frac{A(q_i)}{\omega(q_i)}\exp(S_0)|_C\ , \tag{30}$$

where the subscript C means that only connected contributions should be included. To get one of such contributions we must pair p_1 to one of the q_i and p_2 to one of the remaining q_i. There are 12 ways to do that. For each such way, after rearranging the A's into B's we get

$$\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\rangle_1=\delta(p_1+p_2)\frac{1}{\omega^4(p_1)}\frac{1}{Z_0}\frac{1}{2}g\int[da^*da]2B^2(p_1)\frac{1}{L^4}\sum_q\frac{B(q)}{\omega^2(q)}\exp(S_0) \tag{31}$$

so that finally integrating over the $a^*(p),a(p)$ we find

$$\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\rangle_1=\delta(p_1+p_2)\frac{1}{\omega^4(p_1)}\frac{1}{2}g\frac{1}{L^4}\sum_{q\neq\pm p_1}\frac{1}{\omega^2(q)}\ . \tag{32}$$

The restriction $q\neq\pm p$ has been kept to give an example of a consequence of nilpotency which is irrelevant in the termodynamic limit.

Let us now evaluate to one loop the four-point connected correlation function:

$$\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4)\rangle_1=\frac{1}{Z_0}\frac{1}{2}\frac{g^2}{(4!)^2}\frac{1}{L^8}\sum_{q_i,t_i}\int[da^*da]\delta(q_1+q_2+q_3+q_4)\times\delta(t_1+t_2+t_3+t_4)\prod_{i=1}^{4}\frac{A(p_i)A(q_i)A(t_i)}{\omega(p_i)\omega(q_i)\omega(t_i)}\exp(S_0)|_C\ . \tag{33}$$

The above equation holds for momenta p_i different from one another. Proceeding as above we find that its divergent contribution is

$$\langle \tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4)\rangle_{1\,\mathrm{div}}=\delta(p_1+p_2+p_3+p_4)\prod_{i=1}^{4}\frac{1}{\omega^2(p_i)}\frac{1}{L^4}\frac{3}{16\pi^2}g^2\ln\frac{\Lambda}{m}\ . \tag{34}$$

This result is the same as in the ordinary ϕ^4 theory. Because of the positive sign of the action, however, the counterterm has opposite sign with respect to that of the ordinary theory with the same sign of g, so that the β function

$$\beta(g)=-\frac{3}{16\pi^2}g^2 \tag{35}$$

is negative and the theory is asymptotically free. Note that, because of the positive sign of the action, the four-point function at the tree level is proportional to g, while in the ordinary theory it is proportional to $-g$. In the present model for g positive the coupling is therefore attractive.

In conclusion we have seen that by means of NC variables we can construct models with properties otherwise unattainable. Moreover these variables are not just abstract mathematical entities devoid of any physical interpretation, but describe composites of fermions, so that such models can be of phenomenological relevance.

We have also argued that NC variables are potentially useful in a variety of problems in particle and many-body physics. We think that the present example makes this possibility more likely, showing how they can be used in actual calculations.

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