# Embedded defects

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We give a prescription for embedding classical solutions and, in particular, topological defects in field theories which are invariant under symmetry groups that are not necessarily simple. After providing examples of embedded defects in field theories based on simple groups, we consider the electroweak model and show that it contains the $Z$ string and a one-parameter family of strings called the $W(\alpha)$ string. It is argued that although the members of this family are gauge equivalent when considered in isolation, each member becomes physically distinct when multistring configurations are considered. We then turn to the issue of stability of embedded defects and demonstrate the instability of a large class of such solutions in the absence of bound states or condensates. The $Z$ string is shown to be unstable for all values of the Higgs boson mass when $\theta_W=\pi/4$. $W$ strings are also shown to be unstable for a large range of parameters. Embedded monopoles suffer from the Brandt-Neri-Coleman instability. Finally, we connect the electroweak string solutions to the sphaleron.

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## I. INTRODUCTION

Topological defects are classical solutions of certain field theories and have been known for nearly three decades. These include domain walls, strings, and monopoles. However, few field theories admit the required topology and the standard model of the electroweak interactions [1] lacks any topological defects. Over the last few years, it has been realized [2] that even if the nontrivial topology required for the existence of a defect is absent in a field theory, it may be possible to have defect-like solutions. The idea is simply that topological defects may be "embedded" in such topologically trivial field theories. Embedded defect solutions are very common, and even the electroweak model admits string solutions. It is the properties of these solutions that are discussed in this paper.

A crucial difference between topological and embedded defects is that the stability of the former is guaranteed by topology while embedded defects are generally unstable against small perturbations. Therefore, if embedded defects are to be significant, some mechanism by which they can be stabilized must be found. At least one embedded defect, the semilocal string [3,4], is stable by itself and electroweak strings can also be classically stable [5–7]. A general mechanism for stabilizing embedded defects was proposed in Ref. [16], where it was shown that scalar bound states on electroweak strings vastly improve their stability. It was also argued that fermionic bound states would improve the string stability and that this mechanism of stabilizing solutions would apply to other saddle-point solutions as well. Hence, the possibility that stable embedded defects exist in the real world must be considered.

In this paper we investigate the existence and stability of embedded defects with particular emphasis on defects in the electroweak model. We first consider an arbitrary pattern of symmetry breaking $G\rightarrow H$ and derive the conditions under which embedded defects are possible (Sec. II). In doing this, we clarify the analysis in Ref. [2] where only the case of a simple group $G$ was considered; here we also treat the case when $G$ is not simple. This extension has direct relevance since the electroweak model is based on $G=SU(2)\times U(1)$, which is not simple. We also find that a suitable choice of basis in the Lie algebra of $G$ reduces the six conditions given in Ref. [2] to two conditions. In Sec. III, we offer a few concrete examples, including the simplest embedded defect—a domain wall embedded in a global O(2) model, the $O(3)\rightarrow O(2)$ string of Ref. [2], and the known string solutions in the electroweak model. We provide further insight into electroweak strings and show that there is a one-parameter family of string solutions—the $W(\alpha)$ string, $\alpha$ being the parameter. All the $W$-string solutions are equivalent in isolation but become distinguishable when patched together. It is also pointed out that the $W$ string is superconducting.

We turn to the issue of stability in Sec. IV, first showing that embedded global defects are unstable by constructing a continuous sequence of field configurations of lowering energy. The same construction is then applied to embedded *gauged* defects when the group $G$ is simple and we find that they are unstable provided a certain condition on the group generators is satisfied. This analysis

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immediately shows that the O(4)$\rightarrow$O(3) monopole [2], the O(3)$\rightarrow$O(2) string and the electroweak $Z$ string at $\theta_W=\pi/4$ are all unstable.

Embedded monopoles fall into the class of "nontopological" monopoles considered by Brandt and Neri [8] and by Coleman [9], who showed that such monopoles always suffer from a long-range instability. We relate the Brandt-Neri-Coleman instability to the more general issue of the instability of a constant magnetic field in a non-Abelian gauge theory [10].

It is a little more involved to show that the electroweak $W$ string is unstable. Fortunately, the realization that the sphaleron and electroweak strings are equivalent (discussed in Sec. VII) and also the elegant derivation of the sphaleron instability can be combined to show that the $W$ string is unstable in the absence of bound states (Sec. VI).

In addition to strings, the electroweak model is known to contain a saddle-point solution called the "sphaleron." The sphaleron is an important solution because it mediates baryon-number-violating processes. We discuss the connection of the sphaleron to the electroweak string in Sec. VII. Our arguments show that the sphaleron can be interpreted as a collapsed segment or loop of electroweak string. Finally, we summarize our findings in Sec. VIII.

## II. EMBEDDING SOLUTIONS

In this section we discuss what conditions must be satisfied for embedded solutions to satisfy the equations of motion. A discussion of the equally important question of stability is postponed until the next section. We first state the problem precisely. Consider a field theory in which a gauge symmetry $G$ is broken to a smaller group $H$ by the condensation of a Higgs field $\phi$. In such a theory there exist topologically stable defect solutions when one of the homotopy groups $\pi_k(G/H)$ is nontrivial, with nontrivial $\pi_0(G/H)$, $\pi_1(G/H)$, and $\pi_2(G/H)$ corresponding to topologically stable domain wall, cosmic string, and monopole solutions, respectively. To construct "embedded" defect solutions, one chooses a subgroup $G_{\rm emb}\subset G$ such that $\pi_k(G_{\rm emb}/H_{\rm emb})$ is nontrivial, where $H_{\rm emb}=H\cap G_{\rm emb}$. The idea is to find a smaller theory containing a topologically stable defect solution and then to embed this solution into the larger theory.

The embedded subtheory is defined by the pair $(G_{\rm emb},\mathcal{V}_{\rm emb})$, where $\mathcal{V}_{\rm emb}$ is the linear subspace of $\mathcal{V}$, $\mathcal{V}$ being the vector space in which the Higgs field $\phi$ lives. We shall use $\mathcal{G}$ and $\mathcal{G}_{\rm emb}$ to denote the Lie algebras of $G$ and $G_{\rm emb}$, respectively. We further require that the subspace $\mathcal{V}_{\rm emb}$ be invariant under the action of the subgroup $G_{\rm emb}$. Let $\alpha$ and $I$ denote indices of the Lie algebra of $G_{\rm emb}$ and indices of the vector space $\mathcal{V}_{\rm emb}$, respectively, and $\bar{\alpha}$ and $\bar{I}$ denote orthogonal directions in $\mathcal{G}$ and $\mathcal{V}$, respectively. (For defining $\mathcal{V}_{\rm emb}^{\perp}$, there is a natural choice of inner product for $\mathcal{V}$, uniquely defined up to an overall constant. For defining $\mathcal{G}_{\rm emb}^{\perp}$, when $G$ contains several simple factors, the correct choice of inner product depends on ratios of the gauge coupling constants, as shall later be discussed in more detail.)

When $\pi_k(G_{\rm emb}/H_{\rm emb})$ is nontrivial, it is well known that topological defect solutions of the form

$$\phi^J(x)=\phi^J_{\rm emb}(x),\quad \phi^{\bar{J}}(x)=0\ ,$$
$$A^{\alpha}_{\mu}(x)=A^{\alpha}_{\mu\,{\rm emb}}(x),\quad A^{\bar{\alpha}}_{\mu}(x)=0 \tag{2.1}$$

exist in the smaller theory.

Before proceeding to the equations of motion, we first discuss how the inner product on $\mathcal{G}$ (and thus the directions for the $\bar{\alpha}$ indices) is defined. When $G$ is a product of $R$ simple factors $G=G_1\times\cdots\times G_R$, the Maxwell term appearing in the Lagrangian may be written as

$$\frac{-1}{4}\sum_{f=1}^{R}\frac{1}{g_f^2}F_f^2\ . \tag{2.2}$$

This Maxwell term defines a $G$-invariant inner product $(,)$ on $\mathcal{G}$, parametrized by the coupling constants $g_1,\dots,g_R$, so that the Maxwell term may be expressed as

$$\frac{-1}{4}(F^a_{\mu\nu}T_a,F^{b\mu\nu}T_b)=\frac{-1}{4}F^a_{\mu\nu}F^{b\mu\nu}(T_a,T_b)\ .$$

Let us choose a basis for $\mathcal{G}$ orthonormal with respect to the inner product $(\ ,\ )$ so that $T_\alpha=T_a$ with $a=1,\dots,P$ spans $\mathcal{G}_{\rm emb}$ and $T_{\bar{\alpha}}=T_a$ with $a=P+1,\dots,R$ spans $\mathcal{G}_{\rm emb}^{\perp}$. In this basis the structure constants are $f_{abc}=([T_a,T_b],T_c)$. Invariance of the inner product is expressed by the relation $([T_a,T_b],T_c)+(T_b,[T_a,T_c])=0$. From this it follows that $f^{ab}{}_c=f^{ca}{}_b$, making $f^{ab}{}_c$ antisymmetric in all pairs of indices. (It is only in such a special orthonormal basis that the distinction between upper and lower indices disappears.)

We now write the equations of motion

$$[\mathcal{D}^\mu\mathcal{D}_\mu\phi]^A=-\frac{\partial V[\phi]}{\partial\phi^A}\ , \tag{2.3}$$

$$\eta_{ab}[\mathcal{D}_\mu F^{\mu\nu}]^b=J_a\ , \tag{2.4}$$

where $\eta_{ab}=(T_a,T_b)=\delta_{ab}$ and

$$\mathcal{D}_\mu=\partial_\mu+iA^a_\mu T_a\ ,$$
$$[T_a,T_b]=if_{abc}T_c\ , \tag{2.5}$$
$$J^\mu_a i=i[\phi^\dagger T_a(\mathcal{D}^\mu\phi)-(\mathcal{D}^\mu\phi)^\dagger T_a\phi]\ .$$

We know that the equations of motion for unbarred indices, that is, those indices corresponding to the embedded subtheory, are satisfied. The fact that $\mathcal{G}_{\rm emb}$ acting on $\mathcal{V}_{\rm emb}$ also lies in $\mathcal{V}_{\rm emb}$ ensures that the left-hand side of Eq. (2.3) is nonvanishing only for the unbarred (embedded) directions.

If the condition

$$\frac{\partial V[\phi]}{\phi_\phi\bar{I}}=0\quad\text{for }\phi\in\mathcal{V} \tag{2.6}$$

is satisfied, then the right-hand side of (2.3) vanishes for the barred indices. For the left-hand side of (2.5) the closure of $\mathcal{G}$ ensures that $[\mathcal{D}_\mu F^{\mu\nu}]$ lies entirely in $\mathcal{G}$, and the orthogonality of the barred directions ensures that $\eta_{ab}$ does not mix unbarred into barred directions. For the currents, *a priori* the possibility exists that the barred currents might not vanish despite the fact that both $\phi$ and $[\mathcal{D}_\mu\phi]$ lie in $\mathcal{V}_{\rm emb}$. However, *if*

$$T^{\bar{a}}\phi \in \mathcal{V}^{\perp} \quad \text{for } \phi \in \mathcal{V} \ , \tag{2.7}$$

then the barred currents do indeed vanish.

In this section we have given a prescription for embedding defects in larger theories. First a subgroup $G_{\rm emb} \subset G$ is chosen so that $\pi_k(G_{\rm emb}/H_{\rm emb})$ is nontrivial. Then a linear subspace $\mathcal{V}_{\rm emb} \subset \mathcal{V}$ is chosen so that $\mathcal{V}_{\rm emb}$ is invariant under the action of $G_{\rm emb}$ and a topologically stable solution is found in the smaller subtheory. This solution in the smaller theory is also a solution in the larger theory provided that conditions (2.6) and (2.7) are satisfied.

## III. EXAMPLES

We now present some concrete examples. The first series of examples arises in models with an O($N$) symmetry, with $N$ taking various values. Consider the static energy functional

$$E[\phi(x)] = \int d^3x \left[ \tfrac{1}{2}(\mathcal{D}_k\phi_a)(\mathcal{D}_k\phi_a) + V[\phi] + \mathbf{B}^2 \right] , \tag{3.1}$$

where $a=1,\ldots,N$ and $V[\phi]=\lambda[\phi_a\phi_a-\eta^2]^2$ for specificity.

We form the simplest embedded solution, an embedded domain wall, by setting $N=2$. In this case $G=$O(2). We take $G_{\rm emb}=Z_2$ by restricting $\phi_2=0$, so that there is an embedded domain wall in the smaller theory. Clearly, the potential $V[\phi]=\lambda[\phi_a\phi_a-\eta^2]^2$ satisfies condition (2.6). Furthermore, the embedded domain wall does not generate any nonvanishing sources for the gauge fields.

By setting $N=3$, we construct an embedded string solution [11,12]. Here $G=$SO(3). We choose $G_{\rm emb}=$SO(2) and define $\mathcal{V}_{\rm emb}$ by setting $\phi_3=0$. The embedded string solution is

$$\begin{aligned} &\phi = f_{\rm vor}(r) e^{iT^3\theta} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \\ &A_i^3 = (A_i)_{\rm vor} , \\ &A_i^{\bar{a}} = 0 \ , \quad \bar{a}=1,2 \ , \end{aligned} \tag{3.2}$$

where $r,\theta$ are cylindrical coordinates and the subscript vor indicates the Abrikosov-Nielsen-Olesen vortex solution. It is easily verified that the conditions (2.6) and (2.7) are satisfied by this candidate configuration.

Likewise, by setting $N=4$, one may embed the 't Hooft–Polyakov monopole into a model with $G=$O(4). Here we set $G_{\rm emb}=$O(3) and $H_{\rm emb}=$O(2) so that $\pi_2(G_{\rm emb}/H_{\rm emb})=Z$ and write down the candidate solution

$$\phi = \begin{bmatrix} \vec{\phi}_{\rm tHP} \\ 0 \end{bmatrix}, \quad A_i^{a} = [A_i^a]_{\rm tHP}, \quad A_i^{\bar{a}} = 0 \ , \tag{3.3}$$

where the subscript tHP stands for the 't Hooft–Polyakov monopole solution. It is readily verified that the embedded solution satisfies the equations of motion.

As an example of a nonsimple group $G$, we consider the Weinberg-Salam electroweak model [1,23], with $SU(2)_L \times U(1)_Y \to U(1)_Q$. In the electroweak model we have

$$S = \int d^4x \left[ \frac{-1}{4} W^a_{\mu\nu} W^{\mu\nu a} + \frac{-1}{4} B_{\mu\nu} B^{\mu\nu} + [\mathcal{D}_\mu \Phi]^\dagger [\mathcal{D}^\mu \Phi] - \lambda(\Phi^\dagger\Phi - v^2)^2 \right] , \tag{3.4}$$

where

$$\mathcal{D}_\mu = \partial_\mu + \frac{ig}{2} W^a_\mu \tau^a + \frac{ig'}{2} B_\mu Y \ . \tag{3.5}$$

With $\Phi(x)=\binom{0}{\phi(x)}$, it is convenient to rewrite

$$\begin{aligned} Z_\mu &= \cos\theta_W W^3_\mu - \sin\theta_W B_\mu \ , \\ A_\mu &= \sin\theta_W W^3_\mu + \cos\theta_W B_\mu \ , \end{aligned} \tag{3.6}$$

where $\tan\theta_W = g'/g$. Consequently,

$$\mathcal{D}_\mu = \partial_\mu + i\frac{g}{2}[W^1_\mu \tau^1 + W^2_\mu \tau^2] + \frac{i}{2}\frac{gg'}{\sqrt{g^2+g'^2}}[\tau^3+Y]A_\mu + \frac{i}{2}\frac{g^2\tau^3 - g'^2 Y}{\sqrt{g^2+g'^2}} Z_\mu \ . \tag{3.7}$$

In the electroweak model, there are two types of embedded string solutions—the $Z$ string and the $W$ string. The embedded $Z$ string may be written as [5,13]

$$\begin{aligned} &\Phi(r,\varphi) = v f_{\rm vor}(r) \begin{bmatrix} 0 \\ e^{i\varphi} \end{bmatrix} , \\ &\mathbf{Z} = \frac{v_{\rm vor}(r)}{r}\hat{\mathbf{e}}_\varphi, \quad \mathbf{W}=0, \quad \mathbf{A}=0 \ . \end{aligned} \tag{3.8}$$

In Ref. [2] it was shown that the electroweak model has a $W$-string solution that is not gauge equivalent to the $Z$-string solution. Here we show that there exists a one-parameter continuous family of such solutions. Originally, the $W$-string solution was written as

$$\begin{aligned} \Phi(r,\theta) &= v f_{\rm vor}(r) U(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= v f_{\rm vor}(r) \begin{bmatrix} \cos\theta \\ i\sin\theta \end{bmatrix} , \\ \mathbf{W}^1 &= \frac{v_{\rm vor}(r)}{r}\hat{\mathbf{e}}_\theta, \quad \mathbf{W}^2 = \mathbf{W}^3 = \mathbf{B} = 0 \ , \end{aligned} \tag{3.9}$$

where the subscript vor indicates the familiar Abrikosov-Nielsen-Olesen vortex solution [12] and $U(\theta)=e^{i\theta\tau_1}$.

By making a new choice of gauge in which the magnetic field lies along the $T_3$ direction, we show that the $W$ string is not merely a $Z$ string disguised in a different gauge. This can be accomplished by a global rotation such as $U=\exp[i(\pi/2)T_2]$, so that (3.9) becomes

$$\Phi(r,\theta)=\frac{1}{\sqrt{2}}f_{\rm vor}(r)\begin{bmatrix} e^{i\theta} \\ e^{-i\theta}\end{bmatrix},$$
$$\mathbf{B}=\mathbf{W}^1=\mathbf{W}^2=0,\quad \mathbf{W}^3=\frac{v_{\rm vor}(r)}{r}\hat{\mathbf{e}}_\theta . \tag{3.10}$$

This choice of gauge facilitates comparison with the $Z$ string (3.8). For the $Z$ string the $U(1)_Y$ gauge fields are excited; however, for the $W$ string the $U(1)_Y$ current from the upper component of the Higgs field is exactly canceled by a contribution from the lower component. We note that the $W$ string has a zero mode, leading to the vector type of string superconductivity first discussed by Everett [14]. The orientation of the magnetic field inside the string core is a physical observable with definite gauge transformation and covariant transport properties. Also, it is not invariant under $U(1)_Q$ rotations. The generator of electric charge is $Q(\theta)=U(\theta)Q_0U^{-1}(\theta)$, where $Q_0=\mathrm{diag}(0,1)$. Now it is easy to check that, for example, if $\theta=0$ then

$$e^{i\alpha Q}T_1e^{-i\alpha Q}=\cos\alpha T_1+\sin\alpha T_2 .$$

Therefore, one can construct the one-parameter family of $W$-string solutions, parametrized by $\alpha$, which may be written as

$$\Phi(r,\theta)=vf_{\rm vor}(r)e^{i\alpha Q}e^{i\theta\tau_1}e^{-i\alpha Q}\begin{bmatrix}1\\0\end{bmatrix}$$
$$=vf_{\rm vor}(r)\begin{bmatrix}\cos\theta\\ ie^{-i\alpha}\sin\theta\end{bmatrix},$$
$$\mathbf{W}^1=\cos\alpha\frac{v_{\rm vor}(r)}{r}\hat{\mathbf{e}}_\theta ,$$
$$\mathbf{W}^2=\sin\alpha\frac{v_{\rm vor}(r)}{r}\hat{\mathbf{e}}_\theta , \tag{3.11}$$
$$\mathbf{W}^3=\mathbf{B}=0 ,$$

or as

$$\phi(r,\theta)=\frac{1}{\sqrt{2}}f_{\rm vor}(r)\begin{bmatrix}e^{i\theta}\\ e^{-i(\theta+\alpha)}\end{bmatrix},$$
$$\mathbf{B}=\mathbf{W}^1=\mathbf{W}^2=0,\quad \mathbf{W}^3=\frac{v_{\rm vor}(r)}{r}\hat{\mathbf{e}}_\theta . \tag{3.12}$$

The magnetic field in the core carries a $U(1)_Q$ charge. This expression can be generalized to any $Q(\theta)$.

The angular parameter $\alpha$ is not a mere gauge artifact, because differences in $\alpha$ along the same string, or between adjacent strings, can be measured using gauge-invariant operators. Let $A$ and $A'$ be points in the core of the same string or in the cores of adjacent strings. Let $\hat{\mathbf{n}}(A)$ and $\hat{\mathbf{n}}(A')$ be unit vectors pointing along the string at $A$ and $A'$, respectively. We define the operator

$$\mathcal{O}(A,A')=\epsilon^{ijk}\hat{\mathbf{n}}^i(A)W^{jk}(A)P\exp\left[i\int_{a'}^{A}dx^\mu W_\mu(x)\right]$$
$$\times\epsilon^{i'j'k'}\hat{\mathbf{n}}^{i'}(A')W^{j'k'}(A') , \tag{3.13}$$

where the vector field appearing in the exponential is in the adjoint representation. Here $P$ indicates that the exponential is path ordered. The operator takes values $C\cos(\alpha-\alpha')$, where $C$ is a multiplicative constant depending on the vortex core structure. The angles $\alpha$ and $\alpha'$ indicate the orientation of the $W$ field inside the string. Although the angle $\alpha$ describing the orientation of an isolated string is not well defined and can be changed by a global gauge transformation, as shown above the relative angles for $W$ strings patched together, or for different points along the same $W$ string, are physically measurable, gauge-invariant quantities. As described in Ref. [14], variations in $\alpha$ along the string, so that $\partial_z\alpha\neq0$, where $z$ is the direction along the string, give rise to superconducting currents along the string. This would introduce an $A_z$ gauge field to cancel the gradients of $\alpha$.

## IV. STABILITY

In this section we consider the stability of embedded defects. Although the question of which embedded solutions are stable against small perturbations is not answered in its full generality, a large class of embedded solutions are shown to be unstable by explicitly indicating a particular instability. Qualitatively, this mode may be described as a rotation of the embedded solution into a trivial classical vacuum solution. The instability found here applies only when the embedded gauge group $G_{\rm emb}$ acts trivially on the subspace spanned by $\phi_\perp$ and when the potential is of the form $\lambda[\phi^\dagger\phi-v^2]^2$, although we expect the result to apply to a much larger class of potentials. It should be stressed that *all* global embedded defects are unstable to small perturbations, provided the potential has the proper form.

Let us consider an embedded solution in a model with the energy functional given in (3.1) and with the specific form of the potential:

$$V[\phi]=\lambda[\phi^\dagger\phi-\eta^2]^2 . \tag{4.1}$$

Let $\phi^{(0)}(x)$ and $A_j^{(0)}(x)$ be the embedded defect solution. We consider the sequence of configurations

$$\phi(x;\xi)=\cos\xi\phi^{(0)}(x)+\sin\xi\phi_\perp ,$$
$$A_j(x;\xi)=A_j^{(0)}(x) , \tag{4.2}$$

where $\xi$ varies from 0 to $\pi/2$ and where $\phi_\perp$ is constant and independent of position with $\phi_\perp{}^\dagger\phi_\perp=\eta^2$ and $\phi_\perp{}^\dagger\phi_{(0)}(x)=0$. For $\xi=0$ the configuration is simply the original embedded defect solution; for $\xi=\pi/2$ it is the trivial vacuum. One has

$$\phi^\dagger(x;\xi)\phi(x;\xi)=\cos^2\xi\phi^{(0)\dagger}\phi^{(0)}+\sin^2\xi\eta^2 ,$$
$$\mathcal{D}_i\phi(x;\xi)=\cos\xi[\partial_i+iA_i^\alpha(x)T^\alpha]\phi^{(0)}(x)$$
$$+i\sin\xi A_i^\alpha(x)T^\alpha\phi_\perp, \tag{4.3}$$

so that $V[\phi(x,\xi)]=\cos^2\xi V[\phi_0(x)]$. Moreover, if

$$T^\alpha\phi_\perp=0 \tag{4.4}$$

for the embedded directions, in other words, if the gauge fields do not create covariant gradients in the $\phi_\perp$ vacuum, then

$$[\mathcal{D}_i\phi(x;\xi)]^\dagger[\mathcal{D}_i\phi(x;\xi)]=\cos^2\xi[\mathcal{D}_i\phi^{(0)}(x)]^\dagger[\mathcal{D}_i\phi^{(0)}(x)] \ . \tag{4.5}$$

Consequently,

$$E(\xi)=\cos^2\xi[E_{SG}+E_{pot}]+E_{mag} \ , \tag{4.6}$$

where $E_{SG}$ is the scalar gradient energy, $E_{pot}$ is the potential energy, and $E_{mag}$ is the magnetic energy. It is clear that whenever a vacuum direction $\phi_\perp$ satisfying (4.4) can be found, there exists a quadratic instability. In the gauged case, although strictly speaking the continuous sequence of configurations defined in (4.2) does not connect the embedded defect to the vacuum, it is trivial to show using a variant of Derrick's theorem that the final configuration at $\xi=\pi/2$ is unstable against dilations when the number of spatial dimensions is less than four. Under dilatations, the energy scales as $L^{d-4}$ [15]. Therefore, the curvature rapidly spreads out to infinity. For a global embedded defects condition (4.4) is unnecessary, because there are no gauge field to worry about. Therefore, *all* global embedded defects are unstable.

We point out that the existence of a sequence of configurations of lowering energy as given in (4.2) does not necessarily imply that the defect will decay in a finite amount of time. As written, the sequence of configurations in (4.2) has associated with it an infinite inertia, because the vacuum everywhere in space is being rotated without there being a compensating gauge vector potential. (By "inertia" here, we mean time derivative contributions to the energy associated with changing $\xi$.) In the gauged case, we may cutoff this divergence in the inertia by introducing at large distances a compensating gauge field in the time direction $A_0$ to make the covariant derivative $\mathcal{D}_0\phi$ vanish. The global case is more tricky. It is possible that cutting off the inertia by restricting the rotation to a finite volume might destroy the instability by introducing additional spatial gradient energy.

For the SO($N$) examples considered in the previous section, condition (4.4) is clearly satisfied. The direction $\phi_N$ is invariant under SO($N-1$). We now consider two examples for which the procedure for demonstrating instability described above fails. For the $W$ string, with $G=SU(2)_L\times U(1)_Y$, condition (4.4) cannot be satisfied because $U(1)_Y$ does not annihilate any nonvanishing Higgs vector, so the argument above does not apply. For the embedded $Z$ string the situation is slightly more complicated. The above argument fails, except for the special case $\theta_W=\pi/4$ for which $T_Z=\frac{1}{2}[\tau^3-Y]$, which annihilates

$$\phi_\perp^\dagger=\begin{bmatrix}1\\0\end{bmatrix} \ .$$

This last result is of some interest because the stability analysis of Ref. [6] did not include the case of very low Higgs boson masses and it was not clear from the given stability diagram whether it would be possible to find some value of the Higgs boson mass for which the $Z$ string in the standard model with $\sin^2\theta_W=0.23$ is stable. The result here shows that the $Z$ string is unstable for *all* Higgs boson masses at $\theta_W=\pi/4$, making it extremely unlikely for there to be stable solutions for smaller values of $\theta_W$ (also see Ref. [7]).

So far we have ignored the possibility that there may be bound states on the embedded defects. It has been shown [16] that such bound states can considerably enhance the stability of the defect. The physical reason behind this enhancement is the same as the reason behind the existence of nontopological solitons [17] and is discussed in some detail in Ref. [16]. Mathematically, the introduction of a bound state would result in the presence of additional terms in the varied energy functional (3.1) proportional to $\sin^2\xi$. With these additional terms, it is possible that $\xi=0$ describes a local minimum of the energy and so there is no instability towards increasing $\xi$.

For monopoles, the issue of stability has been addressed by Brandt and Neri [8] and by Coleman [9]. They find that the asymptotic magnetic field of a monopole has an unstable mode unless the monopole is topologically stable. Heuristically, this observation can be related to earlier work on the stability of a constant magnetic field in a non-Abelian gauge theory [10]. Nielsen and Olesen showed that in a non-Abelian gauge theory a constant[1] magnetic field has a classically unstable mode. Quantum mechanically, this corresponds to some of the gauge particles having a negative mass squared. The existence of such a mode can be understood in terms of Landau levels. For the charged vector bosons (*charged* here meaning charged with respect to the direction of the constant magnetic field $B$), one has

$$E_n^2=m^2+p_z^2+(2n+1)eB-2eBs \ , \tag{4.7}$$

where $s$ and $m$ are the spin and the mass of the particle, respectively, and $z$ is the direction of the magnetic field. For massless charged vector bosons whose magnetic moments are aligned with the field, $n=0$ and $p_z<eB$ we have that the energy becomes imaginary, signaling that this mode is unstable. For the 't Hooft–Polyakov monopole, there is a massless U(1) gauge field, and the charged gauge fields are massive. Thus the instability described above is avoided. Now let us embed the 't Hooft–Polyakov monopole in a larger theory, with an SO(4) symmetry, so that the unbroken symmetry group is SO(3). In the embedded theory there are massless charged vector particles. Hence the long-range field of the embedded object can decay by the instability described above.

## V. *W*-STRING INSTABILITY

Here we will show that the bare $W$ string is unstable for a large range of parameters. The proof follows from the observation that the sphaleron and the $W$ string are closely related (see the following section) and so the instability of the sphaleron found by Manton [19] might well

[1]By *constant* we shall mean that the covariant derivative of the gauge curvature vanishes.

apply to the $W$ string also.

Consider the family of configurations parametrized by $\mu$:

$$\Phi(r,\theta,z;\mu)=\sin\mu \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos\mu f_{\rm vor}(r) \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} ,$$
$$A_j(r,\theta,z;\mu)=-iv_{\rm vor}(r)[\partial_j U]U^{-1} , \tag{5.1}$$

where

$$U(r,\theta,z;\mu)=\begin{bmatrix} \sin\mu - i\cos\mu\cos\theta & \cos\mu\sin\theta \\ -\cos\mu\sin\theta & \sin\mu + i\cos\mu\cos\theta \end{bmatrix} . \tag{5.2}$$

Here we smoothly deform the $W$ string, corresponding to $\mu=0$, into the trivial vacuum, corresponding to $\mu=\pi/2$.

The energy per unit length as a function of $\mu$ is

$$E(\mu)=2\pi\int_0^\infty r dr \left[ \cos^2\mu \left\{ \frac{v'^2}{g^2r^2}+f'^2+\frac{f^2(1-g)^2}{r^2} \right\} + \frac{\sin^2\mu\cos^2}{r^2}[v^2(1-f)^2-2v(1-f)f(1-v)]+\cos^4\mu\,\lambda(1-f^2)^2 \right] . \tag{5.3}$$

It is clear that $E$ decreases monotonically as $\mu$ varies from 0 to $\pi/2$ provided that

$$\bar{E}=\sin^2\mu\cos^2\mu \int \frac{dr}{r}[v^2(1-f)^2-2v(1-f)(1-v)] + \cos^2\mu \int \frac{dr}{r} f^2(1-v)^2 \tag{5.4}$$

also decreases monotonically. Let us denote the two integrals in (5.4) by $I_1$ and $I_2$, respectively. $\bar{E}$ decreases monotonically over the desired range if and only if $I_2 > I_1$. We have checked this condition numerically for certain values of the parameters $\lambda$ and $g$ and found it to be satisfied in every case. This demonstrates that the $W$ string is unstable for the parameters that we considered.

We wish to remark that if we could show that

$$s(\xi)\equiv\xi(1-\xi)[v^2(1-f)^2-2vf(1-f)(1-v)] + \xi f^2(1-v)^2 \tag{5.5}$$

is an increasing function of $\xi$ ($\xi=\cos\mu$), then the condition regarding $\bar{E}$ would also be satisfied. Now it is straightforward to show that $s(\xi)$ is maximum at $\xi=1$ if

$$(1+\sqrt{2})f(1-v)\geq v(1-f) \tag{5.6}$$

for all $r$. Numerical evaluations of the Nielsen-Olesen vortex profile have shown that (5.6) is satisfied for almost all $r$ for a large range of values of $\lambda$. The inequality (5.6) is violated only in the large-$r$ region, where the integrands in (5.4) are exponentially small. So the contributions that could change the monotonic increase of $E_1$ with $\xi$ are exponentially suppressed and $E_1(\xi)$ is an increasing function of $\xi$ for a wide range of parameters.[2]

## VI. ELECTROWEAK STRINGS AND THE SPHALERON

In this section, we connect [18] the electroweak string solutions with the sphaleron solution discovered by Manton [19]. For small $\theta_W$, the sphaleron solution is approximately [20]

$$\Phi=f_s(r)\begin{bmatrix} e^{i\phi}\sin\theta \\ \cos\theta \end{bmatrix} \equiv f_s(r)U\begin{bmatrix} 0 \\ 1 \end{bmatrix} ,$$
$$A_\mu\equiv W^a_\mu\tau^a=-iv_s(r)[\partial_\mu U]U^{-1} , \tag{6.1}$$

where

$$U=\exp[i\theta(\sin\phi\tau_1+\cos\phi\tau_2)] = \begin{bmatrix} \cos\theta & \sin\theta e^{i\phi} \\ -\sin\theta e^{-i\phi} & \cos\theta \end{bmatrix} . \tag{6.2}$$

The sphaleron solution in (6.1) necessarily has an accompanying electromagnetic field which can readily be calculated to first order in $\theta_W$ [20].

A comparison of (6.1) with (3.11) suggests that the sphaleron configuration (6.1) is precisely that of a degenerate "twisted" loop of $W$ string in which $\alpha=\phi$. A twisted loop of $W$ string can be collapsed to a single point, thus becoming a sphaleron.

One can also obtain different interpretations of the sphaleron in terms of electroweak strings by considering various slices of (6.1). For example, slicing the sphaleron with the $xz$ or $yz$ planes yields $W$ strings with different values of $\alpha$ ($\alpha=\pi/2$ and $\alpha=0$, respectively). Therefore, stretching the sphaleron along any axis in the $xy$ plane gives finite segments of $W$ strings.

The final interpretation of the sphaleron seems like the most interesting. It is possible to arrive at this interpretation in two ways. The $xy$ slice of the sphaleron, obtained by setting $\theta=\pi/2$ in (6.1), is the $Z$ string defined in (3.8), up to a deformation of the profile functions and a global gauge transformation. If one were to stretch the sphaleron along the $z$ axis, we would get a finite segment of $Z$ string. From the work of Nambu [13] we know that a finite segment of $Z$ string ends on magnetic monopoles. Hence, the sphaleron is equivalent to a monopole sitting adjacent to an antimonopole along the $z$ axis.

This interpretation can be arrived at directly by looking at the Higgs field configuration of the electroweak monopole found by Nambu [13],

[2]Note that a condition similar to (5.6) is assumed to hold in the case of the sphaleron [19].

$$\Phi = f_m(r) \begin{bmatrix} e^{i\phi}\sin\theta/2 \\ \cos\theta/2 \end{bmatrix} , \tag{6.3}$$

and comparing to the sphaleron Higgs field configuration given in (6.1). [The gauge fields for the monopole are given by the same formula as in (6.1).] The two configurations are identical up to the profile functions and, more importantly, up to a factor of 2 wherever $\theta$ appears. Therefore, as $\theta$ varies from 0 to $\pi/2$ in (6.1), the full monopole configuration is mapped out. As $\theta$ subsequently varies from $\pi/2$ to $\pi$, the antimonopole configuration is mapped out.

However, the factor of $\frac{1}{2}$ inside the trignometric functions in (6.3) are important when one finds the gauge field configuration. The derivatives in (6.1) mean that there will be additional factors of $\frac{1}{2}$ that would spoil the complete equivalence of the monopole-antimonopole pair and the sphaleron. This observation agrees with the analysis of Hindmarsh and James [21] in which they find that the sphaleron is a monopole-antimonopole pair but, in addition, some currents are present.

An issue that we have not investigated but feel could be interesting is the possible connection of electroweak strings with the (deformed) sphaleron solutions found by Yaffe [22] for large values of the Higgs boson mass.

## VII. CONCLUSIONS

We summarize our main results.

(1) We described a procedure by which embedded defect solutions may be constructed in Sec. II. The procedure applies whether or not the symmetry group of the theory is simple. We provided some examples. In particular, it was shown that the electroweak model contains a $Z$ string and a one-parameter family of $W$ strings.

(2) In Sec. III we considered the stability of embedded defects in the absence of bound states. By considering a specific perturbation of the embedded defect solution, we showed that embedded global defects are unstable and the embedded gauge defects are unstable if condition (4.4) is satisfied. By an application of this condition, we showed that the electroweak $Z$ string is unstable when $\sin^2\theta_W = \frac{1}{2}$. Using another argument (in Sec. V), we showed that the $W$ string is unstable for a wide range of parameters.

(3) In Sec. VI, we showed that the sphaleron may be reinterpreted as segments or loops of electroweak string.

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