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Gauged WZW models and non-Abelian duality

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We consider WZW models based on the non-semi-simple algebras that were recently constructed as contractions of corresponding algebras for semisimple groups. We give the explicit expression for the action of these models, as well as for a generalization of them, and discuss their general properties. Furthermore we consider gauged WZW models based on these non-semi-simple algebras and we show that they are equivalent to non-Abelian duality transformations on WZW actions. We also show that a general non-Abelian duality transformation can be thought of as a limiting case of the non-Abelian quotient theory of the direct product of the original action and the WZW action for the symmetry gauge group H. In this action there is no Lagrange multiplier term that constrains the gauge field strength to vanish. A particular result is that the gauged WZW action for the coset $(G_k \otimes H_l)/H_{k+l}$ becomes in a certain limit, involving $l \rightarrow \infty$, the dualized WZW action for G_k with respect to the subgroup H.

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I. INTRODUCTION

One of the most striking symmetries in string theory is that of duality. The simplest example of a theory with this symmetry is that of a single boson compactified on a circle of radius R [1]. From the mathematical point of view the partition function of the theory is invariant under the duality transformation $R \rightarrow 1/R$ (in Planck units) for all genera (a redefinition of the constant dilaton is also necessary to ensure the invariance of the string coupling constant) and from a more physical point of view invariance under duality implies that physics at small scales is indistinguishable from physics at large scales and that a smaller distance beyond which probing physics does not make sense should exist [1–3]. Duality is not a property of pointlike objects. Moreover it is exclusively stringy in the sense that it is not also a property of higher than one-dimensional extended objects (p-branes) [4].

This simple case was generalized to arbitrary toroidal compactifications [5,3,6] and subsequently it was realized, at the nonlinear σ-model level, that duality was a symmetry of all string vacua with one [7] or more Abelian isometries [8–10]. In the case of d Abelian isometries the duality transformations enlarged to $O(d,d,R)$ ones relate apparently different curved backgrounds in string theory and can be used to generate new solutions from known ones [11,12]. Also duality plays an important role in discussing cosmology in the context of string theory [13–15]. It has been argued [8] that in the case of d Abelian isometries with compact orbits the duality group $O(d,d,Z)$ [9] interpolates between different backgrounds that are manifestations of the same conformal field theory (CFT) (for the noncompact $SL(2,\mathbb{R})/\mathbb{R}$ case see [16–20]). An important feature is that the dual of such background has also the same number of Abelian isometries and that its dual is the original model. For the relation between Abelian duality transformations and marginal perturbations of string backgrounds see [21] and for review articles see [22,23].

There is a second kind of duality transformation which is much less understood, where the isometries with respect to which the dualization is done form a non-Abelian group [24] (for earlier work see [25]). This has very important consequences. First of all generically the dual model has much less symmetry that the original one and the isometry group usually disappears or gets smaller [24–28] (in fact the local original symmetry seems to manifest itself in a nonlocal way in the dual model [26]). Moreover, it has been argued that the non-Abelian duality transformations interpolate between solutions not of the same CFT but of different ones possibly related by orbifold construction [24,26] (for an extensive treatment of global issues on duality transformations see [27]). The analogue of the duality group of $O(d,d,Z)$ for Abelian duality is not known and in fact because the initial isometry group is not preserved, one does not know how to find the "inverse" transformation.

A common characteristic of all duality transformations is that they can be formulated in a way that the action is that of the original theory written in a gauge invariant way by using gauge fields and a Lagrange multiplier term that constrains, upon integration in the path integral, the gauge field strength to vanish [25], thus giving (after gauge fixing) the original model [8,24]. The dual theory is obtained by integrating instead over the gauge fields. At this point the procedure resembles the one that is being extensively followed in the case of gauged Wess-Zumino-Witten (WZW) models [29], when the original theory is a WZW model [30], or more generally in the case of non-Abelian quotient models where a target spacetime symmetry is being gauged. For instance, similarly to what we have already mentioned for the case of non-Abelian duality transformations, in the case of gauged WZW models the original global symmetry that is being gauged also disappears; i.e., it is being gauge fixed. Therefore one may wonder whether or not this ap-

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parent similarity can be made more precise. It will be explained that such a relationship is found in the context of contraction of certain non-Abelian quotient models and on gauged WZW models based on a particular class of non-semi-simple groups.

Recently WZW models based on non-semi-simple groups have been constructed [31–33]. It was shown in [32] that the symmetry current algebra for these models can be obtained by a contraction procedure on the current algebra of the WZW model for the direct product group $G\otimes H$. A feature all of these models have in common is that their central charge is an integer. A bosonization and computation of the spectrum of the first and the simplest of these theories [31] with $G=\mathrm{SO}(3)$ and $H=\mathrm{SO}(2)$ and denoted by E_2^c can be found in [34]. Gauged WZW models were also constructed by gauging various anomaly-free subgroups of E_2^c [34,35]. In particular the three-dimensional (3D) model in [35] was shown to correspond to a correlated limit of the charged black string background $\mathrm{SL}(2,\mathbb{R})\otimes\mathbb{R}/\mathbb{R}$ of [36,37] both in the semiclassical limit as well as when all α' corrections are included. Moreover, although curvature singularities are still present it was shown that they can be removed via Abelian duality transformations that map this background to a flat spacetime with constant antisymmetric tensor and dilaton fields (for the neutral black string, that is directly related to the two-dimensional (2D) black hole [38], a similar conclusion holds). One of the aims of this paper is to generalize this as much as possible to the general models of [32].

The organization of the paper and some of the main results are as follows. In Sec. II in order to set-up our notation and for completeness first we review the current algebra construction of [32] for WZW models based on a class of non-semi-simple algebras as a particular correlated limit of the current algebra for the direct product $G\otimes H$. Next we derive an explicit form of the corresponding WZW action that reveals all the essential properties of these models, such as the existence of $\dim(H)$ null Killing vectors, by generalizing the work of [33]. We also explicitly show how this action can be obtained from a correlated limit on the WZW action for $G\otimes H$. In Sec. III we formulate gauged WZW models by gauging an anomaly-free subgroup. Before the contraction this corresponds to the gauged WZW models for the cosets $(G\otimes H)/H$ and we show the explicit correspondence. Unlike the original WZW models who have integer central charges the gauged models have in general rational central charges. Section IV contains two main results. First, general non-Abelian duality transformations are shown to correspond to limiting cases of direct product models, where a target spacetime symmetry is being gauged. In these models before the limit is taken the gauge field strength is *not* constrained to vanish. As a particular case non-Abelian transformations on the WZW model for a group G with respect to the vectorial action of a subgroup H can be thought of as a limit of the gauged WZW model for the coset $(G\otimes H)/H$. The second result is that the gauged WZW models of Sec. III are equivalent to non-Abelian duality transformations on the WZW model for the direct product $H\otimes \mathrm{U}(1)^{\dim(G/H)}$. As we shall see, our formulation of non-Abelian duality makes possible to compute the α' corrections to the semiclassical expressions for the σ-model background fields, by making contact with known results from the coset models. We end the main part of the paper with concluding remarks and discussion of our results in Sec. V. In Appendix A we extend the results of Sec. III to the case of axial gauging and we show that the resulting curved backgrounds can be obtained from the flat one with constant antisymmetric tensor and dilaton fields. In Appendix B we extend the construction of [32] to cover more general cases. In Appendix C we present a CFT description for the generalization of the plane wave model of [31] in higher dimensions.

II. WZW MODELS BASED ON NON-SEMI-SIMPLE GROUPS

In this section we will construct WZW models based on *non-semi-simple* groups. This will be done by first reviewing the work of [32] on the construction of the current algebra for such models via a contraction of the current algebra for WZW models based on *semi-simple* groups. Then we will give explicit expressions for the action of the resulting WZW σ models by following [33]. Many other formulas derived in this section will be useful in subsequent ones.

Let us consider the WZW model for the direct product group $G\otimes H$, where G and H are groups (throughout this section they are taken to be compact ones) and where G should contain a subgroup isomorphic to H. The holomorphic currents associated with the current algebra symmetry of the corresponding WZW model are $\hat{g}=\{u_i,R_\alpha\}$, $\hat{h}=\{v_i\}$, where $i=1,2,\ldots,\dim(H)$ and $\alpha=1,2,\ldots,\dim(G/H)$. They obey the operator product expansions (OPE's)[1]

$$
\begin{aligned}
&u_i u_j \sim \frac{i f_{ij}{}^k u_k}{z-w}+\frac{k_G \eta_{ij}}{(z-w)^2}\ , \quad v_i v_j \sim \frac{i f_{ij}{}^k v_k}{z-w}+\frac{k_H \eta_{ij}}{(z-w)^2}\ ,\\
&R_\alpha R_\beta \sim \frac{i M_{\alpha\beta}{}^i u_i + i S_{\alpha\beta}{}^\gamma R_\gamma}{z-w}+\frac{k_G \eta_{\alpha\beta}}{(z-w)^2}\ , \qquad (2.1)\\
&u_i R_\alpha \sim \frac{i M_{i\alpha}{}^\beta R_\beta}{z-w}\ ,
\end{aligned}
$$

where $f_{ij}{}^k$, $M_{\alpha\beta}{}^i$, and $S_{\alpha\beta}{}^\gamma$ are structure constants of the corresponding Lie algebras one can use to compute the Killing metrics $\eta_{ij},\eta_{\alpha\beta}$. The levels k_G and k_H are assumed to be positive integers. The energy-momentum tensor and the central charge of the corresponding

[1]Throughout this paper OPE's that are not written down explicitly are assumed to have only regular terms. Also we will not explicitly mention the various Lie algebras since one can easily read them from the associated OPE's. Moreover, we will use the same symbols for Lie algebra and current algebra generators.

Virasoro algebra are

$$T=\frac{:u^2+R^2:}{2(k_G+g_G)}+\frac{:v^2:}{2(k_H+g_H)}\ ,$$
$$c=\frac{k_G\dim(G)}{k_G+g_G}+\frac{k_H\dim(H)}{k_H+g_H}\ , \tag{2.2}$$

where g_G, g_H are the dual Coxeter numbers for G and H and where the regularization prescription of [39] is assumed in writing current bilinears. As in [32] we let

$$T_i=u_i+v_i\ ,\quad F_i=\epsilon(u_i-v_i)\ ,\quad P_\alpha=\sqrt{2\epsilon}\,R_\alpha\ ,$$
$$k_G=\tfrac{1}{2}(\beta+\alpha/\epsilon)\ ,\quad k_H=\tfrac{1}{2}(\beta-\alpha/\epsilon)\ , \tag{2.3}$$

and rewrite (2.1) in the basis $\{J_A\}=\{P_\alpha,T_i,F_i\}$. In the limit $\epsilon\to 0$ we discover a new current algebra *not* equivalent to (2.1), because the transformation (2.3) is not invertible in that limit. Let us also note that $\beta\in\mathbb{R}^{\dagger}$ and $\alpha\in\mathbb{R}$. The OPE's of the new current algebra one obtains are

$$T_iT_j\sim\frac{if_{ij}{}^kT_k}{z-w}+\frac{\beta\eta_{ij}}{(z-w)^2}\ ,\quad T_iP_\alpha\sim\frac{iM_{i\alpha}{}^\beta P_\beta}{z-w}\ ,$$
$$T_iF_j\sim\frac{if_{ij}{}^kF_k}{z-w}+\frac{\alpha\eta_{ij}}{(z-w)^2}\ , \tag{2.4}$$
$$P_\alpha P_\beta\sim\frac{iM_{\alpha\beta}{}^iF_i}{z-w}+\frac{\alpha\eta_{\alpha\beta}}{(z-w)^2}\ .$$

We will denote this current algebra by $\hat{g}_h^c$ and the corresponding Lie algebra by g_h^c (for the group G_h^c will be used). The holomorphic stress tensor and the central charge of the corresponding Virasoro algebra obtained from (2.2), using (2.3) in the $\epsilon\to 0$ limit, read

$$T=\frac{:P^2+2FT:}{2\alpha}-\frac{\beta+g_G+g_H}{2\alpha^2}:F^2:\ ,$$
$$c=\dim(G)+\dim(H)\ . \tag{2.5}$$

Of course with respect to the above energy-momentum tensor all currents are primary fields of conformal dimension one. The OPE's in (2.4) define a quadratic form

$$\Omega_{AB}=\begin{array}{c|ccc} & P_\beta & T_j & F_j\\ \hline P_\alpha & \alpha/\beta\eta_{\alpha\beta} & 0 & 0\\ T_i & 0 & \eta_{ij} & \alpha/\beta\eta_{ij}\\ F_i & 0 & \alpha/\beta\eta_{ij} & 0\end{array}\ , \tag{2.6}$$

which is symmetric, i.e., $\Omega_{AB}=\Omega_{BA}$, a group invariant, i.e., $f^D_{AB}\Omega_{CD}+f^D_{AC}\Omega_{BD}=0$ and the inverse matrix

$$\Omega^{AB}=\begin{array}{c|ccc} & P_\beta & T_j & F_j\\ \hline P_\alpha & \eta^{\alpha\beta} & 0 & 0\\ T_i & 0 & 0 & \eta^{ij}\\ F_i & 0 & \eta^{ij} & -\beta/\alpha\eta^{ij}\end{array}\ \beta/\alpha\ , \tag{2.7}$$

obeying $\Omega^{AB}\Omega_{BC}=\eta_C^A$ exists. The above properties of the quadratic form (2.6) are a consequence of the fact that the current algebra (2.4) is a contraction of (2.1) for which the Killing metric η_{AB}, sharing all of these properties, is taking as the quadratic form.[2] Nevertheless one may explicitly verify them.

A form for the WZW action whose symmetry algebra is (2.4) was given for the general model in [32]. However, it is not very explicit (in particular it involves two Wess-Zumino terms) and extracting general conclusions from it is rather difficult. Here we will follow the method, applied explicitly for the case of E_d^c [in our notation $E_d^c=\mathrm{SO}(d+1)^c_{\mathrm{SO}(d)}$] models in [33], which involves an explicit parametrization of the group element $g\in G_h^c$. The latter can be generally parametrized as (summation over repeated indices is implied)

$$g=e^{ia\cdot P}e^{iv\cdot F}h_x\ ,\quad a\cdot P=a^\alpha P_\alpha\ ,\quad v\cdot F=v^iF_i\ , \tag{2.8}$$

where the group element $h_x\in H$ parametrizes $\dim(H)$ parameters x^μ and the a^α's and v^i's the remaining ones. It will be useful at this point to further establish our notation by introducing some useful matrices:

$$C_{ij}(h_x)=\mathrm{Tr}(T_ih_xT_jh_x^{-1})=\beta/\alpha\,\mathrm{Tr}(F_ih_xT_jh_x^{-1})\ ,$$
$$C_{ik}C_j^k=\eta_{ij}\ ,$$
$$L_\mu^i=-i\,\mathrm{Tr}(T^ih_x^{-1}\partial_\mu h_x)\ , \tag{2.9}$$
$$R_\mu^i=-i\,\mathrm{Tr}(T^i\partial_\mu h_xh_x^{-1})=C^i{}_jL^j_\mu\ ,$$
$$M_{ij}=C_{ij}-\eta_{ij}\ ,\quad m_{i\alpha}=M_{i\beta\alpha}a^\beta\ ,\quad n_{ij}=f_{ij}{}^kv_k\ .$$

We will also denote by L_i^μ, R_i^μ the inverses of the matrices L^i_μ, R^i_μ, respectively, and by m^t the transpose matrix of m. Using Duhamel's formula

$$de^H=\int_0^1 ds\, e^{sH}dH\,e^{(1-s)H} \tag{2.10}$$

one can compute the left-invariant Maurer-Cartan form

$$g^{-1}dg=i(da\cdot P'-\tfrac{1}{2}da^\alpha m_\alpha{}^iF_i')+i\,dv\cdot F'+iR^i_\mu dx^\mu T_i'\ , \tag{2.11}$$

where the generators $J_A'=h_x^{-1}J_Ah_x$ satisfy the same commutation relations as the J_A's. Similarly we compute the right-invariant Maurer-Cartan form

$$dg\,g^{-1}=i(da\cdot P+\tfrac{1}{2}da^\alpha m_\alpha{}^iF_i)$$
$$+iR^i_\mu dx^\mu e^{ia\cdot P}e^{iv\cdot F}T_ie^{-iv\cdot F}e^{-ia\cdot P}+i\,dv\cdot F\ . \tag{2.12}$$

To obtain an explicit expression we note that $dg\,g^{-1}=-g\,dg^{-1}$ and then we make use of (2.11) with $(a_i\to -a_i, v_i\to -v_i, h_x\to h_x^{-1})$ and $(P_i\to P_i', F_i\to F_i')$ with the additional contribution of the terms $a\cdot dP'$ and $v\cdot dF'$ (these terms contribute when derivatives with respect to the parameters x^μ of h_x are taken). The result-

[2]For the case of $G=\mathrm{SO}(3)$ and $H=\mathrm{SO}(2)$ the Lie algebra that (2.4) defines appeared before in the context of contraction of Lie groups [40] and in studies of $(1+1)$-dimensional gravity [41]. For the same case the quadratic form (2.6) had appeared in [42].

ing expression is

$$dg\,g^{-1}=i\,da^{\alpha}(P_\alpha+\tfrac{1}{2}m_\alpha{}^iF_i)+i\,dv\cdot F + iR^i_\mu dx^\mu[T_i+m_i{}^\alpha P_\alpha+(\tfrac{1}{2}m_i{}^\alpha m_\alpha{}^j-n_i{}^j)F_j]\ . \tag{2.13}$$

Using the previous expression for the right-invariant Maurer-Cartan form we compute the following matrix defined as $dg\,g^{-1}=i\,dX^ME_M{}^AJ_A$, where $X^M=\{x^\mu,a^\alpha,v^i\}$,

$$E_M{}^A=\begin{array}{c c} & \begin{array}{ccc} P_\beta & T_i & F_i \end{array} \\ \begin{array}{c} x^\mu \\ a^\alpha \\ v^j \end{array} & \begin{pmatrix} R^k_\mu m_k{}^\beta & R^i_\mu & R^k_\mu(\tfrac{1}{2}m_k{}^\gamma m_\gamma{}^i-n_k{}^i) \\ \delta_\alpha{}^\beta & 0 & \tfrac{1}{2}m_\alpha{}^i \\ 0 & 0 & \delta_j{}^i \end{pmatrix} \end{array} \tag{2.14}$$

and its inverse

$$E_A{}^M=\begin{array}{c c} & \begin{array}{ccc} x^\mu & a^\alpha & v^j \end{array} \\ \begin{array}{c} P_\beta \\ T_i \\ F_i \end{array} & \begin{pmatrix} 0 & \delta_\beta{}^\alpha & -\tfrac{1}{2}m_\beta{}^j \\ R_i^\mu & -m_i{}^\alpha & n_i{}^j \\ 0 & 0 & \delta_i{}^j \end{pmatrix} \end{array}\ . \tag{2.15}$$

The zero modes of the holomorphic currents J_A can be constructed as first order differential operators acting on the group parameter space of G using $J_A=iE_A{}^M\partial_M$. Their explicit form is

$$P_\alpha=i\partial_{a^\alpha}-\frac{i}{2}m_\alpha{}^j\partial_{v^j}\ ,$$
$$T_i=iR_i^\mu\partial_{x^\mu}-im_i{}^\beta\partial_{a^\beta}+in_i{}^j\partial_{v^j}\ ,\quad F_i=i\partial_{v^i}\ . \tag{2.16}$$

Using the fact that $iR_i^\mu\partial_{x^\mu}$ separately obey $\mathcal{L}(H)$, i.e., the Lie algebra of the group H, one may explicitly verify, with the aid of the Jacobi identities that the various structure constants obey, that the operators (2.16) indeed generate the Lie algebra g_h^c. The WZW action whose current algebra symmetry is (2.4) is defined as

$$S(g)=\frac{\beta}{2\pi}\int_\Sigma d^2z\,\mathrm{Tr}(\partial g^{-1}\bar\partial g\Omega) + \frac{\beta}{6\pi}\int_B \mathrm{Tr}(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\Omega)\ , \tag{2.17}$$

where $\Omega_{AB}=\mathrm{Tr}(T_AT_B\Omega)$ and $\Sigma=\partial B$. Using a procedure analogous to the one in [33] or by using the Polyakov-Wiegman identity [43] (the latter is valid in our case because of the properties of the quadratic form Ω_{AB}) and (2.13) we evaluate

$$S(g)=\beta I_0(h_x)+\frac{\alpha}{2\pi}\int_\Sigma d^2z[\partial a\cdot\bar\partial a+(2\partial v_i+m_{i\alpha}\partial a^\alpha)R^i_\mu\bar\partial x^\mu]\ , \tag{2.18}$$

where $I_0(h_x)$ is the WZW action for the group element $h_x\in H$. It is clear that the above action corresponds to string backgrounds, in $\dim(G)+\dim(H)$ space-time dimensions, with $\dim(H)$ null Killing vectors corresponding to the coordinates v^i, thus generalizing the similar statement made previously for the cases of the WZW models for E_2^c [31] and E_d^c [33]. The signature of the spacetime has for $\alpha>0$ ($\alpha<0$) $\dim(G)$ positive (negative) entries and $\dim(H)$ negative (positive) ones as it can be seen from the eigenvalues of the quadratic form (2.6). To obtain models with one timelike coordinate we should take $\alpha>0$ and consider only one-dimensional subgroups H. If among the set of generators $\{R_\alpha\}$ there is an invariant subset with corresponding structure constants $M_{\alpha\beta}{}^i=0$, i.e., if the subgroup H is not the maximum one, then after the contraction (2.3) they correspond to commuting generators or to free decoupled fields in the WZW action (2.18) and therefore we can safely ignore them (for an example see Appendix C). The correct measure in the path integral for the action (2.18) is the Haar measure for the group H times the flat measure for the sets of coordinates $\{v^i\}$ and $\{a^\alpha\}$, as one can see by explicitly computing the square root of the determinant of the metric corresponding to (2.18). Even though the constant β as defined in (2.3) can be any positive real number it should be chosen to be a positive integer in order to have a well-defined path integral for (2.18) [30]. It is also a quite straightforward computation to verify, using the stress tensor (2.5) and the Lie algebra differential operators (2.16), that the same metric (up to a shift $\beta\to\beta+g_G+g_H$), and a constant dilaton arises using the operator algebraic method of [19,44].[3]

Being a WZW action (2.18) has a number of "obvious" global symmetries corresponding to the transformations

$$h_x\to h_x\Lambda\ ,$$
$$\{h_x\to Sh_x, v\to SvS^{-1}, a\to SaS^{-1}\}\ , \tag{2.19}$$
$$v\to v+N\ ,$$

where Λ, S are constant group elements of H and N is a constant matrix in $\mathcal{L}(H)$. The first two transformations in (2.19) represent "left" and "right" global transformations of the group element in (2.8) and the third one is due to the fact that there exist Killing vectors along the directions v^i. One might use these symmetries to generate new solutions via duality transformations.

We could have obtained (2.4)–(2.6) with a different logic (this was effectively used for the cases of E_2^c and E_d^c in

[3]The forementioned shift in the value of the constant β is a direct consequence of the regularization prescription used in (2.5) and obviously does not affect the vanishing of the β functions in conformal perturbation theory (see for instance [45]). Also notice that the fact that the central charge in (2.5) does not depend at all on β means that no matter what regularization scheme we use all loop contributions to it should vanish.

[31,33]). If one starts with the Lie algebra of the group G performs an Inönü-Wigner contraction [46], i.e., $R_\alpha \to (1/\epsilon)R_\alpha$, then one discovers that the WZW action one writes down using the Killing metric cannot be anything else but that of the WZW model for the subgroup H itself (this is because the Killing metric is degenerate due to the non-semi-simplicity of the corresponding contracted algebra). If one insists that all the currents are primary fields of conformal weight one then the minimal resolution to the problem (for a different and more involved one see Appendix B) is to introduce extra generators $\{F_i\}$ with the OPE's given in a unique way by (2.4) which correspond to the nondegenerate quadratic form (2.6), thus making it possible to write the corresponding WZW action. However, this approach is not as nice as the one in [32] since it does not make contact with already existing models. If the condition that all the currents must be primaries is relaxed, then we can still have a consistent current algebra given by (2.4) with the F_i's taken formally to zero. Then one can show that there exist a stress tensor and a corresponding central charge given by

$$\begin{aligned} T &= \frac{:P^2:}{2\alpha} + \frac{:T^2:}{2(\beta+g_H)} , \\ c &= \dim(G) + \frac{g_G - 2g_H}{\beta+g_H}\dim(H) . \end{aligned} \tag{2.20}$$

However, only the T_i's and not the P_α's are primary fields with conformal dimension one with respect to that stress tensor. Because of this there can be no WZW action with the symmetry algebra we just described. A similar conclusion arises from the fact that the quadratic form for this theory [given by the relevant entries in (2.6)] even though it is invertible and symmetric it is not a group invariant and therefore the Wess-Zumino term, necessary for any WZW model, cannot be defined. Nevertheless, because (2.20) is a solution of the Master equation [47,48] (a new one, to the best of our knowledge) it is conceivable that an explicit form for the corresponding action using the results of [49–51], can also be found.

One may wonder whether or not it is possible to obtain the action (2.18) through a limiting procedure taken directly at the level of the WZW action for the direct product group $G\otimes H$. After all this is how one obtains the OPE's in (2.4) in a natural way. Let us start with the action for the $G\otimes H$ (with a general parametrization for the group elements):

$$S = k_G I_0(th_{x'}) + k_H I_0(h_x) , \quad t = e^{ia\cdot R} , \tag{2.21}$$

where t belongs in the left coset, i.e., $t\in(G/H)_L$, and $h_{x'}$ and h_x contain the remaining subgroup variables. Next we expand the subgroup element $h_{x'}$ near the corresponding element h_x as $h_{x'}=(I+2i\epsilon v\cdot u)h_x$, where by the v^i's we collectively call the result of various, generically more complicated, shifts and rescalings of the original parameters x^μ and[4] also we scale the parameters $a^\alpha \to \sqrt{2\epsilon}\, a^\alpha$. Then by writing $th_{x'} \equiv t_\epsilon h_x$ where the matrix t_ϵ and its inverse are

$$\begin{aligned} t_\epsilon &= I + i\sqrt{2\epsilon}\, a\cdot R + \epsilon[2iv\cdot u - (a\cdot R)^2] + O(\epsilon^{3/2}) , \\ t_\epsilon^{-1} &= I - i\sqrt{2\epsilon}\, a\cdot R - \epsilon[2iv\cdot u + (a\cdot R)^2] + O(\epsilon^{3/2}) , \end{aligned} \tag{2.22}$$

one can compute, in powers of ϵ, the corresponding left- and right-invariant Maurer-Cartan forms and the WZW action for the group element t_ϵ:

$$\begin{aligned} t_\epsilon^{-1}dt_\epsilon &= i\sqrt{2\epsilon}\, da\cdot R + i\epsilon[2\,dv\cdot u - da^\alpha m_\alpha{}^i u_i + \alpha^\alpha da^\beta S_{\alpha\beta}{}^\gamma R_\gamma] + O(\epsilon^2) , \\ dt_\epsilon t_\epsilon^{-1} &= i\sqrt{2\epsilon}\, da\cdot R + i\epsilon[2\,dv\cdot u + da^\alpha m_\alpha{}^i u_i - \alpha^\alpha da^\beta S_{\alpha\beta}{}^\gamma R_\gamma] + O(\epsilon^2) , \\ I_0(t_\epsilon) &= \frac{\epsilon}{\pi}\int_\Sigma d^2z\, \partial a\cdot\bar\partial a + O(\epsilon^{3/2}) . \end{aligned} \tag{2.23}$$

Also the formulas

$$\begin{aligned} &k_G\,\mathrm{Tr}[u_i(th_{x'})^{-1}\partial(th_{x'})] + k_H\,\mathrm{Tr}(v_i h_x^{-1}\partial h_x) = i\beta L^i_\mu \partial x^\mu + i\alpha(\partial v_j + \tfrac{1}{2}m_{ja}\partial a^\alpha)C^{ji} , \\ &k_G\,\mathrm{Tr}[u_i\bar\partial(th_{x'})(th_{x'})^{-1}] + k_H\,\mathrm{Tr}(v_i\bar\partial h_x h_x^{-1}) = i[\beta I + \alpha(n + \tfrac{1}{2}mm^t)]_{ij} R^j_\mu \bar\partial x^\mu + i\alpha(\bar\partial v_i - \tfrac{1}{2}m_{ia}\bar\partial a^\alpha) , \end{aligned} \tag{2.24}$$

valid in the limit $\epsilon\to 0$, will prove useful in subsequent sections. Using the above expansion formulas, the Polyakov-Wiegman identity, as well as the redefinitions of k_G and k_H in (2.3) one can see that in the limit $\epsilon\to 0$ the action (2.21) reduces to that in (2.18).

III. COSET MODELS G^c_h/H

In this section we use the results of the previous one to construct gauged WZW models based on non-semi-simple groups. In particular we gauge the subgroup generated by the subset of generators $\{T_i\}$. We explicitly show how all the relevant formulas at the algebraic as well as at the action level can be obtained as limiting cases of the corresponding ones for the $(G\otimes H)/H$ gauged WZW models.

Let us consider the gauged WZW model one obtains by gauging the diagonal subgroup, generated by $\{u_i+v_i\}$, of

[4]If we write $h_{x'}=e^{ix'\cdot u}$ (and similarly for h_x) and let $x_i'=x_i+2i\epsilon\phi_i$ we can easily show using (2.10) and (2.9) that, in the limit $\epsilon\to 0$,

$$v^i = \phi^j \int_0^1 ds\, C^i{}_j(h_{sx}) , \quad h_{sx} = e^{isx\cdot u} ,$$

where C_{ij} is the matrix defined in (2.9). Obviously only for Abelian subgroups $v^i=\phi^i$.

the direct product group $G\otimes H$ (the axial gauging case is considered in Appendix A). For the corresponding CFT coset model $(G\otimes H)/H$ the energy momentum tensor and central charge of the Virasoro algebra are given by

$$T=\frac{:u^2+R^2:}{2(k_G+g_G)}+\frac{:v^2:}{2(k_H+g_H)}-\frac{:(u+v)^2:}{2(k_G+k_H+g_H)}\ ,$$
$$c=\frac{k_G\dim(G)}{k_G+g_G}+\frac{k_H\dim(H)}{k_H+g_H}-\frac{(k_G+k_H)\dim(H)}{k_G+k_H+g_H}\ . \qquad (3.1)$$

Then we use (2.3) and take the limit $\epsilon\to 0$. The energy momentum tensor for the resulting coset theory, denoted by G_h^c/H, reads

$$T=\frac{:P^2+2FT:}{2\alpha}-\frac{\beta+g_G+g_H}{2\alpha^2}:F^2:-\frac{:T^2:}{2(\beta+g_H)}\ ,$$
$$c=\dim(G)+\frac{g_H\dim(H)}{\beta+g_H}\ . \qquad (3.2)$$

Of course one may obtain the same result by directly considering the gauging of the subgroup generated by $\{T_i\}$, of the non-semi-simple group G_h^c. We also see that the central charge for the gauged WZW G_h^c/H model is no longer an integer as it was in the case of the WZW model for G_h^c, unless the subgroup H is an Abelian one.

To obtain an explicit form for the gauged WZW action we start with [29]

$$S=\beta[I_0(h^{-1}g\bar h)-I_0(h^{-1}\bar h)]\ , \qquad (3.3)$$

where $h,\bar h$ are group elements in the subgroup H generated by $\{T_i\}$ and the group element $g\in G_h^c$ is parametrized as in (2.8). Writing explicitly the above action with the aid of (2.18) we obtain

$$S=\beta I(h_x,A)+\frac{\alpha}{2\pi}\int_\Sigma d^2z[Da\cdot\bar Da+i(Da)^\alpha m_\alpha{}^i(\bar D h_x h_x^{-1})_i-2i\,\mathrm{Tr}(Dv\bar D h_x h_x^{-1}-vF_{z\bar z})]\ , \qquad (3.4)$$

with $I(h_x,A)$ being the usual gauged WZW action for H/H,

$$I(h_x,A)=I_0(h_x)+\frac{1}{\pi}\int_\Sigma d^2z\,\mathrm{Tr}(A\bar\partial h_x h_x^{-1}-\bar A h_x^{-1}\partial h_x+h_x^{-1}Ah_x\bar A-A\bar A)\ , \qquad (3.5)$$

and where the gauge fields $A,\bar A$ take values in $\mathcal{L}(H)$, and the corresponding field strength $F_{z\bar z}$ and the covariant derivatives are defined as

$$A=\partial h h^{-1}\ ,\quad \bar A=\bar\partial\bar h\bar h^{-1}\ ,\quad F_{z\bar z}=\partial\bar A-\bar\partial A-[A,\bar A]\ ,\quad D=\partial-[A,\cdot]\ ,\quad \bar D=\bar\partial-[\bar A,\cdot]\ . \qquad (3.6)$$

Naively one might have expected that (3.3) would have been given just by the sum of the gauged WZW for the coset H/H, i.e., $I_0(h_x,A)$, and the covariantized terms one obtains by simply replacing ordinary derivatives by covariant ones in the terms involving a^α and v^i in (2.18). In fact, such an action is gauge invariant by itself. However, only by the inclusion of the term involving the field strength $F_{z\bar z}$ one is able to rewrite it as a sum of two *independent* WZW actions, as in (3.3), that guarantees conformal invariance. The action (3.4) is invariant under the infinitesimal gauge transformations

$$\delta h_x=[h_x,i\epsilon]\ ,\quad \delta a^\alpha=-m^\alpha{}_j\epsilon^j\ ,\quad \delta v^i=n^i{}_j\epsilon^j\ ,\quad \delta A=-iD\epsilon\ ,\quad \delta\bar A=-i\bar D\epsilon\ , \qquad (3.7)$$

where $\epsilon=\epsilon^i T_i$. To obtain the σ model one integrates over the gauge fields. A straightforward computation gives[5]

$$\tilde S_\sigma=\beta\Bigg\{I_0(h_x)+\frac{1}{2\pi}\int_\Sigma d^2z\{\partial a\cdot\bar\partial a+(2\partial v_i+m_{i\alpha}\partial a^\alpha)R^i_\mu\bar\partial x^\mu-2[L^i_\mu\partial x^\mu+(\partial v_k+\tfrac12 m_{k\alpha}\partial a^\alpha)C^{ki}]$$
$$\times[M+(n+\tfrac12 mm^t)C]^{-1}_{ij}[(I+n+\tfrac12 mm^t)^j{}_l R^l_\nu\bar\partial x^\nu+\bar\partial v^j-\tfrac12 m^j{}_\beta\bar\partial a^\beta]\}\Bigg\}\ . \qquad (3.8)$$

There is also a dilaton field induced from the finite part of the determinant one obtains by integrating out the gauge fields in (3.4):

$$\tilde\Phi=\ln\det[M+(n+\tfrac12 mm^t)C]+\Phi_0\ . \qquad (3.9)$$

The above forms for the action and the dilaton can be simplified considerably by using the properties of the matrix C_{ij} in (2.9). The final expressions are

$$\tilde S_\sigma=\beta\left[-I_0(h_x)+\frac{1}{2\pi}\int_\Sigma d^2z[\partial\alpha\cdot\bar\partial a+2(R^i_\mu\partial x^\mu+\partial v^i+\tfrac12 m^i{}_\alpha\partial a^\alpha)(M^t-n-\tfrac12 mm^t)^{-1}_{ij}(L^j_\nu\bar\partial x^\nu+\bar\partial v^j-\tfrac12 m^j{}_\beta\bar\partial a^\beta)]\right] \qquad (3.10)$$

[5]Throughout this paper whenever an action or a dilaton is denoted with a tilded symbol will contain the rescaled $v^i\to\beta/\alpha v^i$ and $\alpha^\alpha\to\sqrt{\beta/\alpha}\,a^\alpha$.

and

$$\tilde{\Phi}=\ln\det(M^t-n-\tfrac{1}{2}mm^t)+\Phi_0\ . \qquad (3.11)$$

The signature of the spacetime described by (3.10) will have (if G and H are compact groups) for $\alpha>0$ $\dim(G/H)$ positive and $\dim(H)$ negative entries and for $\alpha<0$ $\dim(G)$ negative and no positive ones (analytically continuing β to negative values renders a positive signature spacetime). Obviously for models with one timelike coordinate the subgroup H has to be one dimensional and $\alpha,\beta>0$. If we allow the use of noncompact groups more possibilities exist. We find that one timelike coordinate is also possible if (1) $\beta>0$, $H=G$ and G is a noncompact group with one timelike generator, i.e., $G=\mathrm{SO}(1,d)_{-\beta}$, (2) $\alpha>0$, $\beta<0$, H a compact group, and G a noncompact one such that the coset G/H gives rise to a spacetime with one timelike coordinate. All such models have been classified in [16]. The complete list is $(\mathrm{SU}(p,q),\mathrm{SU}(p)\otimes\mathrm{SU}(q))$, $(\mathrm{SO}(p,2)\mathrm{SO}(p))$, $(\mathrm{Sp}(2p,\mathbb{R}),\mathrm{SU}(p))$, $(\mathrm{SO}^*(2p),\mathrm{SU}(p))$, $(\mathrm{E}_6,\mathrm{SO}(10))$, and $(\mathrm{E}_7,\mathrm{E}_6)$, where the first entry in the parentheses refers to the group G and the second one to the subgroup H. The lowest dimensional examples have $D=6$, i.e., $(G,H)=(\mathrm{SO}(2,2),\mathrm{SO}(2))$ or $(\mathrm{SO}^*(4),\mathrm{SU}(2))$.

The actions (3.8) and (3.10) as well as the corresponding dilaton fields are still invariant under the transformations (3.7). A convenient gauge choice would be to set to zero all the parameters in h_x except those corresponding to the Cartan subalgebra of H which cannot be gauged away due to the existence of a nontrivial isotropy (for discussions relevant to this point see also [24,26]). That still leaves a number of parameters equal to the rank of H to be fixed among the remaining ones $\{a^\alpha\}$ and $\{v^i\}$ [in the second set those with an H-Cartan subalgebra index are inert under the remaining gauge transformations in (3.7)]. As in the case of the WZW action (2.18) that follows directly from (2.21) through the limiting procedure described in the previous section, one can also show that (3.4) can be obtained from the usual gauged WZW action for the coset $(G\otimes H)/H$ (the gauge group is the total subgroup generated by u^i+v^i),

$$S=k_G[I_0(h^{-1}th_{x'}\bar{h})-I_0(h^{-1}\bar{h})]$$
$$+k_H[I_0(h^{-1}h_x\bar{h})-I_0(h^{-1}\bar{h})]\ , \qquad (3.12)$$

via a similar procedure. In addition to the expansion formulas (2.22), (2.23), and (2.24) the following one

$$\mathrm{Tr}(u_it_\epsilon h_xu_jh_x^{-1}t_\epsilon^{-1})=C_{ij}+\epsilon(2n+mm^t)_i{}^k(C_{kj}+O(\epsilon^2)\ , \qquad (3.13)$$

should also be used.

Let us also briefly discuss how possible $\alpha'\sim 1/\beta$ corrections to our σ model (3.10), (3.11) can arise, by considering the effective action for it. Again the connection with the original $(G\otimes H)/H$ proves useful. The effective action for the latter models is [52,53] (we ignore possible field renormalizations since they give rise to nonlocal terms in the final σ model [54])

$$S=(k_G+g_G)I_0(h^{-1}th_{x'}\bar{h})+(k_H+g_H)I_0(h^{-1}h_x\bar{h})$$
$$-(k_G+k_H+g_H)I_0(h^{-1}\bar{h})\ , \qquad (3.14)$$

where all the definitions were given in[6] (3.6). Then the limit $\epsilon\rightarrow 0$ this reduces to the effective action for our G_h^c/H coset models:

$$\Gamma=(\beta+g_G+g_H)I(h_x,A)+\frac{\alpha}{2\pi}\int_\Sigma d^2z[Da\cdot\bar{D}a+i(Da)^\alpha m_\alpha{}^i(\bar{D}h_xh_x^{-1})_i-2i\,\mathrm{Tr}(Dv\bar{D}h_xh_x^{-1}-vF_{z\bar{z}})]+g_GI_0(h^{-1}\bar{h})\ . \qquad (3.15)$$

The above action although local in the h fields is not local in the gauge fields A and $\bar{A}$, due to the presence of the last term. Of course in the limit of large β it reduces to the action given in (3.4). Next, one is to solve for the gauge fields via their equation of motion and identify the local part containing the second derivative terms as the final σ model. We will not repeat this procedure here as it has been discussed extensively in [53,54] and especially in [57] and because the final expressions for the σ model are quite complicated. We will only mention that for the metric and the dilaton there are $1/\beta$ corrections (in the standard "conformal" scheme [57]) in agreement with what is expected from the different shiftings of β in the F^2 and the T^2 terms in the expression of the stress tensor (3.2). For the antisymmetric tensor it was shown in [57] that there are two natural prescriptions, consistent with gauge invariance, for extracting it. The one which was called "corrected" gives $1/\beta$ corrections to the semiclassical result for the antisymmetric tensor. However, the second prescription gives just the semiclassical result. At this point we should mention that for the case of the coset $E_2^c/\mathrm{U}(1)$ the explicit expressions containing all quantum $1/\beta$ corrections were derived in [35], both in the axial and the vector gauging cases. In the case of the axial gauging both of the prescriptions mentioned give the semiclassical result for the antisymmetric tensor. Being derived via correlated limit from the 3D charged black string [36,37] its conformal invariance has already been checked up to two loops in conformal perturbation theory in [57].

Notice also that in (3.15) the coefficient of the last term which is responsible for all $1/\beta$ corrections cannot vanish even if $H=G$ (nevertheless in such cases the σ-model expressions will further simplify since $m_{i\alpha}=0$). In contrast with the case of a G/G coset model where we can gauge out all degrees of freedom, except those corresponding to

[6]We should emphasize that (3.14) does not belong to the "induced-type" actions for gravity (see for instance [55]), in which the level shifts are different. For further details on this distinction between "effective" actions see [54,56].

the Cartan torus that constitute a free field theory, in the case of a G_h^c/G coset model the resulting σ model is a nontrivial one. This fact receives more significance in view of the results that will follow and specifically to the relation to non-Abelian duality transformations considered in the next section.

IV. ON NON-ABELIAN DUALITY

In this section we formulate the non-Abelian duality transformations [24] as a particular limiting case of a larger class of models where an anomalous free symmetry of the target spacetime geometry is being gauged. Moreover we show how the gauged WZW models corresponding to the cosets G_h^c/H, considered in the previous section, can be thought of as non-Abelian duality transformations on the $H\otimes \mathrm{U}(1)^{\dim(G/H)}$ WZW model with respect to the vectorial action of H.

A. Non-Abelian duality transformations and gauged WZW models

In the usual Lagrangian formulation (for a Hamiltonian one see [28]) of non-Abelian duality [24,26–28] one starts with an action $S(X)$ corresponding to some CFT, denoted by $\mathcal{A}$, and then gauges a nonanomalous symmetry corresponding to a group H and adds a Lagrange multiplier term (of course the same procedure is applicable to any nonlinear, not necessarily conformal, σ model with a global symmetry). The total action reads

$$S(X,A,\bar{A},v)=S(X,A,\bar{A})+\frac{\alpha}{\pi}\int_\Sigma d^2z\, i\,\mathrm{Tr}(vF_{z\bar{z}}) \ . \quad (4.1)$$

Each term in the above action is invariant under gauge transformations which for A, $\bar{A}$ and the v^i's were given in (3.7). The "matter" fields X's transform in some representation of the gauge group H in such a way that the action $S(X,A,\bar{A})$ is gauge invariant by itself. In the path integral for $S(X,A,\bar{A},v)$ integration over the Lagrange multipliers v^i's forces the gauge fields to be pure gauges, i.e., $A=\partial hh^{-1}$, $\bar{A}=\bar{\partial}hh^{-1}$. Choosing the gauge condition $h=I$ one recovers the original model with the action $S(X)$. If instead one integrates out the gauge fields one obtains the dual model. In the case of non-Abelian duality the original and the dual model do not necessarily correspond to the same CFT [24,26,27] as in the case of the Abelian duality for compact groups [8]. Instead it has been conjectured that non-Abelian duality transformations interpolate between solutions of different CFT's possibly related by an orbifold construction [24,26]. Next we will show how (4.1) can be thought of as a limiting case of a more general gauged theory. Consider the direct product of the CFT's $\mathcal{A}$ and the current algebra for H at level k_H. In order to gauge the H symmetry the appropriate action is

$$\mathcal{S}(X,A,\bar{A},h_x)=S(X,A,\bar{A})+k_H I(h_x,A) \ , \quad (4.2)$$

where $I(h_x,A)$ is the gauged WZW action for H/H as in (3.5). Let us consider the above action in the limit of very large k_H, or equivalently the limit $\epsilon\to 0$ with $k_H=\alpha/\epsilon$ and α a finite constant. In order to obtain a finite contribution in that limit we consider elements $h_x\in H$ close to the identity element.[7] If we parametrize $h_x=I+i\epsilon v\cdot u$ then we can easily show that in the limit $\epsilon\to 0$ the action (4.2) becomes identical to the action (4.1). Namely, we have proved

$$\begin{aligned} &\mathcal{S}(X,A,\bar{A},h_x)|_{\epsilon\to 0}=S(X,A,\bar{A},v) \ , \\ &k_H=\frac{\alpha}{\epsilon} \ , \quad h_x=I+i\epsilon v\cdot u \ . \end{aligned} \quad (4.3)$$

It is worth noticing that the Lagrange multiplier term in (4.1) is nothing but the dimensionally reduced non-Abelian Chern-Simons action (v^iu_i is the third component of the A field).[8] It should be emphasized that the relation (4.3) holds up to a total derivative which, however, is important when one discusses global issues [8,9,26,27], which for non-Abelian duality still remain to be resolved. In this paper we are only concerned with the local (short-distance) conformal properties of the models for which the relation (4.3) [and also (4.4) below] is unambiguous. Let us be more specific and consider for the CFT $\mathcal{A}$ the one corresponding to the WZW for a group G at level k. What we have proved is that

$$\left.\frac{G_k\otimes H_{k_H}}{H_{k+k_H}}\right|_{k_H\to\infty}$$

$$=\text{dual of } G \text{ with respect to } H \text{ (vector)} \ , \quad (4.4)$$

where the limit is taken in a correlated way, as in (4.3). Our formulation of non-Abelian duality as a limiting case of a larger class of gauged models makes it possible to work out explicitly α' corrections to the semiclassical expressions. We can illustrate this briefly by considering the case of duality transformations on a WZW model for a simple group G with respect to its subgroup H. In that case the effective action replacing (4.1) is [cf. (3.15)] (again we ignore possible field redefinitions)

$$\begin{aligned} \Gamma=&(k+g_G)I(g,A)+g_G I_0(h^{-1}\bar{h}) \\ &+\frac{\alpha}{\pi}\int_\Sigma d^2z\, i\,\mathrm{Tr}(vF_{z\bar{z}}) \ , \end{aligned} \quad (4.5)$$

where $g\in G$ is parametrized in terms of x^μ, $\mu=1,2,\ldots,\dim(G)$. We will not present the derivation of the exact σ-model background fields here. The method is identical to the one followed in [53,54,57] for the case of G/H coset models with only some

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[7] We assumed that $h_x\in H$ belongs to an irreducible representation. In that case all fixed points of the gauge transformation (points $h_0\in H$ such that $[h_0,u_i]=0, \forall u_i\in\mathcal{L}(H)$) are proportional to the identity element according to Schur's first Lemma.

[8] Abelian versions of that term were also derived by [38] in showing that the gauged WZW action for the coset model $\mathrm{SL}(2,\mathbb{R})/\mathbb{R}$ close to the $uv=1$ "singularity" looks like a topological field theory and by [58] in the derivation, from a Chern-Simons theory, of the Verlinde formula that counts the number of conformal blocks of a rational CFT.

modifications. The result is

$$S(x)=\frac{1}{\pi\alpha'}\int_{\Sigma}d^2z(G_{MN}+B_{NN})\partial x^M\bar{\partial}x^N\ ,$$
$$x^M=\{x^\mu,v^i\}\ ,\quad \alpha'=\frac{2}{k+g_G}\ ,\qquad (4.6)$$

where the metric is

$$G_{MN}=G^{(s)}_{MN}-2b(\tilde{\mathcal{V}}^{-1}\mathcal{M}^t\mathcal{M}^{-1})_{ij}\mathcal{L}^i_{(m}\mathcal{R}^j_{N)}-b(\tilde{\mathcal{V}}^{-1})_{ij}$$
$$\times\mathcal{L}^i_M\mathcal{L}^j_N-b(\mathcal{V}^{-1})_{ij}\mathcal{R}^i_M\mathcal{R}^j_N\ ,\qquad (4.7)$$

with $b\equiv -g_G/(k+g_G)$, the antisymmetric tensor is

$$B_{MN}=B^{(s)}_{MN}-2b^2[\tilde{\mathcal{V}}^{-1}\mathcal{M}^t\mathcal{M}^{-1}(\mathcal{M}-\mathcal{M}^t)\mathcal{V}^{-1}]_{ij}\mathcal{L}^i_{[M}\mathcal{R}^j_{N]}$$
$$-b^2[\tilde{\mathcal{V}}^{-1}(\mathcal{M}-\mathcal{M}^t)\tilde{\mathcal{V}}^{-1}]_{ij}\mathcal{L}^i_{[M}\mathcal{L}^j_{N]}$$
$$-b^2[\mathcal{V}^{-1}(\mathcal{M}-\mathcal{M}^t)\mathcal{V}^{-1}]_{ij}\mathcal{R}^i_{[M}\mathcal{R}^j_{N]}\qquad (4.8)$$

and the dilaton is

$$\Phi=\tfrac{1}{2}\ln\det\mathcal{V}+\Phi_0\ .\qquad (4.9)$$

The semiclassical expressions, in the limit $b\rightarrow 0$, are

$$G^{(s)}_{MN}=G_{0MN}-2\mathcal{M}^{-1}_{ij}\mathcal{L}^i_{(M}\mathcal{R}^j_{N)}\ ,$$
$$B^{(s)}_{MN}=B_{0MN}-2\mathcal{M}^{-1}_{ij}\mathcal{L}^i_{[M}\mathcal{R}^j_{N]}\ ,\qquad (4.10)$$
$$\Phi^{(s)}=\ln\det\mathcal{M}\ .$$

G_{0MN} and B_{0MN} are the original WZW (group space) couplings,

$$G_{0MN}=\eta_{AB}L^A_M L^B_N\ ,$$
$$3\partial_{[K}B_{0MN]}=iL^A_K L^B_M L^C_N f_{ABC}\ ,\qquad (4.11)$$

and the various matrices are defined as

$$\mathcal{M}_{ij}=M_{ij}+n_{ij}\ ,\quad \mathcal{V}=\mathcal{M}\mathcal{M}^t-b(\mathcal{M}+\mathcal{M}^t)\ ,$$
$$\tilde{\mathcal{V}}=\mathcal{M}^t\mathcal{M}-b(\mathcal{M}+\mathcal{M}^t)\ ,\qquad (4.12)$$

where we have rescaled $v^i\rightarrow(k+g_G)/\alpha v^i$ and

$$\mathcal{L}^i_M\partial x^M=L^i_\mu\partial x^\mu+\partial v^i\ ,$$
$$\mathcal{R}^i_M\bar{\partial}x^M=R^i_\mu\bar{\partial}x^\mu+\bar{\partial}v^i\ .\qquad (4.13)$$

Because of the gauge invariance we should fix dim(H) of the parameters among the x^M's. The remaining dim(G) variables will be the string coordinates of the dual space of G under non-Abelian duality with respect to a subgroup H. Thinking of them as H-subgroup invariants one can determine their range of values using group theoretic methods [59]. If $H\neq G$ then there is generically no isometry and all of the parameters to be gauge fixed can be chosen entirely within the group element $g\in G$. Notice that similarly to the gauged models of the previous section there are quantum corrections in (4.6) even if $H=G$. For such cases there is nonvanishing isometry and the remarks of the paragraph just before (3.12), about a possibly convenient gauge choice are equally applicable. Let us note that the exact expression for the antisymmetric tensor (4.8) was obtained with the "corrected" prescription of [57]. As it was explained also in Sec. III there is a second prescription that leads to the semiclassical result in (4.10) for the exact antisymmetric tensor. Both prescriptions give the same metric and dilaton, are consistent with the gauge invariance of (4.6) (before gauge fixing) and with results available from conformal perturbation theory [57]. The σ model corresponding to the case of $G=H=\mathrm{SU}(2)$ was previously considered, in the semiclassical limit $k\rightarrow\infty$, in [26]. What we have proved is that it is a limit of the semiclassical σ model for the coset $\mathrm{SU}(2)_{k_1}\otimes\mathrm{SU}(2)_{k_2}/\mathrm{SU}(2)_{k_1+k_2}$ considered[9] in [60].

Since the relation (4.3) involves a *singular* limit the final dual theory is not necessarily equivalent to the original corresponding to the action $S(X)$ even though some quantities like the central charge in the case of (4.4) are the same (equal to the central charge of G_k). In the case of duality transformations with respect to an Abelian subgroup we can avoid taking any singular limit. This is the case because under the vector gauge transformations (3.7) any point in the group element $h_x=e^{ix\cdot u}$ is a fixed point, i.e., $\delta x^i=0,\forall i$. Then if in

$$I(h_x,A)=\int d^2z(\tfrac{1}{2}\partial x_i\bar{\partial}x^i+iA_i\bar{\partial}x^i-i\bar{A}_i\partial x^i)$$

we shift $A_i\rightarrow A_i+i/4\partial x_i$ and $\bar{A}_i\rightarrow\bar{A}_i-i/4\bar{\partial}x_i$ (this is not a gauge transformation) we can absorb the bilinear in the x's term and then (4.2) takes the form of (4.1) with $\alpha=k_H$ and $v^i=x^i$. The redefined $A,\bar{A}$ have the same transformation properties as the old ones and they are the ones to be used in $S(X,A,\bar{A})$.

B. Relation of G^c_h/H to non-Abelian duality

Let us uncover the relation of the gauged WZW model for the coset G^c_h/H that we considered in Sec. III to the class of models one obtains via non-Abelian duality transformations performed on the background corresponding to the WZW model for the direct product group $H\otimes\mathrm{U}(1)^l$. In particular we will consider non-Abelian duality transformations with respect to the group H itself. Since we do not want to leave the coordinates corresponding to the factor $\mathrm{U}(1)^l$ inert under the duality transformation we embed the model into a larger one for the group G that contains H as a subgroup. In this way the gauge transformations are as in (3.7), with the a^α's parametrizing the original U(1) factors whose number is obviously restricted to be $l=\dim(G/H)$. The gauged invariant action we start with is [cf. (4.1)]

$$S=\beta I_0(h_x,A)+\frac{\alpha}{2\pi}\int_{\Sigma}d^2z[Da\cdot\bar{D}a+2i\,\mathrm{Tr}(vF_{z\bar{z}})]\ ,\qquad (4.14)$$

[9]This coset was also considered in [61,62] for the case $k_1=k_2$ where the geometry of the resulting σ model is easier to interpret. Obviously, in this case correspondence with non-Abelian duality cannot be made.

where all the necessary definitions are given by (3.6). Integrating over the gauge fields one obtains the dual action

$$\tilde{S}_{\rm dual}=\beta\left[I_0(h_x)+\frac{1}{2\pi}\int_\Sigma d^2z[\partial\alpha\cdot\bar\partial a-2(L^i_\mu\partial x^\mu+\partial v^i+\tfrac{1}{2}m^i{}_\alpha\partial a^\alpha)(M+n+\tfrac{1}{2}mm^t)^{-1}_{ij}(R^j_\nu\bar\partial x^\nu+\bar\partial v^j-\tfrac{1}{2}m^j{}_\beta\bar\partial a^\beta)]\right] . \tag{4.15}$$

The induced dilaton is

$$\tilde\Phi=\ln\det(M+n+\tfrac{1}{2}mm^t)+\Phi_0 . \tag{4.16}$$

A quick inspection (and taking into account the rescaling as described in footnote 5) shows that if in (4.15) and (4.16) we send $\beta\to-\beta$ and $h_x\to h_x^{-1}$ we obtain precisely (3.10) and (3.11). In fact correspondence can also be made for the "original" models one obtains by integrating over the Lagrange multipliers v^i's in (3.4) and (4.14). For the latter case the result is, as we have already mentioned, the WZW model for $H\otimes U(1)^{\dim(G/H)}$:

$$S=\beta I_0(h_x)+\frac{\alpha}{2\pi}\int_\Sigma d^2z\,\partial a\cdot\bar\partial a . \tag{4.17}$$

Varying (3.4) with respect to the v^i's and using the fact that

$$F_{z\bar z}=D(\bar A-\tilde A) , \quad A=\partial hh^{-1} , \quad \tilde A=\bar\partial hh^{-1} , \tag{4.18}$$

we obtain the equation

$$\bar D h_x h_x^{-1}+\bar A-\tilde A=0 . \tag{4.19}$$

Choosing the gauge fixing condition $h=I$ one obtains $A=\tilde A=0$ and $\bar A=-h_x^{-1}\bar\partial h_x$. After a little algebra (3.4) becomes

$$S=-\beta I_0(h_x^{-1})+\frac{\alpha}{2\pi}\int_\Sigma d^2z\,\partial a\cdot\bar\partial a , \tag{4.20}$$

thus revealing the same relationship between the two models we have already uncovered by comparing (3.10) and (3.11) top (4.15) and (4.16). It appears as if the central charge corresponding to the model (4.20) is not the same as in (3.2); i.e., "naively" it is $c=\dim(G)+g_H\dim(H)/(\beta-g_H)$. However, in order to correctly compute it, one has to take into account the nontrivial Jacobian arising from changing variables from $(A,\bar A)\to(h,\bar h)$ in the path integral functional before the gauge-fixing condition, $h=I$, is imposed. The Jacobian regularized in a gauged invariant way [63] gives a factor

$$\mathcal{J}=e^{2g_H I_0(h_x^{-1})}(\det\partial\bar\partial)^{\dim(H)} \tag{4.21}$$

which shifts the value of $\beta\to\beta+2g_H$ in (4.20). Thus the central charge is given by

$$c=\frac{-(\beta+2g_H)\dim(H)}{-(\beta+2g_H)+g_H}+\dim(G/H)$$
$$+2\dim(H)-2\dim(H) ,$$

which produces the correct central charge as given by (3.2). In the previous expression the second term is due to the a^α's and the last two terms due to the contributions of the v^i's and the factor $\det(\partial\bar\partial)^{\dim(H)}$ in (4.21) [they cancel because each factor corresponds to $\dim(H)$ independent (b,c) systems of conformal weight (0,1) but of opposite statistics]. Therefore we have proved the relation

$$G^c_h/H\leftrightarrow\text{dual of } H\otimes U(1)^{\dim(G/H)}$$
$$\text{with respect to } H \text{ (vector)} , \tag{4.22}$$

with the specific relation between the central extension parameters that have already mentioned. The equivalence relation (4.22) is very similar to (4.4). In fact it seems that it can be directly deduced from it, without having to go through the explicit computations of this subsection. However, this is not true because there is a double correlated limit to be taken on the left-hand side of (4.4) instead of the simple one $k_H\to\infty$. Also (4.22) holds as a true equivalence relation (at least locally) between the two theories on the left- and the right-hand sides, whereas (4.4) [and (4.3)] should be thought of only as a way to reproducing the starting action for non-Abelian duality transformations (4.1) from actions corresponding to better understood theories. Let us also stress that (4.22) is nontrivial in the sense that the actions describing the two sides of it, namely, (3.4) and (4.14) appear quite different from one another.

V. CONCLUDING REMARKS AND DISCUSSION

In this paper we have been studying limiting cases of WZW and gauged WZW models based on simple groups both at the algebraic and the σ-model action level. The resulting models after the limiting procedure is performed are WZW and gauged WZW models based on a certain class of non-semi-simple groups. We have seen that there is an intimate connection between these models as well as non-Abelian quotient models, and the models one obtains via non-Abelian duality transformations. This correspondence facilitates a lot the systematic computation of the α' corrections to the semiclassical results of the non-Abelian duality transformations, as we have already seen. A more difficult question to be answered is how the spectrum of the dual theory is related to that of the original one. In that case relations such as (4.3), (4.4), and (4.22) are potentially useful.

A motivation for studying gauged WZW models based on non-semi-simple groups was that according to the results of [35] for the coset $E^c_2/U(1)$, the σ-model background fields corresponding to the G^c_h/H cosets might be possible to be mapped to nonsingular ones, even though themselves may have curvature singularities. In other words, even though the original background for the coset $(G\otimes H)/H$ cannot be dual to a nonsingular one (for instance, the charged 3D black string is dual to the neutral one and vice versa [64], but both backgrounds have curvature singularities), the one corresponding to the "contracted" coset G_h^c/H can. This would have been a possible and quite general way string theory deals with gravitational singularities (although the latter are not peculiar from a CFT point of view [38,10]). We partially succeeded toward this goal. In the case of an Abelian subgroup H of a general group G it is shown in Appendix A that it is possible to map the curved singular backgrounds to flat spacetimes with constant antisymmetric tensor and dilaton fields under an Abelian duality transformation. The reason that for the non-Abelian case we were not able to make a similar statement is that, in view of the connection to non-Abelian duality transformations, such an "inverse" transformation is not known how to be performed and as we have already mentioned the symmetry of the dual background is much less than that of the original one. Thus finding a way to define the "inverse" non-Abelian duality transformation will give a definite answer to whether or not such a desired mapping is possible.

A natural question is whether or not the construction of [32] and the similar one in Appendix B exhaust all possible non-semi-simple algebras with a sensible action description. It will be desirable, for instance, to find the action corresponding to the CFT whose Virasoro construction is given by (2.20), using possibly the actions of [49–51]. We should also mention that in [65] a formalism for constructing WZW models based on non-semi-simple groups that generically give rise to noninteger values for the central charge was proposed. For the models of [32] this formalism corresponds to a shift of the constant β defined in (2.3) and it gives no new results (the essential reason for that is the fact that $\Omega^{T_iT_j}=0$). It will be of interest to search for models where the results of [65] are applicable in a nontrivial way.

Note added

After the completion of this work we received Ref. [66] which proves that there are no Sugawara-like constructions (with respect to which all currents are primary fields with conformal weight one) based on non-semi-simple algebras that give rise to noninteger values for the central charge along the lines of [65].

We would like also to suggest a possible explanation for the origin of a problem with non-Abelian duality that was noted in [28]. In that paper the three-dimensional symmetry algebra of the spatial part of the metric in which the non-Abelian duality transformation is performed admits no Sugawara-like construction. One can easily prove that by a direct computation or by using the result of [66] according to which the only such three-dimensional algebras are su(2), sl(2), and $u(1)^3$. That will give rise to a conformal anomaly [28] when we perform the non-Abelian duality transformation since the assumption that the currents coupled to the gauged fields in the action scale with dimension one is not satisfied at the quantum level. A better and detailed understanding of it is important but beyond the scope of this note. I would like to thank R. Ricci for motivating me to think about this problem.

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APPENDIX A: AXIAL GAUGING

In this appendix we consider the case of the axial gauging. This is anomaly-free when the gauge group is Abelian, i.e., isomorphic to $U(1)^d$. We will show how the action for the gauged WZW models G_h^c/H in the axial gauging can be obtained directly from the action of the gauged WZW models (in the axial gauging as well) for the $G\otimes U(1)^d/U(1)^d$ through a limiting procedure and also how duality transformations can be used to map the final σ models (with curvature singularities in general) to *flat spacetimes* with constant antisymmetric tensor and dilaton fields. For the former models the group element is parametrized as in (2.8) with $h_x=e^{iT\cdot x}$. In the axial gauging the corresponding action is given by (see for instance [38,61,10])

$$S=I_0(h^{-1}g\bar{h})-I_0(h\bar{h}) , \tag{A1}$$

where $h,\bar{h}$ are groups elements in $H\simeq U(1)^d$. Defining $A=\partial hh^{-1}$ and $\bar{A}=\bar{\partial}\bar{h}\bar{h}^{-1}$ and shifting $\bar{A}_i\to\bar{A}_i-i\bar{\partial}x_i$ (this will, among other things, effectively change the coefficients $\beta_i\to-\beta_i$ below) we obtain with the help of (2.11) and (2.13) the action

$$S_{\rm axial}=\frac{1}{2\pi}\int_\Sigma d^2z\{\alpha\partial a\cdot\bar{\partial}a-\beta_i\partial x_i\bar{\partial}x_i+iA_i[\alpha(2\bar{\partial}v_i-m_{i\alpha}\bar{\partial}a^\alpha)-2\beta_i\bar{\partial}x_i]$$
$$-i\bar{A}_i[\alpha(2\partial v_i+m_{i\alpha}\partial a^\alpha)+2\beta_i\partial x_i]+A_i[\alpha(mm^t)_{ij}+4\beta_i\delta_{ij}]\bar{A}_j\} , \tag{A2}$$

where the summation over repeated indices is implied. This action is invariant under the infinitesimal gauge transformations [in fact (A2) can be cast in a similar to (3.4) form, i.e., as the first two terms with covariant derivatives replacing the ordinary ones plus the Lagrange multiplier term]

$$\delta x_i=2\epsilon_i , \quad \delta a^\alpha=m^\alpha{}_j\epsilon^j , \quad \delta v_i=0 , \quad \delta A_i=i\partial\epsilon_i , \quad \delta\bar{A}_i=i\bar{\partial}\epsilon_i . \tag{A3}$$

Let us next consider the action for the axially gauged WZW models $(G\otimes U(1)^d)/U(1)^d$:

$$S=kI_0(g)+\frac{k}{\pi}\int_\Sigma d^2z[A\bar{\partial}gg^{-1}-\bar{A}g^{-1}\partial g+Ag\bar{A}g^{-1}+A\bar{A}]+\frac{1}{\pi}\int_\Sigma d^2zk_i[\tfrac{1}{2}\partial\gamma_i\bar{\partial}\gamma_i+iA_i\bar{\partial}\gamma_i-i\bar{A}\partial\gamma_i+2A_i\bar{A}_i]. \tag{A4}$$

where the γ_i's parametrize the $\mathrm{U}(1)^d$ factor in $G\otimes \mathrm{U}(1)^d$. The above action is invariant under the infinitesimal axial gauge transformations

$$\delta g=\{g,i\epsilon\}\ ,\quad \delta\gamma_i=2\epsilon_i\ ,\quad \delta A_i=i\partial\epsilon_i\ ,\quad \delta\bar{A}_i=-i\bar{\partial}\epsilon_i\ . \tag{A5}$$

The group element $g\in G$ is parametrized as

$$g=e^{i\sqrt{2\epsilon}\,a\cdot R}e^{2\epsilon i v\cdot u}\ . \tag{A6}$$

Using the expansions in powers of ϵ [very similar to the ones in (2.23)],

$$\begin{aligned}
&g^{-1}\partial g=i\sqrt{2\epsilon}\,\partial a\cdot R+i\epsilon[2\partial v\cdot u-\partial a^\alpha m_\alpha{}^i u_i+\alpha^\alpha\partial a^\beta S_{\alpha\beta}{}^\gamma R_\gamma]+O(\epsilon^2)\ ,\\
&\bar{\partial}g g^{-1}=i\sqrt{2\epsilon}\,\bar{\partial}a\cdot R+i\epsilon[2\bar{\partial}v\cdot u+\bar{\partial}a^\alpha m_\alpha{}^i u_i-\alpha^\alpha\bar{\partial}a^\beta S_{\alpha\beta}{}^\gamma R_\gamma]+O(\epsilon^2)\ ,\\
&\mathrm{Tr}(Ag\bar{A}g^{-1})=A_i\bar{A}_i+\epsilon A_i(mm^t)_{ij}\bar{A}_j+O(\epsilon^2)\ ,\quad I_0(g)=\frac{\epsilon}{\pi}\int_\Sigma d^2z\,\partial a\cdot\bar{\partial}a+O(\epsilon^{3/2})\ ,
\end{aligned} \tag{A7}$$

choosing the gauge-fixing condition $\gamma_i=0$, $i=1,2,\dots,d$ and letting

$$k=\frac{\alpha}{2\epsilon}\ ,\quad k_i=\beta_i-\frac{\alpha}{2\epsilon}\ , \tag{A8}$$

we find that the action (A4) in the limit $\epsilon\to 0$ becomes the action (A2) in the gauge $x_i=0$, $i=1,2,\dots,d$. Notice that in (A2) the v^i's enter as Lagrange multipliers (up to an important total derivative [8,9]) for Abelian duality transformations in agreement with the general discussion in Sec. IV. Moreover, since the duality transformations are Abelian it is apparent that the final σ model one obtains by integrating out the gauge fields will have $\dim(H)$ Killing vectors along the v^i directions; i.e., it will be invariant under the constant shifts $v^i\to v^i+\epsilon^i$ (in the non-Abelian case this symmetry gets replaced by a nonlocal one [26]). By gauging this symmetry or in other words by performing the inverse duality transformation one obtains the background, in *flat spacetime*, described by (A2) after setting the gauge fields to zero (for a nice proof that two successive duality transformations corresponding to the same Abelian isometries lead to the original model see [26]). This is generalization of a similar statement made in [35] at the σ-model level (after integrating over the gauge fields) for the case of $E_2^c/\mathrm{U}(1)$ which was shown to be related to the $D=3$ black string $\mathrm{SL}(2,\mathbb{R})\otimes\mathbb{R}/\mathbb{R}$.

APPENDIX B: CONTRACTION OF $G\otimes H\otimes H'$

In this appendix we construct new WZW models based on non-semi-simple groups via a similar to the Sec. II limiting procedure. Let us consider the WZW model for $G\otimes H\otimes H'$ where G, H, and H' (H,H' are assumed isomorphic) are groups and as before G should contain a subgroup isomorphic to H and H'. We choose a basis for the currents: $\hat{g}=\{u_i,R_\alpha\}$, $\hat{h}=\{v_i^+,v_i^-\}$, where $i=1,2,\dots,\dim(H)$ and $\alpha=1,2,\dots,\dim(G/H)$. Above $\hat{g}$ and $\hat{h}$ belong to the current algebras associated with the WZW models for the groups G and $H\otimes H'$, respectively. The OPE's for $\hat{g}$ are given by the corresponding ones in (2.1), whereas those for $\hat{h}$ are

$$\begin{aligned}
&v_i^\pm v_j^\pm\sim\frac{if_{ij}{}^k v_k^+}{z-w}+\frac{k^+\eta_{ij}}{(z-w)^2}\ ,\\
&v_i^+v_j^-\sim\frac{if_{ij}{}^k v_k^-}{z-w}+\frac{k^-\eta_{ij}}{(z-w)^2}\ .
\end{aligned} \tag{B1}$$

The corresponding energy momentum tensor and the associated central charge are

$$\begin{aligned}
&T=\frac{:u^2+R^2:}{2(k_G+g_G)}+\frac{:v_1^2:}{2(k_1+g_H)}+\frac{:v_2^2:}{2(k_2+g_H)}\ ,\\
&c=\frac{k_G\dim(G)}{k_G+g_G}+\frac{k_1\dim(H)}{k_1+g_h}+\frac{k_2\dim(H)}{k_2+g_H}\ ,
\end{aligned} \tag{B2}$$

where the currents $(v_1)_i=(v_i^++v_i^-)/2$ and $(v_2)_i=(v_i^+-v_i^-)/2$ generate two commuting copies of the Kac-Moody algebra with levels $k_1=(k^++k^-)/2$ and $k_2=(k^+-k^-)/2$, respectively. Next we define

$$\begin{aligned}
&T_i=u_i+v_i^+\ ,\quad F_i=\epsilon(u_i-v_i^+)\ ,\\
&P_\alpha=\sqrt{2\epsilon}\,R_\alpha\ ,\quad S_i=\sqrt{2\epsilon}\,v_i^--\frac{\gamma}{\alpha}F_i\ ,\\
&k_G=\tfrac{1}{2}(\beta+\alpha/\epsilon)\ ,\quad k^+=\tfrac{1}{2}(\beta-\alpha/\epsilon)\ ,\\
&k^-=\frac{1}{\sqrt{2\epsilon}}\gamma\ ,
\end{aligned} \tag{B3}$$

and take the singular limit $\epsilon\to 0$. In this limit we discover a new current algebra *not* equivalent to the original one because the transformation (B3) is not invertible in that limit. The OPE's of the new current algebra one obtains this way are given by (2.4) and the additional ones

$$\begin{aligned}
&S_iS_j\sim\frac{-if_{ij}{}^kF_k}{z-w}+\frac{-a\eta_{ij}}{(z-w)^2}\ ,\\
&T_iS_j\sim\frac{if_{ij}{}^kS_k}{z-w}\ .
\end{aligned} \tag{B4}$$

We see that the current generators S_i although they have a different index structure are very similar to the P_α's and that the constant γ drops out completely. The corresponding energy momentum tensor and the associated central charge are

$$\begin{aligned}
&T=\frac{:P^2-S^2+2FT:}{2\alpha}-\frac{\beta+g_G+2g_H}{2\alpha^2}:F^2:\ ,\\
&c=\dim(G)+2\dim(H)\ .
\end{aligned} \tag{B5}$$

The OPE's in (B4) define the quadratic form

$$\Omega_{AB}=\begin{array}{c|cccc} & P_\beta & T_j & S_j & F_j \\ \hline P_\alpha & \alpha/\beta\eta_{\alpha\beta} & 0 & 0 & 0 \\ T_i & 0 & \eta_{ij} & 0 & \alpha/\beta\eta_{ij} \\ S_i & 0 & 0 & -\alpha/\beta\eta_{ij} & 0 \\ F_i & 0 & \alpha/\beta\eta_{ij} & 0 & 0 \end{array}\ , \tag{B6}$$

which by construction shares all three properties that are necessary if it is to be used to write down a WZW action (it is symmetric, a group invariant and invertible). Sparing the details we will give the final expression for such an action. In the parametrization where

$$g=e^{ib\cdot S}e^{ia\cdot P}e^{iv\cdot F}h_x \tag{B7}$$

the WZW action, whose symmetry current algebra is given in (2.4) and (B4), reads

$$S(g)=\beta I_0(h_x)+\frac{\alpha}{2\pi}\int_\Sigma d^2z[\partial a\cdot\bar\partial a-\partial b\cdot\bar\partial b+(2\partial v_i+m_{i\alpha}\partial a^\alpha-b_{ij}\partial b^j)R^i_\mu\bar\partial x^\mu]\ , \tag{B8}$$

where $b_{ij}=f_{ikj}b^k$. Similarly to (2.18) the action (B8) describes string backgrounds in $\dim(G)+2\dim(H)$ spacetime dimensions with $\dim(H)$ null Killing vectors associated with the coordinates v^i. The coordinates b^i and a^α enter the action in a similar way although they have a different index structure. If H is Abelian the theory becomes equivalent to $G_h^c\otimes\mathbb{R}^{\dim(H)}$. It is better to analytically continue $S_i\to iS_i$. In that case and for G,H compact groups ($\alpha,\beta>0$) the signature of the spacetime of (B8) has $\dim(G)+\dim(H)$ positive and $\dim(H)$ negative entries.

APPENDIX C: PLANE WAVE SOLUTIONS

Let us, in (2.18), consider the case of $G=SO(d+2)$ and $H=SO(2)$. In this case there is only one timelike coordinate. Also to agree with widely accepted conventions in the literature we use the symbol u for the single parameter in $h_x\in SO(2)$. The invariant subgroup $SO(d)$ after the contraction gives rise to $d(d-1)/2$ decoupled free fields in the action (2.18) which we shall ignore. After a shifting in v, to absorb the $\partial u\bar\partial u$ term, and a rescaling we obtain the action

$$S=\frac{\beta}{2\pi}\int_\Sigma d^2z[2\partial v\bar\partial u+\partial a^i_\alpha\bar\partial a^i_\alpha+\epsilon_{\alpha\beta}\partial a^i_\alpha a^i_\beta\bar\partial u]\ , \tag{C1}$$

where $\alpha,\beta=1,2$ and $i=1,2,\ldots,d$ and as usual summation over repeated indices is implied. Of course for $d=1$ this is the result of [31] and the corresponding CFT is the current algebra for E_2^c. Also notice that (C1) is *not* the action for just the direct product of d E_2^c models. A straightforward extension of the change of variables used in [31]

$$\begin{aligned} &a^i_1=x^i_1+x^i_2\cos u\ ,\quad a^i_2=x^i_2\sin u\ ,\\ &v\to v+\tfrac{1}{2}x^i_1x^i_2\sin u\ , \end{aligned} \tag{C2}$$

gives the final form of the action

$$\begin{aligned} S=\frac{\beta}{2\pi}\int_\Sigma d^2z[&2\partial v\bar\partial u+\partial x^i_1\bar\partial x^i_1+\partial x^i_2\bar\partial x^i_2\\ &+2\cos u\,\partial x^i_1\bar\partial x^i_2]\ . \end{aligned} \tag{C3}$$

This belongs to a class of plane wave-type exact string solutions with a covariantly constant null Killing vector that have been discussed extensively in the literature (see for instance [67,68]). However, what is important here is that there is also a CFT description for (C3) given by the current algebra for $SO^*(d+2)^c_{SO(2)}$, where the star implies that we neglect the $d(d-1)/2$ free decoupled fields, as we have already stated. For completeness we give the corresponding OPE's [cf. (2.4)]

$$\begin{aligned} &JP^i_\alpha\sim\frac{i\epsilon_{\alpha\beta}P^i_\beta}{z-w}\ ,\quad JF\sim\frac{1}{(z-w)^2}\ ,\\ &P^i_\alpha P^j_\beta\sim\frac{i\delta^{ij}\epsilon_{\alpha\beta}F}{z-w}+\frac{\delta^{ij}\delta_{\alpha\beta}}{(z-w)^2}\ , \end{aligned} \tag{C4}$$

and the Virasoro algebra stress tensor and central charge:

$$T=\tfrac{1}{2}:(P^i_\alpha P^i_\alpha+2JF-dF^2):\ ,\quad c=2(d+1)\ . \tag{C5}$$

Finally, let us note that we may replace the trigonometric function $\cos u$ in the last term in (C3) by $\sin u$ for some of the x^i_α's (an arbitrary number of them). This can be achieved by performing, for these x^i_α's, the transformation

$$\begin{aligned} &x^i_1\to x^i_1+\frac{x^i_2}{\sin u}\ ,\quad x^i_2\to -x^i_2\cot u\ ,\\ &v\to v+\tfrac{1}{2}x^i_2x^i_2\cot u\ . \end{aligned} \tag{C6}$$

Moreover we can replace the action (C3) by the more general (and still conformal) one

$$\begin{aligned} S=\frac{\beta}{2\pi}\int_\Sigma d^2z[&2\partial v\bar\partial u+\partial x^i_1\bar\partial x^i_1+\partial x^i_2\bar\partial x^i_2\\ &+2\cos(c_iu+d_i)\partial x^i_1\bar\partial x^i_2]\ , \end{aligned} \tag{C7}$$

where c_i,d_i, $i=1,2,\ldots,d$ are arbitrary constants and where summation over all repeated indices is, as usual, implied.

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