Zero-slope limit of the compactified closed bosonic string

R. Marotta
Dipartimento di Scienze Fisiche, Università di Napoli Mostra d'Oltremare, Pad. 19, I-80125 Napoli, Italy and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Mostra d'Oltremare, Pad. 20, I-80125 Napoli, Italy

F. Pezzella
Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Mostra d'Oltremare, Pad. 20, I-80125 Napoli, Italy
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We investigate, in the low-energy limit of the closed bosonic string theory, the role of the toroidal compactification, in which the extra spatial coordinates are circular with radius R. We explicitly show that the double limit $\alpha' \to 0$ and $R \to 0$, performed on the tree scattering amplitude of four massless scalar particles, leads to four-dimensional amplitudes describing a diffusion due to the exchange of a scalar, spin-1, and spin-2 particle and that it is not influenced by the compactification procedure adopted.

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I. INTRODUCTION

String theories have to reproduce, at the "low-energy" limit in which the slope α' goes to zero, the ordinary field theories describing the fundamental interactions formulated in a number of space-time dimensions coincident with the dimensionality D of the space-time in which the string is embedded ($D = 26$ for the bosonic string). In order to reduce this dimension to $D = 4$ a compactification scheme must be adopted. The aim of our work is to investigate the role of compactification in the low-energy limit of the theory. In particular we explicitly show, on the one hand, how, with a suitable choice of the compactification procedure, string amplitudes reproduce, in the above-mentioned limit, four-dimensional field theory amplitudes and, on the other one, that the low-energy limit is not influenced by the compactification scheme adopted.

We will consider here the "toroidal compactification" [1–3] of the bosonic closed string, in which 22 spatial coordinates are compactified into circles with radius R and the compactified space becomes a lattice which is required to be Lorentzian, self-dual, and even in order to have consistency with the properties of the bosonic string theory.

Constructing such a lattice yields the introduction of Lie algebra lattices where the massless states which arise from the toroidal compactification lie on the root lattice and belong to the adjoint representation of the gauge group relative to the algebra.

After having endowed the theory with the above compactification scheme, the limit $\alpha' \to 0$ makes all the massive modes uncouple giving rise to a description in terms of only massless states.

In particular we consider tree scattering amplitudes of four massless scalar particles. We perform the double limit $\alpha' \to 0$ and $R \to 0$, keeping the ratio $a = R/\sqrt{\alpha'}$ fixed; in so doing, we obtain amplitudes corresponding to the diffusion of four scalar particles due to the exchange of a scalar particle ($\lambda\phi^3$ theory), a spin-1 particle (scalar electrodynamics), and a spin-2 particle. In particular, this latter amplitude is coincident with the one obtained in the framework of linearized quantum gravity. An analogous computation was performed in the context of the generalized dual Virasoro model [4].

The article is organized as follows. In Sec. II we recall some generalities about the toroidal compactification and its connections with Lie algebra lattices; we mainly stress the properties enjoyed by the lattice on which the closed bosonic string theory is compactified. In Sec. III we give the definition of compactified vertices. In particular we write them for massless scalar particles. By using this helpful version of vertex operators, we compute the tree scattering amplitude of four scalar particles in the compactified space-time, on which it is then performed the double limit $\alpha' \to 0$ and $R \to 0$.

II. TOROIDAL COMPACTIFICATION AND LIE ALGEBRA LATTICES

The toroidal compactification consists in associating the d internal extracoordinates of the string to d circles having radius R_i with $i = 1, \ldots, d$; this can be done by identifying the points of the internal space as follows:

$$X^I \equiv X^I + 2\pi L^I ,$$

where $I = 1, \ldots, d$ and

$$L^I = \left[\frac{1}{2}\right]^{1/2} \sum_{i=1}^{d} n_i R_i e_i^I$$

$n_i \in Z$ being a so-called "winding number." The vectors $\hat{\mathbf{e}}_i \equiv (e_i^1, \ldots, e_i^d)$ are linearly independent and normalized as follows:

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = 2 .$$

The quantities L^I's can be thought as components of a vector defined on a d-dimensional lattice Λ^d which admits as a basis the set of vectors $\{\sqrt{\tfrac{1}{2}} R_i \hat{\mathbf{e}}_i\}$ with $i = 1, \ldots, d$.

It follows that the torus on which we compactify is the quotient space:

$$T^d = \frac{\mathbf{R}^d}{2\pi\Lambda^d} .$$

One gets the following mode expansion for the compactified string field X^I [2,3]:

$$X^I(z,\bar{z}) = X^I_L(z) + X^I_R(\bar{z}) ,$$

with

$$X^I_L(z) = x^I_L - i\frac{\alpha'}{2}p^I_L \ln z + i\left[\frac{\alpha'}{2}\right]^{1/2}\sum_{n\neq 0}\frac{1}{n}\alpha^I_n z^{-n} , \qquad (1)$$

$$X^I_R(\bar{z}) = x^I_R - i\frac{\alpha'}{2}p^I_R \ln \bar{z} + i\left[\frac{\alpha'}{2}\right]^{1/2}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}^I_n \bar{z}^{-n} , \qquad (2)$$

where

$$x^I_L = \frac{1}{2}x^I + \frac{\alpha'}{2}Q^I ,$$

$$x^I_R = \frac{1}{2}x^I - \frac{\alpha'}{2}Q^I ,$$

Q^I being the operator canonically conjugate to L^I, here introduced in order to define completely independent left and right sectors [3]; furthermore,

$$p^I_R = p^I - \frac{L^I}{\alpha'}, \quad p^I_L = p^I + \frac{L^I}{\alpha'} .$$

The following commutation relations hold:

$$[x^I_L, p^J_L] = [x^I_R, p^J_R] = i\delta^{IJ}$$

with all the other commutators vanishing. The compactification of the internal spatial coordinates implies that also the momenta p^I's, representing the translation operators of those coordinates, lie on a d-dimensional lattice that is the dual of the lattice Λ^d and it is denoted by $(\Lambda^d)^*$: i.e.,

$$p^I = \sqrt{2}\sum_{i=1}^{d}\frac{m_i}{R_i}e^{*I}_i , \qquad (3)$$

the vector $\hat{\mathbf{e}}^*_i$ being the dual of $\hat{\mathbf{e}}_i$. A basis on such a lattice is given by the vectors $\{(\sqrt{2}/R_i)\hat{\mathbf{e}}_i\}$. In this compactification scheme, in which we consider $R_i = R$ $\forall i = 1, \ldots, d$, the constraint conditions of the bosonic closed string become

$$L_0 - 1 = 0 \Longleftrightarrow \frac{\alpha'}{4}m^2 = \frac{\alpha'}{4}p^2_R + N - 1 , \qquad (4)$$

$$\bar{L}_0 - 1 = 0 \Longleftrightarrow \frac{\alpha'}{4}m^2 = \frac{\alpha'}{4}p^2_L + \bar{N} - 1 . \qquad (5)$$

Equations (4) and (5) can be rewritten as

$$\frac{\alpha'}{2}m^2 = \frac{\alpha'}{4}(p^2_R + p^2_L) + N + \bar{N} - 2 , \qquad (6)$$

$$N - \bar{N} = \frac{\alpha'}{4}(p^2_L - p^2_R) . \qquad (7)$$

From here it is possible to observe that the bivector

$$\hat{\mathbf{P}} \equiv \left[\left[\frac{\alpha'}{2}\right]^{1/2}p_R, \left[\frac{\alpha'}{2}\right]^{1/2}p_L\right]$$

lies on an even lattice $\Gamma_{d,d}$, after having chosen the metric of the lattice to be of the form $[(+1)^d, (-1)^d]$ (Lorentzian lattice); furthermore, modular invariance forces such a lattice to be self-dual too.

Another condition that must be imposed on the lattice comes from the following considerations.

By analyzing the lattice $\Gamma_{d,d}$ it turns out that the right and the left components of the bivector $\hat{\mathbf{P}}$ can be written as

$$\left[\frac{\alpha'}{2}\right]^{1/2}p_R = \sum_{i=1}^{d}m_i\left[\frac{\sqrt{\alpha'}}{R}\right]\hat{\mathbf{e}}^*_i - \frac{1}{2}\sum_{i=1}^{d}n_i\left[\frac{R}{\sqrt{\alpha'}}\right]\hat{\mathbf{e}}_i , \qquad (8)$$

$$\left[\frac{\alpha'}{2}\right]^{1/2}p_L = \sum_{i=1}^{d}m_i\left[\frac{\sqrt{\alpha'}}{R}\right]\hat{\mathbf{e}}^*_i + \frac{1}{2}\sum_{i=1}^{d}n_i\left[\frac{R}{\sqrt{\alpha'}}\right]\hat{\mathbf{e}}_i . \qquad (9)$$

Equations (8) and (9) can be generalized by adding a constant background antisymmetric tensor field B_{ij} to the usual action of the bosonic string: this operation is necessary to get more general and larger gauge groups [2,3]. Taking into account this generalization we can rewrite the components of the bivector $\hat{\mathbf{P}}$ as

$$\left[\frac{\alpha'}{2}\right]^{1/2}p^I_R = \sum_{i=1}^{d}m_i\left[\frac{\sqrt{\alpha'}}{R}\right]\hat{\mathbf{e}}^{*I}_i - \frac{1}{2}\sum_{i=1}^{d}n_i\left[\frac{R}{\sqrt{\alpha'}}\right]\hat{\mathbf{e}}^I_i - \sum_{ij=1}^{d}B_{ij}n_j\left[\frac{\sqrt{\alpha'}}{R}\right]e^{*I}_i ,$$

$$\left[\frac{\alpha'}{2}\right]^{1/2}p^I_L = \sum_{i=1}^{d}m_i\left[\frac{\sqrt{\alpha'}}{R}\right]\hat{\mathbf{e}}^{*I}_i + \frac{1}{2}\sum_{i=1}^{d}n_i\left[\frac{R}{\sqrt{\alpha'}}\right]\hat{\mathbf{e}}^I_i - \sum_{ij=1}^{d}B_{ij}n_j\left[\frac{\sqrt{\alpha'}}{R}\right]e^{*I}_i .$$

The basis vectors in $\Gamma_{d,d}$ are evidently $(\sqrt{\alpha'}/R)\hat{\mathbf{e}}^*_i$ and the dual ones are $(R/\sqrt{\alpha'})\hat{\mathbf{e}}_i$.

In the double limit $\alpha' \to 0$ and $R \to 0$, p_L and p_R will be well-defined quantities only if the ratio $a = R/\sqrt{\alpha'}$ is kept fixed [5]; in particular we choose $a = 1$ [1]. This choice leads to a rational lattice.

In conclusion, the lattice on which the theory is compactified must be Lorentzian, self-dual, even, and rational.

It is known that a large class of such lattices can be constructed in $\mathbf{R}^{d,d}$ considering the set of all vectors of the form (v_1, v_2) so that v_1 and v_2 belong to the same conjugacy class of a semisimple Lie algebra of rank d [3].

By evaluating the double limit $R = \sqrt{\alpha'} \to 0$ on the equations (6) and (7) one has that the only particles with finite masses which survive are massless particles for which the norm of the components of $\hat{\mathbf{P}}$ is null or equal to 2. In particular, these latter are by definition lattice roots. Roots include in any case those of the Lie algebra used in constructing the lattice, but in some cases there may be additional norm 2 vectors in the other conjugacy classes.

We are going to compactify on a lattice where all the norm 2 vectors belong to the root lattice of a simply laced Lie algebra. On this kind of lattices the scalar product between the vectors, which survive after performing the double limit, takes integer values. This will greatly simplify our computation.

A possible lattice satisfying the above requirements is $\Gamma_{d,d}=\Gamma_d\otimes\Gamma_d$ with $\Gamma_d=E_8\otimes E_8\otimes E_6$; however, there exists a large class of Lorentzian, self-dual, even, and rational lattices that well suit our problem.

III. COMPACTIFIED VERTICES AND LOW-ENERGY LIMIT OF SCATTERING AMPLITUDES

We are going to consider scattering amplitudes involving scalar particles. These come from the levels reported in Table I, together with the corresponding norms of p_R and p_L.

In order to compute those scattering amplitudes, we are going to consider a compactified version of the vertex operators $V_\psi(z,\bar z)$ providing the amplitude for the emission of a state ψ in the string spectrum.

Compactified vertices can be obtained from the ordinary ones through simple correspondences; for example, the compactified vertex relative to the scalar particles belonging to the level $N=\bar N=0$ is the naive version of the compactified tachyon vertex:

$$V_0(k,k_R,k_L;z,\bar z)=:e^{ik\cdot X(z,\bar z)}e^{ik_R\cdot X_R(z)}e^{ik_L\cdot X_L(\bar z)}:$$

where $X^\mu(z,\bar z)$ is the usual string field, with $\mu=1,\ldots,26-d$; X_R^I and X_L^I, with $I=1,\ldots,d$ are the field defined in (2). The state corresponding to V_0 is defined, as usual, through the limit

$$\lim_{z,\bar z\to 0} V_0(k,k_R,k_L;z,\bar z)|\text{vacuum}\rangle\ .$$

Conformal invariance requires V_0 to be a conformal field with dimensions $\Delta=\bar\Delta=1$; since the following operator product expansion holds between V_0 and the stress energy tensor,

$$T(z)V_0(w,\bar w)=\left[\frac{1}{z-w}\partial_w+\frac{(k^2+k_R^2)\frac{\alpha'}{4}}{(z-w)^2}\right]V_0(w,\bar w)\ ,$$

one has

$$k^2+k_R^2=\frac{4}{\alpha'}$$

with a similar relation holding for the antiholomorphic sector. This shows again that for massless scalar particles one has $k^2=0$ and $k_R^2=k_L^2=4/\alpha'$.

These considerations suggest to introduce a general vertex, which is nothing but a linear combination of the vertices relative to the massless scalar particles introduced in Table I:

$$V_s=:e^{ik\cdot X(z,\bar z)}[e^{ik_R\cdot X_R(z)}+\xi\cdot\partial_z X_R(z)][e^{ik_L\cdot X_L(\bar z)}+\bar\xi\cdot\partial_{\bar z}X_L(\bar z)]: \tag{10}$$

where ξ and $\bar\xi$ are polarization vectors defined in the compactified space.

By using the version (10) of the vertex operators it is straightforward to compute the tree scattering amplitude of four scalar particles in D dimensions, by using the operatorial formalism of the N-string vertex [6] (see Fig. 1):

$$A=N_0\int dz\,d\bar z\,z^{(\alpha'/2)k_3\cdot k_4+n_{34}}(1-z)^{(\alpha'/2)k_2\cdot k_3+n_{23}}\bar z^{(\alpha'/2)k_3\cdot k_4+\bar n_{34}}(1-\bar z)^{(\alpha'/2)k_2\cdot k_3+\bar n_{23}}\ , \tag{11}$$

where N_0 is a suitable normalization constant dictated by unitarity and given explicitly by

$$N_0=\frac{2}{\pi}g_D^2$$

with g_D being the D-dimensional coupling constant [8–10], related to the 26-dimensional one and to the compactification radius by

$$g_D^2=g_{26}^2(2\pi R)^{-26+D}\ .$$

Furthermore,

$$n_{ij}=\frac{\alpha'}{2}k_{R,i}\cdot k_{R,j}\ ,$$

$$\bar n_{ij}=\frac{\alpha'}{2}k_{L,i}\cdot k_{L,j}\ .$$

Equation (11) has been obtained by performing an average on the polarizations and a sum on all the possible values of the Lie algebra roots is understood; both of these operations make the sum of all the terms involving ξ and $\bar\xi$ null.

Equation (11) can be considered as a generalization of the dual Virasoro amplitude; its dependence on the lattice is entirely contained in the variables n_{ij}. These are characterized by taking integer values, as we have previously seen as a consequence of our choice of the lattice. On the other hand, conservation laws of the winding numbers and the compactified momenta [7] imply the constraints

TABLE I. Levels providing scalar particles and corresponding norms of the right p_R and left p_L momenta.

N	$\bar N$	p_R^2	p_L^2
0	0	$4/\alpha'$	$4/\alpha'$
0	1	$4/\alpha'$	0
1	0	0	$4/\alpha'$
1	1	0	0

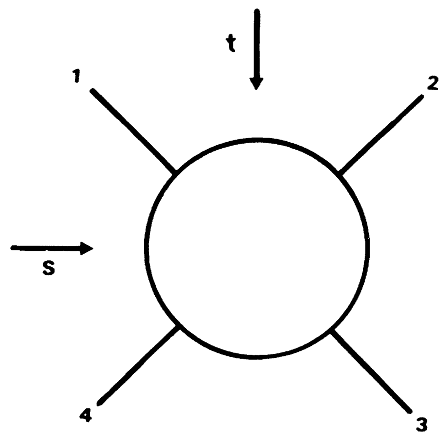

FIG. 1. Labeling of the four interacting scalar particles.

$$\hat{\mathbf{k}}_{R,1}+\hat{\mathbf{k}}_{R,2}+\hat{\mathbf{k}}_{R,3}+\hat{\mathbf{k}}_{R,4}=0$$

from which

$$n_{13}+n_{23}+n_{34}=-2 \tag{12}$$

and

$$n_{ij}=\frac{\alpha'}{2}\hat{\mathbf{k}}_{R,i}\cdot\hat{\mathbf{k}}_{R,j}\le\frac{\alpha'}{2}|\hat{\mathbf{k}}_{R,i}|^2=2$$

i.e.,

$$-2\le n_{ij}\le 2 \ . \tag{13}$$

Analogously $-2\le\bar{n}_{ij}\le 2$. Furthermore, since the lattice can be chosen in such a way that the vectors $\hat{\mathbf{k}}_{R,i}$ are roots of the Lie algebra used in constructing the lattice, the scalar product between two of them must be integer.

By using standard techniques it is possible to write Eq. (11) in the form

$$A_{\text{tree}}(s,t,u)=N_0\pi\frac{\Gamma\left[-\frac{\alpha'}{4}s+n_{23}+1\right]}{\Gamma\left[\frac{\alpha'}{4}s-\bar{n}_{23}\right]}\times\frac{\Gamma\left[-\frac{\alpha'}{4}t+n_{34}+1\right]}{\Gamma\left[\frac{\alpha'}{4}t-\bar{n}_{34}\right]}\times\frac{\Gamma\left[-\frac{\alpha'}{4}u+\bar{n}_{13}+1\right]}{\Gamma\left[\frac{\alpha'}{4}u-n_{13}\right]} \ . \tag{14}$$

Equation (14) holds if at least two of the differences $n_{ij}-\bar{n}_{ij}$ are non-negative. Otherwise, one gets

$$A_{\text{tree}}(s,t,u)=N_0\pi\frac{\Gamma\left[-\frac{\alpha'}{4}s+\bar{n}_{23}+1\right]}{\Gamma\left[\frac{\alpha'}{4}s-n_{23}\right]}\times\frac{\Gamma\left[-\frac{\alpha'}{4}t+\bar{n}_{34}+1\right]}{\Gamma\left[\frac{\alpha'}{4}t-n_{34}\right]}\times\frac{\Gamma\left[-\frac{\alpha'}{4}u+n_{13}+1\right]}{\Gamma\left[\frac{\alpha'}{4}u-\bar{n}_{13}\right]} \ . \tag{15}$$

Equation (15) corresponds to Eq. (14) in which the variables n_{ij} and $\bar{n}_{ij}$ are interchanged.

The amplitude (14) has poles compatible with the mass formula (4). Analogously the amplitude (15) has poles consistent with the constraint (5).

We are now interested in performing the limit $\alpha'\to 0$ of the scattering amplitudes (14) [or, equivalently, of (15)]. Taking into account the possible values of the variables n_{ij} and by using the analytic properties of the Γ function, it is straightforward to obtain the following result in the limit $\alpha'\to 0$:

$$A_{\text{tree}}(s,t,u)=A_{\text{tree}}^{(\lambda\phi^4)}+A_{\text{tree}}^{(\lambda\phi^3)}+A_{\text{tree}}^{(\text{spin 1})}+A_{\text{tree}}^{(\text{spin 2})}$$

with

$$A_{\text{tree}}^{(\lambda\phi^4)}\sim g_D^2 \ , \tag{16}$$

$$A_{\text{tree}}^{(\lambda\phi^3)}=-2\frac{4}{\alpha'}g_D^2\left[\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right] , \tag{17}$$

$$A_{\text{tree}}^{(\text{spin 1})}=2g_D^2\left[\frac{u+t}{s}+\frac{s+u}{t}+\frac{s+t}{u}\right] , \tag{18}$$

$$A_{\text{tree}}^{(\text{spin 2})}=2\frac{\alpha'}{4}g_D^2\left[\frac{tu}{s}+\frac{su}{t}+\frac{st}{u}\right] . \tag{19}$$

Equations (17), (18), and (19) represent amplitudes of scalars interacting through the exchange, respectively, of a scalar ($\lambda\phi^3$ theory, with $\lambda^2\equiv 2(4/\alpha')g_d^2$), spin-1 ("scalar electrodynamics"), and spin-2 particle (quantum gravity). The constant value (16) can be interpreted as a tree diagram of an "effective" $\lambda\phi^4$ theory, coming from a tree diagram of a $\lambda\phi^3$ theory on which the double limit produces an ultralocal limit of the propagator.

By comparing the amplitude (19) with the analogous one computed in quantum gravity, we can obtain a relationship between the string coupling constant g_D and the gravitational coupling constant G_N in $D=4$ dimensions:

$$g_4^2 = \frac{16\pi G_N}{\alpha'} .$$

This expression coincides with the one already known in literature [10].

In conclusion we have explicitly shown that compactification of the closed bosonic string theory reproduces only in the double limit $\alpha' \to 0$ and $R \to 0$, at the tree level, the ordinary field theories. We would like here to stress that in our work specifying the lattice has resulted to be unnecessary: hence, at least at this level, compactification does not influence the low-energy limit.

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