Classical open string models in four-dimensional Minkowski spacetime

Paweł Węgrzyn*

Department of Field Theory, Institute of Physics, Jagellonian University, 30-059 Cracow 16, Poland

(Received 17 February 1994)

Classical bosonic open string models in four-dimensional Minkowski spacetime are discussed. Special attention is paid to the choice of edge conditions, which can follow consistently from the action principle. We consider Lagrangians that can depend on second order derivatives of world sheet coordinates. A revised interpretation of the variational problem for such string theories is given. We derive a general form of a boundary term that can be added to the open string action to control edge conditions and modify conservation laws. An extended boundary problem for minimal surfaces is examined. Following the treatment of this model in the geometric approach, we obtain that classical open string states correspond to solutions of a complex Liouville equation. In contrast with the Nambu-Goto case, the Liouville potential is finite and constant at world sheet boundaries. The phase part of the potential defines topological sectors of solutions.

PACS number(s): 11.25.−w

I. INTRODUCTION

There is a common conviction that in order to gain more insight into the dynamical structure of QCD we need most likely to use some string representation of this theory. This is suggested by the topological nature of $1/N$ expansion [1], the area confinement law found in the strong coupling lattice expansion [2], the success of dual models in the description of Regge phenomenology, and the existence of flux-line solutions in confining gauge theories [3, 4]. More arguments are presented in recent reviews [5, 6].

In spite of numerous works, there is still a state of confusion about the existence of an exact, or even approximate, stringy reformulation of four-dimensional QCD at all distance scales. Even at any specific scale, it is not evident what the adequate set of string variables and fields is and how they correspond to QCD gauge fields. Referring only to a long-distance scale we usually adopt the naive, but lucid, picture of a flux tube regime. A pair of quarks in the confining phase is joined by a color flux concentrated in a thin tube. If these quarks are kept sufficiently far apart, the flux tube behaves like a vibrating string. Using string variables as collective coordinates, one should in principle find flux tube excitations by some quantization of the string action. The question of what kind of the string action should be employed to represent the flux tube has yet to be answered. It is conceivable that to the lowest order the action is just given by the Nambu-Goto action, which decribes an infinitely thin relativistic bosonic string with constant energy per unit length. As is well known, we cannot be satisfied with this first approximation because of some unacceptable features of the quantized Nambu-Goto string. Apart from the problems with conformal anomaly outside the critical dimension or tachyons and undesirable massless states in quantum spectra (which are presumably less embarrassing at the long-distance scale [7]), all standard quantizations give the incorrect number of degrees of freedom if we confront it with QCD predictions [8].

Basically, there are two ways to modify the four-dimensional Nambu-Goto action. In the first approach, keeping the conformal invariance we can place additional fields on the world sheet [9, 10]. The conformal anomaly can be saturated due to the contribution of new conformal fields. Since we can hardly justify the assumption that only massless degrees of freedom are important at the hadronic scale, so the respecting of conformal symmetry is here rather a compromise to make our theory mathematically tractable. The second kind of modification of the Nambu-Goto action, advocated in many papers (e.g., [11]), is to introduce new action terms representing interactions between transverse string modes. The fact that Regge trajectories derived directly from fundamental quark models [12–14] depart somewhat from straight lines is a strong argument that vibrating string modes cannot be considered as free. Next, some couplings between these modes (short-distance interactions) would cause preferable smooth string world sheets, leading to a well-defined quantum theory. Unfortunately, all such string "self-interaction" terms involve higher order derivatives in their Lagrangians. Theories with higher order derivatives usually reveal embarrassing pathological features, such as a lack of the energy bound, tachyons already on the classical level, and unitarity violation due to the presence of negative norm states. Presumably, it means that one must regard any particular effective theory of this type with a limited range of validity. From the technical point of view, such theories of strings are nonlinear and cannot be linearized by a suitable choice of gauge. Subsequently, a string cannot be described as an infinite set of oscillators and there is no analogue of the Virasoro algebra. The conformal symmetry is usu-

*Electronic address: wegrzyn@ztc386a.if.uj.edu.pl

ally spoiled. All that makes the evaluation of physical observables technically difficult.

In this paper we discuss possible modifications of the Nambu-Goto model (or any other specific bosonic string model) by the change of boundary conditions for open strings. This aspect is not well explored in the literature, even though the choice of boundary conditions can be crucial for defining relevant open string models. Let us give some examples.

Taking the usual hadronic string picture, we assume that quarks live only at the opposite end points of the string and communicate through their couplings to the string between. Then, to some extent the choice of world sheet boundary conditions determines quark trajectories. For instance, in the classical Nambu-Goto model they are rather peculiar, being boosted periodic lightlike (null) curves. Undoubtedly, the unsolved problem of how the quark masses and quantum numbers (spin, color) couple to the string variables partly lies in the proper specification of string edge conditions.

It is obvious that any internal symmetry of the world sheet is necessarily broken when world sheet boundaries are included. Conformal transformations or the full set of all reparametrizations are examples of that. Correspondingly, conformal field theories defined on surfaces with boundaries are usually endowed with only one copy of Virasoro algebra (instead of two, as for closed surfaces). Recently [10], on the same basis the chiral symmetry breaking mechanism has been included in hadronic string models. This simple observation that the existence of boundaries restricts the group of local world sheet symmetries indicates that physical observables can essentially depend on fields or currents evaluated on string boundaries.

One of the straightforward calculations to test some open string models against QCD expectations is to evaluate the static interquark potential. Asympotically at the long-distance scale, this potential is linear and its slope can be related to the string tension. The first quantum corrections give a universal Coulomb term [15] (Casimir effect), being the function of the number of world sheet fields and of their boundary conditions. In an approximation of the flux-tube action by some conformal string theory, one can represent the boundary conditions by the set of relevant conformal operators inserted at the boundaries [16]. The physical states are now constructed with the help of both bulk and boundary operators. The Coulomb term depends on the effective conformal anomaly [17], being the total conformal anomaly diminished by the weight of the lowest state. This weight is sensitive to the choice of boundary operators [9].

The influence of world sheet boundaries on critical string field theories has been discussed in recent papers (see [18, 19] and references therein). In the framework of the Becchi-Rouet-Stora-Tyutin (BRST) formalism in the critical dimension, we can consider either Neumann-type (e.g., standard Nambu-Goto edge equations) or Dirichlet-type boundary conditions imposed on world sheet coordinates. With Dirichlet conditions we have no physical open strings, but the closed-string theory is radically modified, particularly the massless spectrum. Instead of characteristic exponential fall-off of fixed-angle scattering amplitudes for string models at high energies, we obtain for Dirichlet strings powerlike behavior, like for parton models. We see here that the special type of world sheet boundaries, where these boundaries are mapped to single spacetime points, implies that some pointlike structure may appear at high energies.

In this paper we restrict ourselves to discussing open bosonic string models defined by local Lagrangian densities that can depend on second order derivatives of world sheet coordinates. In Sec. II we present general formulas suitable to perform classical analysis of such string models. In comparison with earlier works on this subject, a different interpretation of the variational problem for string actions with second order derivatives is given. Moreover, all derived classical formulas are explicitly covariant with respect to reparametrization transformations. In Sec. III we derive a general form of a boundary term that can be added to the action, allowed by requirements of Poincaré and reparametrization invariances. Such a term can modify edge conditions for open strings while bulk equations of motion are preserved. Canonical conserved quantities are modified by some edge contributions. Section IV is devoted to the classical analysis of the string model defined by the Nambu-Goto action with some new boundary terms added. It is argued that such an open string model can be a suitable modification of the Nambu-Goto model as far as hadronic string interpretation is concerned. We carry out the classical analysis using the geometric approach, which is particularly convenient for our purposes. The classical open string configurations that extremize the extended action correspond to solutions of a complex Liouville equation. The relevant edge conditions for a Liouville field are derived. The edge values are constant and finite there. Some preliminary discussion about physical consequences is made. In the Appendix, the notation used throughout the paper is introduced and some basic mathematical definitions and equations of surface theory are collected.

II. STRING LAGRANGIANS WITH SECOND ORDER DERIVATIVES

In this section we introduce some general formulas pertaining to the classical analysis of string models defined by Lagrangians which depend on second order derivatives of a world sheet radius vector. In comparison with previous papers (e.g., [20, 21]), all formulas presented below are explicitly covariant with respect to the reparametrization, and especially with the correct derivation of edge conditions for open strings.

Let us consider the general form of the bosonic string action:

$$S = \int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma \mathcal{L}_{\text{string}} . \tag{1}$$

It is convenient to represent the Lagrangian density as

$$\begin{aligned} \mathcal{L}_{\text{string}} &= \mathcal{L}_{\text{string}}(X_{\mu,a}; X_{\mu,ab}) \\ &= \sqrt{-g}\mathcal{L}(g^{ab}; X_{\mu,a}; \nabla_a \nabla_b X_\mu) , \end{aligned} \tag{2}$$

where $\mathcal{L}$ is some scalar function made up of its specified arguments. Having the string Lagrangian with second order derivatives written down in the above form we can much easier perform mathematical calculations and keep the explicit reparametrization invariance in all following steps.

To derive the classical equations of motion, we are to evaluate the variation of the string action under the infinitesimal change of the world sheet. Usually, the following boundary conditions are assumed:

$$\delta X_\mu(\tau_i,\sigma) = \delta \dot{X}_\mu(\tau_i,\sigma) = 0 \ , \quad i=1,2 \ . \tag{3}$$

There is some subtle problem at this point. The above requirements suggest the different interpretation of the variational problem in comparison with the usual Nambu-Goto case. In (3), not only initial and final string positions are fixed, but also the initial and final velocities of string points. Therefore, if we consider some string at the time τ_1, another string at the time τ_2, and some string trajectory being a solution of Euler-Lagrange equations which interpolates between them, the solution does not extremize the string action unless we restrict possible deviations of the world sheet to those that do not change its tangent vectors at the initial and final positions. In other words, the string instant state is specified not only by its position, but also by its velocities. In fact, this modified interpretation is not true as the boundary conditions (3) are not quite proper for the string variational problem with second order derivatives. This point will be clarified below.

The classical equations of motion following from (1) can be presented in the explicitly covariant form

$$\sqrt{-g}\nabla_a \Pi^a_\mu = 0 \ , \tag{4}$$

where Π^a_μ is given by the formula

$$\Pi^a_\mu = -\mathcal{L}\nabla^a X_\mu - \frac{\partial\mathcal{L}}{\partial X^\mu_{,a}} + 2\frac{\partial \mathcal{L}}{\partial g^{bc}} g^{ab}\nabla^c X_\mu + \nabla_b\left[\frac{\partial\mathcal{L}}{\partial(\nabla_a\nabla_b X^\mu)}\right] \ . \tag{5}$$

For open strings, the edge conditions at $\sigma = 0,\pi$ must be satisfied:

$$\sqrt{-g}\Pi^1_\mu + \partial_0\left[\sqrt{-g}\frac{\partial\mathcal{L}}{\partial(\nabla_0\nabla_1 X^\mu)}\right] = 0 \ , \tag{6}$$

$$\sqrt{-g}\frac{\partial\mathcal{L}}{\partial(\nabla_1\nabla_1 X^\mu)} = 0 \ . \tag{7}$$

For the sake of more convenient notation, here and throughout the paper we define and calculate the variational derivatives of $\mathcal{L}$ with the formal assumption that g^{01} and g^{10}, $\nabla_0\nabla_1 X^\mu$, and $\nabla_1\nabla_0 X^\mu$ are independent variables. Thus, all variational derivatives on the right-hand side (RHS) of (5) are tensor objects with respect to the reparametrization invariance. The covariance of edge conditions becomes easy to check if we remind the reader that in the presence of the world sheet boundary any reparametrization transformation $\sigma^a \to \tilde{\sigma}^a(\tau,\sigma)$ must satisfy

$$\tilde{\sigma}(\tau,0) = 0 \ , \quad \tilde{\sigma}(\tau,\pi) = \pi \ . \tag{8}$$

It is necessary in order to preserve the condition that the string parameter σ belongs to the interval $[0,\pi]$. In another case, performing the variation of the string action we are forced to implement the variations due to the change of the σ interval, and the fact that the set of allowed reparametrization transformations is restricted for open strings manifests in additional Euler-Lagrange equations.

The derivation of (4) from the standard Euler-Lagrange variational equations is straightforward, so let us only cite the following identities used in this derivation:

$$\frac{\partial\mathcal{L}}{\partial(\nabla_a\nabla_b X^\mu)} X^\mu_{,c} = 0 \ . \tag{9}$$

To prove the above identities for Lagrangians which include only scalar constant parameters, it is enough to notice that the scalar (with respect to both reparametrization and Poincaré transformations) function $\mathcal{L}$ can be composed of the "building blocks"

$$g^{ab} \ , \quad \epsilon^{\mu\nu\rho\sigma}(\nabla_a\nabla_b X_\mu)(\nabla_c\nabla_d X_\nu) X_{\rho,e} X_{\sigma,f} \ , \quad \nabla_a\nabla_b X^\mu \nabla_c\nabla_d X_\mu \ ,$$

and refer to the trivial identities

$$(\nabla_a\nabla_b X^\mu) X_{\mu,c} = 0 \ .$$

In general, the origin of identities (9) lies in the reparametrization invariance of the string action (1). The full set of all Noether identities [see (27)–(29)] following from the reparametrization invariance of the string action with second order derivatives has been derived in [22].

Let us return to the problem of boundary conditions (3) imposed on the variations of the world sheet. If we assumed only that

$$\delta X_\mu(\tau_i,\sigma) = 0 \ , \quad i=1,2 \ , \tag{10}$$

then using equations of motion (4) together with edge conditions (6) and (7) we would obtain the following result for the variation of the string action:

$$\delta S = \int_0^\pi d\sigma \sqrt{-g}\frac{\partial\mathcal{L}}{\partial(\nabla_0\nabla_0 X^\mu)}\delta\dot{X}^\mu\Big|_{\tau=\tau_1}^{\tau=\tau_2} \ . \tag{11}$$

If $g = 0$ or the surface is locally flat then the following term vanishes, or else we can choose parametrization in such a way that the four-vectors $(\dot{X}_\mu, X'_\mu, \ddot{X}_\mu, \dot{X}'_\mu)$ are linearly independent at the point of the world sheet with $\tau = \tau_i$. Then, we can write down the general form of $\delta\dot{X}_\mu$ as the linear combination of these vectors:

$$\delta\dot{X}_\mu = a_1\dot{X}_\mu + a_2 X'_\mu + a_3\ddot{X}_\mu + a_4\dot{X}'_\mu \ . \tag{12}$$

On the other hand, the variation $\delta\dot{X}_\mu$ induced by the change of parametrization $\sigma^a \to \sigma^a + \delta\sigma^a$ is given by

$$\delta\dot{X}_\mu = -\ddot{X}_\mu\delta\sigma^0 - \dot{X}'_\mu\delta\sigma^1 \ . \tag{13}$$

It means that the variations of $\dot{X}_\mu$ in the directions of $\ddot{X}_\mu$ and $\dot{X}'_\mu$ are not important, because they can be removed by the change of parametrization. In turn, if we restrict ourselves to the "physical" variations of the world sheet, then with the help of identities (9) we conclude that the term (11) vanishes.

Therefore, there are two ways to define properly the variational problem for string action functionals which depend on second order derivatives. One way is to assume boundary conditions (10) together with the additional requirements that the variations $\delta\dot{X}_\mu$ in the directions of $\ddot{X}_\mu$ and $\dot{X}'_\mu$ vanish, which in light of (13) means that the choice of the parametrization of the world sheet is locally fixed at boundary points $\tau=\tau_i$. Another way is to take only the boundary conditions (10), as in the Nambu-Goto case, and together with relevant equations of motion and edge conditions we obtain additional equations

$$\sqrt{-g}\frac{\partial\mathcal{L}}{\partial(\nabla_0\nabla_0X^\mu)}=0 \quad \text{for } \tau=\tau_1,\tau_2\ , \tag{14}$$

which have no dynamical content and impose only some boundary constraints on the choice of world sheet parametrization. Recapitulating, the interpretation of the variational problem for string actions with second order derivatives is the same as in the usual Nambu-Goto case. To derive the classical dynamics of strings from the variational principle it is just enough to consider the boundary conditions (10), i.e., to assume that the initial and final string positions are fixed. The appearance of the term (11) in the action variation and resulting equations reflect only the fact that the geometrical definitions of the *initial* and *final* string positions are not invariant.

One more comment on the derivation of edge conditions should be made. They are an integral part of equations of motion. They arise because in the variational problem for open string world sheets the whole boundary of the world sheet is not fixed (like in an ordinary Plateau problem for two-dimensional surfaces), but only a part of this being composed of the initial and final string positions. The other part of the world sheet boundary, defined by trajectories of string end points, is not fixed (the ends of open strings are free). However, we can use another equivalent method for the derivation of edge conditions. In the variational problem we can dispense with considering the edge variations (assuming that the whole world sheet boundary is fixed), and the edge conditions are produced when we demand that there is no flow of the canonical Noether invariants through the string ends. In distinction with the Nambu-Goto case, for strings with second order derivatives it is not enough to assure only that the canonical momentum is conserved. We must check the same independently for the angular momentum, because of its "spin part" induced by higher order derivatives. The comment on the latter method of the edge conditions derivation is relevant to the recent work of Boisseau and Letelier [23]. They make use of the internal geometrical description of world sheets to study models of strings with second order derivatives. In this approach, they gain some new insight into the content of dynamical equations. However, their formalism should be corrected for open strings. The set of edge conditions derived from the conservation of total energy-momentum should be supplemented by additional conditions associated with the total angular momentum conservation. In particular, it changes some results of the work [23]. For example, the prediction that the end points of the Polyakov rigid string can travel with a speed less than the velocity of light is not valid. Just taking into account the missing set of edge conditions, we check again that these velocities must be lightlike, which agrees with the independent proof of this fact given in [22].

In the last part of this section we write down formulas for Noether invariants. The total momentum reads

$$P_\mu=\int_0^\pi d\sigma p_\mu\ , \tag{15}$$

where

$$\begin{aligned}p_\mu&=-\frac{\partial\mathcal{L}_{\text{string}}}{\partial X^\mu_{,0}}+\partial_0\left(\frac{\partial\mathcal{L}_{\text{string}}}{\partial X^\mu_{,00}}\right)\\&=\sqrt{-g}\Pi^0_\mu-\partial_1\left[\sqrt{-g}\frac{\partial\mathcal{L}}{\partial(\nabla_0\nabla_1X^\mu)}\right]\ .\end{aligned}$$

The total angular momentum can be calculated from the formula

$$M_{\mu\nu}=\int_0^\pi d\sigma m_{\mu\nu}\ , \tag{16}$$

where

$$\begin{aligned}m_{\mu\nu}&=x_{[\mu}p_{\nu]}+\frac{\partial\mathcal{L}_{\text{string}}}{\partial X^{[\mu}_{,0a}}X_{\nu],a}\\&=\sqrt{-g}X_{[\mu}\Pi^0_{\nu]}-\sqrt{-g}X_{[\mu,a}\frac{\partial\mathcal{L}}{\partial(\nabla_a\nabla_0X^{\nu]})}\\&\quad-\partial_1\left[\sqrt{-g}X_{[\mu}\frac{\partial\mathcal{L}}{\partial(\nabla_0\nabla_1X^{\nu]})}\right]\ .\end{aligned}$$

III. BOUNDARY TERMS FOR STRING ACTIONS

We discuss the general string action functional with some boundary term added:

$$S=\int d^2\sigma\mathcal{L}^{\text{bulk}}_{\text{string}}-\int d^2\sigma\partial_aV^a\ . \tag{17}$$

The stationarity of this action results in some equations for the interior of the string following from $\mathcal{L}^{\text{bulk}}_{\text{string}}$, and the role of the second action term is to ensure a more general set of edge conditions for an open string case. Below, we will find the general form of this term allowed by requirements of the locality, Poincaré, and reparametrization invariance. We restrict ourselves to string Lagrangians which depend on not higher than second order derivatives, which implies that

$$\frac{\partial V^a}{\partial X^\mu_{,bc}}X^\mu_{,abc}=0\ . \tag{18}$$

The above identities give immediately the equations

$$\frac{\partial V^0}{\partial X^\mu_{,00}} = \frac{\partial V^0}{\partial X^\mu_{,11}} + 2\frac{\partial V^1}{\partial X^\mu_{,01}} = \frac{\partial V^1}{\partial X^\mu_{,11}} = \frac{\partial V^1}{\partial X^\mu_{,00}} + 2\frac{\partial V^0}{\partial X^\mu_{,01}} = 0 \ , \quad (19)$$

and their general solution is of the form

$$V^a = \epsilon^{ab}\tilde{A}^c_\mu X^\mu_{,bc} + \tilde{B}^a \ , \quad (20)$$

where $\tilde{A}^c_\mu$ and $\tilde{B}^a$ are some arbitrary functions which depend on X_μ and their first derivatives. The translational invariance of the action requires that

$$0 = \frac{\partial(\partial_a V^a)}{\partial X^\mu} = \partial_a\left(\frac{\partial V^a}{\partial X^\mu}\right) \ ; \quad (21)$$

therefore, there exists the function $\Lambda_\mu(X_\nu; X_{\nu,a})$ such that

$$\frac{\partial V^a}{\partial X^\mu} = \epsilon^{ab}\partial_b\Lambda_\mu = \epsilon^{ab}\left(\frac{\partial\Lambda_\mu}{\partial X^\nu}X^\nu_{,b} + \frac{\partial\Lambda_\mu}{\partial X^\nu_{,c}}X^\nu_{,bc}\right) \ . \quad (22)$$

Comparing (22) with (20) we obtain

$$\frac{\partial\Lambda_\mu}{\partial X^\nu_{,a}} = \frac{\partial\tilde{A}^a_\nu}{\partial X^\mu} \ , \quad (23)$$

$$\epsilon^{ab}\frac{\partial\Lambda_\nu}{\partial X^\mu}X^\mu_{,b} = \frac{\partial\tilde{B}^a}{\partial X^\nu} \ . \quad (24)$$

The above equations are consistent provided that

$$\frac{\partial\Lambda_\mu}{\partial X^\nu} - \frac{\partial\Lambda_\nu}{\partial X^\mu} = F_{\mu\nu} \ , \quad (25)$$

where $F_{\mu\nu}$ is some constant antisymmetric tensor. Consequently, there exists a scalar function $\lambda(X_\mu; X_{\mu,a})$ such that

$$\Lambda_\mu = \frac{1}{2}F_{\mu\nu}X^\nu + \frac{\partial\lambda}{\partial X^\mu} \ .$$

Inserting this result in (23) and (24), after some straightforward steps we get the general form of V^a:

$$V^a = \epsilon^{ab}\partial_a\lambda + \tfrac{1}{2}F_{\mu\nu}X^\mu X^\nu_{,b} + \epsilon^{ab}A^c_\mu X^\mu_{,bc} + B^a \ , \quad (26)$$

where new arbitrary functions A^c_μ and B^a depend now only on the first derivatives of X_μ. The first term on the RHS of (26) can be omitted, as it does not contribute to $\partial_a V^a$.

The next step is to assure that the string action boundary term in (17) defined with the general functional V^a of the form (26) is reparametrization invariant. For this purpose, it is convenient to use the Noether theorem for strings with second order derivatives, namely, that the string action functional is invariant under the reparametrization transformations if and only if the Lagrangian satisfies the set of identities [22]

$$\frac{\partial\mathcal{L}_{\text{string}}}{\partial X^\mu_{,ab}}X^\mu_{,c} = 0 \ , \quad (27)$$

$$\frac{\partial\mathcal{L}_{\text{string}}}{\partial X^\mu_{,a}}X^\mu_{,b} + \frac{\partial\mathcal{L}_{\text{string}}}{\partial X^\mu_{,ad}}X^\mu_{,bd} + \frac{\partial\mathcal{L}_{\text{string}}}{\partial X^\mu_{,aa}}X^\mu_{,ab} - \mathcal{L}_{\text{string}}\delta^a_b = 0 \ , \quad (28)$$

$$\left[\frac{\partial\mathcal{L}_{\text{string}}}{\partial X^\mu} - \partial_d\left(\frac{\partial\mathcal{L}_{\text{string}}}{\partial X^\mu_{,d}}\right) + \partial_0^2\left(\frac{\partial\mathcal{L}_{\text{string}}}{\partial X^\mu_{,00}}\right) + \partial_0\partial_1\left(\frac{\partial\mathcal{L}_{\text{string}}}{\partial X^\mu_{,01}}\right) + \partial_1^2\left(\frac{\partial\mathcal{L}_{\text{string}}}{\partial X^\mu_{,11}}\right)\right]X^\mu_{,a} = 0 \ , \quad (29)$$

where fixed indices a, b, c can take values 0 or 1 while the summation over d is assumed. Substituting (26) into the above equations we end up with some final general solution for V^a, which leads to the following general form of the Lagrangian density:

$$\partial_a V^a = \tfrac{1}{2}\alpha\sqrt{-g}R + \beta\sqrt{-g}N + \mathcal{L}_{\text{ext}} \ , \quad (30)$$

where α and β are some dimensionless constants, and $\mathcal{L}_{\text{ext}}$ stands for boundary Lagrangians which include Poincaré vector or tensor constants, i.e., describe some open systems with external fields. For such Lagrangians we have infinitely many possibilities; let us only give some examples:

$$\tfrac{1}{2}\epsilon^{ab}F_{\mu\nu}X^\mu_{,a}X^\nu_{,b} \ ,$$

$$\sqrt{-g}A^\mu\Delta X_\mu \ ,$$

$$\sqrt{-g}\nabla_a\left(\frac{A^\rho\nabla^a X_\rho}{\sqrt{1 + (T_{\mu\nu}g^{bc}X^\mu_{,b}X^\nu_{,c})^2}}\right) \ , \quad \text{etc.}$$

The first term can be interpreted as the coupling of the charged string end points with the external electromagnetic field [30].

There are present only two string self-interaction terms in (30). The relevant coefficients B^a in (26) for these terms vanish, and the coefficients A^a_μ can be calculated from the equations

$$\frac{\partial A^a_\mu}{\partial X^\nu_{,b}} - \frac{\partial A^b_\nu}{\partial X^\mu_{,a}} = \frac{\alpha}{\sqrt{-g}}\epsilon^{ab}G_{\mu\nu} + \beta g^{ab}\tilde{t}_{\mu\nu} \ . \tag{31}$$

Note that the scalar density requirement on $\partial_a V^a$ does not imply that V^a behaves like a vector density under the reparametrization transformations. The considered two self-interaction terms exemplify the case.

Let us summarize the results of this section. We proved that the generic local term which can be added to any specific string action to modify edge conditions for open strings, provided that bulk equations of motion are preserved, has the form (30). We have obtained this conclusion considering only Poincaré and reparametrization invariance, and restricting ourselves to local Lagrangians with not higher than second derivatives. We did not presume that this term should be polynomial or analytical in fields and no "power-counting" arguments for renormalizability of the quantized theory were applied. Thus, our result derived from a small set of very fundamental assumptions has a general significance.

Remarkably, the only two self-interaction terms displayed on the RHS of (30) are polynomial and well known in literature. In Euclidean four-dimensional space, they are topological and related to Euler characteristics and the numbers of self-intersections of two-dimensional surfaces.

IV. MINIMAL OPEN STRING MODELS

In this section we examine the string action functional for minimal timelike surface models, defined by the Lagrangian

$$\mathcal{L}_{\text{string}} = -\gamma\sqrt{-g} - \tfrac{1}{2}\alpha\sqrt{-g}R - \beta\sqrt{-g}N \ . \tag{32}$$

The first term is the Nambu-Goto Lagrangian; γ stands for the string tension. The parameters α and β are dimensionless. Let us also introduce an angle parameter $\theta \in [-\pi, \pi]$ defined as

$$\tan\frac{\theta}{2} = \frac{\beta}{\alpha} \ . \tag{33}$$

According to the discussion in the previous section, the Lagrangian (32) defines the most general model for free open strings, in which world sheets represent minimal timelike surfaces of zero mean curvature.

Both new terms displayed on the RHS of (32) can be relevant for the definition of the hadronic string action. The first boundary term is related to Euler characteristics in its Euclidean version. The genus factors that appear in the Polyakov quantum sum over surfaces [24] can be interpreted as a result of adding such a term to the string action. On the other hand, we will show in this section that this self-interaction term acts like a "mass" term and prevents string ends from propagating with lightlike velocities. It may help to couple consistently quark masses to hadronic strings. The relevance of the second boundary term in (32) to the QCD string has been also pointed out in many papers. Polyakov [11] suggested that the inclusion of the term that weights world sheets according to the number of self-intersections could assure the existence of a "smooth" phase of surfaces. In other works [25–27], this term has been used to reproduce an effect of QCD θ vacua in string models. The exact correspondence between the moduli space of the maps associated with a surface theory and the moduli space of the instanton sector of QCD (or any other Yang-Mills theory) has been elaborated in [28]. Exact instanton solutions in the string model with the self-intersection term have been considered in [29]. Finally, in paper [27] it is argued that this term is necessary for the QCD string also with respect to having quark spins included. Below, we will see on the classical level that the Minkowski version of the self-intersection term induces the topological sectors of solutions, which could correspond to the degenerated vacuum.

Equations of motion following from (32) are the same as in the usual Nambu-Goto theory,

$$\Delta X_\mu = 0 \ , \tag{34}$$

but supplementary edge equations for string end points $\sigma = 0, \pi$ are now affected by additional terms, and have the more general form

$$\gamma\sqrt{-g}\nabla^1 X_\mu - \alpha\partial_0\left(\frac{1}{\sqrt{-g}}\nabla_0\nabla_1 X_\mu\right) - \beta\partial_0\left(\tilde{t}_{\mu\nu}\nabla^0\nabla_0 X^\nu\right) = 0 \ , \tag{35}$$

$$\frac{\alpha}{\sqrt{-g}}\nabla_0\nabla_0 X_\mu - \beta\tilde{t}_{\mu\nu}\nabla^1\nabla_0 X^\nu = 0 \ . \tag{36}$$

We will investigate the string dynamical problem given by the system of equations (34)–(36). The best way is to use the geometrical approach [31, 30], i.e., to express the content of these equations in terms of world sheet curvature coefficients. Then, the differential equations transform into algebraic ones. Equation (34) says that the mean curvature is zero at any point of the world sheet: namely,

$$g^{ab}K^i_{ab} = 0 \ . \tag{37}$$

Edge conditions (35) and (36) can be integrated with respect to world sheet time τ and, after projections onto tangent and normal planes, respectively, they yield

$$\frac{\alpha}{\sqrt{-g}}K^i_{00} + \beta\epsilon^{ij}K^{j1}_0 = 0 \ , \tag{38}$$

$$\frac{\alpha}{\sqrt{-g}}K^i_{01} - \beta\epsilon^{ij}K^{j0}_0 = w^i \ , \tag{39}$$

$$\gamma\sqrt{-g} - w^i K^i_{01} = 0 \ , \tag{40}$$

where w^i are arbitrary functions satisfying

$$D_0 w^i \equiv \partial_0 w^i - \epsilon^{ij}\omega_0 w^j = 0 \ . \tag{41}$$

Let us choose one of the string end points, specified by $\sigma = 0$ or $\sigma = \pi$. We have here seven linear algebraic equations for local values of six curvature coefficients K^i_{ab}, so we can easily find that the solution exists only if the following condition is satisfied:

$$\alpha w^i w^i = \gamma(\alpha^2 + \beta^2) \ . \tag{42}$$

From (41) follows that the expression $w^i w^i$ is time independent, which fact is compatible with the relation (42). Next, we see that the classical solutions exist only for the positive sign of α:

$$\alpha > 0 \ . \tag{43}$$

It is also interesting to note that the classical model defined by the action composed only of the Nambu-Goto and "self-interaction" terms ($\gamma, \beta \neq 0; \alpha = 0$) is inconsistent.

If the relation (42) is satisfied, then the edge values of the curvature cofficients are easily calculable from (37)–(40): namely,

$$K^i_{00} = \frac{\beta g g^{11} \epsilon^{ij} w^j}{\alpha^2 + \beta^2} \ , \tag{44}$$

$$K^i_{01} = \frac{\sqrt{-g}(\alpha w^i + \beta \sqrt{-g} g^{01} \epsilon^{ij} w^j)}{\alpha^2 + \beta^2} \ , \tag{45}$$

$$K^i_{11} = -\frac{2\alpha\sqrt{-g} g^{01} w^i + \beta[1 + (\sqrt{-g} g^{01})^2]\epsilon^{ij} w^j}{(\alpha^2 + \beta^2) g^{11}} \ . \tag{46}$$

One can verify that the formulas (44)–(46) are covariant with respect to both the world sheet reparametrization and local orthogonal rotation transformations. The scalar functions R and N take the following constant values at the boundary of the world sheet:

$$\frac{R}{2} = \frac{\gamma}{\alpha} \frac{\beta^2 - \alpha^2}{\alpha^2 + \beta^2} = -\frac{\gamma}{\alpha} \cos\theta \ ,$$

$$N = -\frac{2\beta\gamma}{\alpha^2 + \beta^2} = -\frac{\gamma}{\alpha} \sin\theta \ .$$

Using the above results one can check that the Lagrangian density (32) vanishes at the string end points, that is a general feature of bosonic open string models.

Now, let us turn into the investigation of classical solutions, satisfying equations of motion together with pertinent edge conditions. As usual we choose the conformal gauge

$$\dot{X}^2 + X'^2 = \dot{X} X' = 0 \ , \tag{47}$$

which makes Eq. (34) linear and the general solution reads

$$X_\mu(\tau, \sigma) = X_{L\mu}(\tau + \sigma) + X_{R\mu}(\tau - \sigma) \ . \tag{48}$$

Let us denote

$$\ddot{X}_L^2 = -\tfrac{1}{4} q_+^2 \ , \quad \ddot{X}_R^2 = -\tfrac{1}{4} q_-^2 \ , \quad q_\pm = q_\pm(\tau \pm \sigma) \ ,$$
$$q_\pm \geq 0 \ . \tag{49}$$

Accordingly,

$$(\ddot{X} \pm \dot{X}')^2 = -(K^i_{00} \pm K^i_{01})(K^i_{00} \pm K^i_{01}) = -q_\pm^2 \ . \tag{50}$$

One can introduce new variables (we follow here [30]):

$$\sqrt{-g} = e^{-\phi} \ , \quad \psi = \alpha_+ - \alpha_- \ ,$$
$$K^1_{00} \pm K^1_{01} = q_\pm \cos\alpha_\pm \ , \quad K^2_{00} \pm K^2_{01} = q_\pm \sin\alpha_\pm \ . \tag{51}$$

In the geometrical approach, the role of dynamical equations plays Gauss-Peterson-Codazzi-Ricci equations (see Appendix), being the embedding conditions for the world sheet embedded in enveloping Minkowski spacetime. Referring to (A2)–(A4), one can evaluate

$$e^\phi(\ddot{\phi} - \phi'') = 2e^{2\phi} q_+ q_- \cos\psi \ , \tag{52}$$

$$\dot{\alpha}_\pm + \omega_0 = \pm(\alpha'_\pm + \omega_1) \ , \tag{53}$$

$$\omega'_0 - \dot{\omega}_1 = -e^\phi q_+ q_- \sin\psi \ . \tag{54}$$

Peterson-Codazzi equations (53) allow us to eliminate torsion coefficients. Two other equations have a nice geometrical interpretation. Gauss equation (52) relates the internal curvature scalar R [LHS of (52)] to the scalar built of the external curvature coefficients [RHS of (52)]. The internal curvature scalar is built of the connections Γ^a_{bc}, introduced for the tangent local frame bundle with defined reparametrization transformations. Thus, Gauss equation (52) describes an immersion of the tangent bundle. Similarly, the LHS of Ricci equation (54) is a scalar expression built of the connections $\epsilon^{ij}\omega_a$ defined on the orthogonal local frame bundle, endowed with local SO(2) transformations. Looking at the RHS of (54) (up to a constant it is equal to the scalar N), we can the interpret the Ricci equation as the immersion of the orthogonal bundle. We see that Gauss and Ricci equations couple "internal" with "external" geometry, describing immersions of tangent and orthogonal two-dimensional local frame bundles in four-dimensional Minkowski spacetime. Remarkably, both scalars R and N constructed from disposable connections and displayed in the immersion equations have been used in (32).

After eliminating the extrinsic torsion, the Gauss and Ricci equations read

$$\ddot{\phi} - \phi'' = 2e^\phi q_+ q_- \cos\psi \ , \tag{55}$$

$$\ddot{\psi} - \psi'' = 2e^\phi q_+ q_- \sin\psi \ . \tag{56}$$

The above equations can be written as one equation on a complex function $\Phi \equiv \phi + i\psi$ (see [30]):

$$\ddot{\Phi} - \Phi'' = 2q_+ q_- e^\Phi \ . \tag{57}$$

The gauge choice (47) leaves the residual symmetry

$$\tau \pm \sigma \to h_\pm(\tau \pm \sigma) \ , \tag{58}$$

where $h_\pm$ are arbitrary monotonic functions. Taking

$$h_\pm(\tau) = \int_{\tau_0}^{\tau} d\tau' q_\pm(\tau') \ ,$$

($h_\pm$ are monotonical due to $q_\pm \geq 0$), Eq. (57) rewritten in the new variables (58) takes the standard form of the Liouville equation:

$$\ddot{\Phi} - \Phi'' = 2e^{\Phi} . \tag{59}$$

As has been proved, the classical Nambu-Goto dynamics (minimal surface problem) reduces to a complex Liouville equation (59). The functions $q_\pm$ are arbitrary and their choice saturates the gauge freedom associated with the reparametrization invariance. Unlike other gauge theories, in the minimal string model the gauge can be completely fixed without breaking the Lorentz invariance. Obviously, the simplest gauge choice complementary to (47) is

$$q_\pm = 1 . \tag{60}$$

Later, we will show that this gauge choice is also allowed when edge conditions for world sheets with boundaries are taken into account. Here, let us note that if we restrict ourselves to reparametrization transformations which preserve world sheet boundaries (8), which means that

$$h_+(\tau) = h_-(\tau) = h_+(\tau - 2\pi) + 2\pi ,$$

then the gauge choice (60) is possible provided that

$$q_+(\tau) = q_-(\tau) = q_+(\tau + 2\pi) . \tag{61}$$

Assuming (60), the general solution of (57) reads

$$\Phi = \ln\left(\frac{-4f'(\tau+\sigma)g'(\tau-\sigma)}{[f(\tau+\sigma) - g(\tau-\sigma)]^2}\right) , \tag{62}$$

where f and g are arbitrary complex functions (not necessarily single valued, only Φ should be single valued). The function Φ is left invariant when f and g are changed by modular transformation:

$$f \to \frac{af+b}{cf+d} , \quad g \to \frac{ag+b}{cg+d} , \quad ad - bc = 1 . \tag{63}$$

It is helpful to know how to translate a given solution Φ of the Liouville equation into the explicit radius-vector representation of the string world sheet $X_\mu(\tau,\sigma)$. In order to achieve it we need to integrate the Gauss-Weingarten equation (A1). For this purpose, it is convenient to introduce the reference system composed of two real k_μ, l_μ and one complex a_μ null vectors:

$$k^2 = l^2 = a^2 = ka = la = 0 , \quad kl = -a\bar{a} = 2 . \tag{64}$$

As a result of the integration of the Gauss-Weingarten equations we obtain (function arguments are omitted)

$$\dot{X}_{L\mu} = \frac{1}{4|f'|}(|f|^2 k_\mu - f a_\mu - \bar{f}\bar{a}_\mu + l_\mu) , \tag{65}$$

$$\dot{X}_{R\mu} = \frac{1}{4|g'|}(|g|^2 k_\mu - g a_\mu - \bar{g}\bar{a}_\mu + l_\mu) . \tag{66}$$

In particular, we can choose

$$k_\mu = (1,0,0,1) , \; l_\mu = (1,0,0,-1) , \; a_\mu = (0,1,i,0) .$$

Then,

$$\dot{X}_L^\mu = \frac{1}{4|f'|}(1+|f|^2, f+\bar{f}, i(f-\bar{f}), 1-|f|^2) , \tag{67}$$

$$\dot{X}_R^\mu = \frac{1}{4|g'|}(1+|g|^2, g+\bar{g}, i(g-\bar{g}), 1-|g|^2) . \tag{68}$$

As it could be expected, the modular transformations (63) coincide with Lorentz transformations of X_μ. The integration of Gauss-Weingarten equations gives also results for n^i_μ variables: namely (here $\partial_\pm = \partial_0 \pm \partial_1$),

$$n^1_\mu + in^2_\mu = \frac{i}{e^\phi \sin\theta}\left[\partial_+\left(e^\phi \dot{X}_{L\mu}\right)e^{i\alpha_-} - \partial_-\left(e^\phi \dot{X}_{R\mu}\right)e^{i\alpha_+}\right] .$$

So far we have proved that the Nambu-Goto equations together with the complete Poincaré-invariant gauge-fixing conditions (47) and (60) are equivalent to the problem defined by a complex Liouville equation (without any additional constraints). To examine the open string case, let us proceed with the derivation of boundary conditions for the Liouville complex field Φ equivalent to edge conditions (35) and (36) for X_μ following from the Lagrangian (32). It is straightforward to convince ourselves that the edge conditions, see (44)–(46), are satisfied if and only if

$$e^{-\phi} = \sqrt{\frac{\alpha}{\gamma}}q_+ \quad \text{for } \sigma = 0,\pi , \tag{69}$$

$$\psi = \pi - \theta \bmod 2\pi \quad \text{for} \quad \sigma = 0,\pi , \tag{70}$$

$$\psi' = 0 \quad \text{for} \quad \sigma = 0,\pi , \tag{71}$$

$$q_+(\tau) = q_-(\tau) , \quad q_+(\tau+2\pi) = q_+(\tau) . \tag{72}$$

We see that the edge equations (72) are exactly the same as the conditions (61). It means that the gauge choice (60) is allowed for open strings as well.

Summarizing, the classical open string equations following from the Lagrangian (32) are equivalent [in conformal gauge (47) supplemented by complementary conditions (60)] to the complex Liouville equation

$$\ddot{\Phi} - \Phi'' = 2e^{\Phi} , \tag{73}$$

with constant Dirichlet boundary conditions for the real part $\phi = \mathrm{Re}\Phi$,

$$e^\phi = \sqrt{\frac{\gamma}{\alpha}} \quad \text{for} \quad \sigma = 0,\pi , \tag{74}$$

and periodic boundary conditions for the imaginary part $\psi = \mathrm{Im}\Phi$,

$$\psi = \pi - \theta \bmod 2\pi , \psi' = 0 \quad \text{for } \sigma = 0,\pi . \tag{75}$$

We have evaluated the general form of boundary conditions which can follow consistently from the string action for isolated open strings with no higher than second derivatives. In this paper we do not develop the thoroughgoing analysis of classical string states. We restrict ourselves to indicate that some essential differences appear while we are comparing the above-defined extended boundary problem for minimal world sheets with the ordinary Nambu-Goto case.

First, let us consider the case $\beta = 0$ ($\theta = 0$). The lowest state solution of the Nambu-Goto model that corresponds to the stationary (soliton) solution for the Liouville field represents the rotating rigid rod:

$$X_\mu = \frac{1}{\lambda^2}\left\{\lambda\tau, \cos(\lambda\tau)\sin\left[\lambda\left(\sigma - \frac{\pi}{2}\right)\right], \sin(\lambda\tau)\sin\left[\lambda\left(\sigma-\frac{\pi}{2}\right)\right], 0\right\} . \tag{76}$$

It is also a solution of our extended boundary problem (73)–(75), but now the string end points are no longer forced to travel with lightlike velocities. The parameter λ is subject to the equation (for Nambu-Goto configurations $\lambda = 1$)

$$\left[\frac{\lambda}{\cos(\lambda\pi/2)}\right]^2 = \sqrt{\frac{\gamma}{\alpha}} . \tag{77}$$

For $\beta \neq 0$ ($\theta \neq 0$), there are no solitonic solutions of the Liouville equation. The imaginary part of the Liouville field ψ, that is an angle variable [see definition (51)], cannot be trivial (i.e., to be constant everywhere on the string). The mapping $e^{i\psi} : [0,\pi] \to S^1$ provides us with some topological winding number, classifying possible solutions.

At the end of this section we will comment on the possible extension of string action (32) by adding the rigidity term [11]

$$\mathcal{L}_{\text{rig}} = \kappa\sqrt{-g}(\Delta X_\mu)^2 . \tag{78}$$

Obviously, the extended boundary problem for rigid strings is much more complicated. However, it is nice to note that all classical open string solutions defined by the system (73)–(75) are still exact solutions when the extended boundary problem is formulated with the rigidity action term taken into account. Moreover, all these solutions carry the same energy and angular momentum in both models (the rigidity term does not influence conservation laws for this class of solutions). Presumably, these are the only open rigid string solutions around which a sensible semiclassical quantization can be performed.

ACKNOWLEDGMENT

This work was supported in part by the KBN under Grant No. 2 P302 049 05.

APPENDIX

In the appendix we introduce notation and gather mathematical equations of surface theory used throughout this paper. The string world sheet is denoted by $X_\mu(\sigma^a) = X_\mu(\tau,\sigma)$ ($\sigma \in [0,\pi]$), its derivatives either by $X_{\mu,a}$ ($a = 0,1$) or by the dot and the prime for the derivatives over τ and σ parameters, respectively. The following causality conditions are imposed on the world sheet:

$$\dot{X}^2 \geq 0 , \dot{X}_0 > 0 , X'^2 \leq 0 .$$

The induced metric is $g_{ab} = X^\mu_{,a}X_{\mu,b}$, its determinant g ($g \leq 0$). Christoffel coefficients Γ^a_{bc}, covariant derivative ∇_a, and raising and lowering indices [denoted by lower case roman letters $(a,b,c,...)$] are defined with respect to the induced metric. The Riemann-Christoffel tensor R_{abcd} is also defined as usual:

$$R_{abcd} = \partial_c\Gamma_{bda} - \partial_d\Gamma_{bca} + \Gamma^e_{bc}\Gamma_{ade} - \Gamma^e_{bd}\Gamma_{ace} ,$$

and the internal curvature scalar R is introduced together with the relation

$$R^c_{acb} = \tfrac{1}{2}g_{ab}R .$$

At any point of the world sheet two orthonormal vectors n^i_μ ($i = 1,2$) can be introduced:

$$n^i_\mu X^\mu_{,a} = 0 , n^i_\mu n^{j\mu} = -\delta^{ij} ,$$

$$\frac{1}{\sqrt{-g}} \in_{\mu\nu\rho\sigma} \dot{X}^\mu X'^\nu n^{1\rho} n^{2\sigma} = +1 .$$

The last condition fixes the orientation of the local frame. There is still some arbitrariness in a choice of orthonormal vectors n^i_μ; namely, one can perform a local SO(2) rotation in a normal plane (M stands for the rotation matrix about the angle ϕ):

$$n^i_\mu \to M^{ij}(\phi)n^j_\mu , \phi = \phi(\tau,\sigma) ,$$

$$\omega_a \to \omega_a + \partial_a\phi .$$

This freedom can be considered as a local symmetry of the system described in the geometric approach. Therefore, for practical purposes it is convenient to use the "double-covariant" derivative D_a, i.e., the derivative covariant with respect to both reparametrization change and local orthogonal rotation. This derivative is defined with the help of respective connections Γ^c_{ab} and $\in^{ij}\omega_a$.

The projection operator onto the normal plane is denoted by $G_{\mu\nu}$:

$$G_{\mu\nu} = \eta_{\mu\nu} - g^{ab}X_{\mu,a}X_{\nu,b} ,$$

and antisymmetric tensor $t_{\mu\nu}$ is introduced as usual:

$$t_{\mu\nu} = \frac{1}{\sqrt{-g}} \in^{ab} X_{\mu,a}X_{\nu,b} , \quad \tilde{t}^{\mu\nu} = \tfrac{1}{2}\in^{\mu\nu\rho\sigma} t_{\rho\sigma} .$$

Let us also define the covariant tensor N_{abcd} as

$$N_{abcd} = \tilde{t}^{\mu\nu}(\nabla_a\nabla_b X_\mu)\nabla_c\nabla_d X_\nu$$

and the scalar function N together with the relation

$$N^c_{acb} = \sqrt{-g}\, \epsilon_{ab}\, N \; .$$

The external curvature K^i_{ab} and torsion ω_a coefficients are defined with Gauss-Weingarten equations (in parentheses we give their form in "double-covariant" notation):

$$X_{\mu,ab} = \Gamma^c_{ab} X_{\mu,c} + K^i_{ab} n^i_\mu \quad (D_a D^b X_\mu = K^{ib}_a n^i_\mu), \tag{A1}$$

$$\partial_a n^i_\mu = K^{ib}_a X_{\mu,b} + \epsilon^{ij} \omega_a n^j_\mu \quad (D_a D^b n^i_\mu = K^{ib}_a X_{\mu,b}) \; .$$

Instead of using radius vector coordinates we can represent the surface (up to Poincaré transformations) by induced metric and external curvature and torsion coefficients, which satisfy the identities (being the compatibility conditions for Gauss-Weingarten equations)

$$R_{abcd} = K^i_{ad} K^i_{bc} - K^i_{ac} K^i_{bd} \quad \text{(Gauss equations)} \; , \tag{A2}$$

$$\nabla_a K^i_{bc} - \nabla_b K^i_{ac} = \epsilon^{ij} (\omega_a K^j_{bc} - \omega_b K^j_{ac})$$

$$\text{(Peterson-Codazzi equations)} \; , \tag{A3}$$

$$\partial_a \omega_b - \partial_b \omega_a = \epsilon^{ij} g^{cd} K^i_{ac} K^j_{bd} \quad \text{(Ricci equations)} \; . \tag{A4}$$

All of the above equations are covariant with respect to both reparametrization change and local orthonormal rotation. The Peterson-Codazzi equations and Ricci equations in "double-covariant" notation have the form

$$D_a K^i_{bc} = D_b K^i_{ac} \; ,$$

$$[D_a, D_b] K^i_{cd} = -\sqrt{-g}\, \epsilon_{ab} \epsilon^{ij}\, N K^j_{cd} \; .$$

[1] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[2] K. Wilson, Phys. Rev. D **8**, 2445 (1974).
[3] H.B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).
[4] M. Baker, J.S. Ball, and F. Zachariasen, Phys. Rev. D **37**, 1036 (1988).
[5] D. Gross, "Some new/old approaches to QCD," Report No. PUPT 1355, 1992 (unpublished).
[6] J. Polchinski, in *Black Holes, Membranes, Wormholes and Superstrings,* Proceedings of the International Symposium, Houston, Texas, 1992, edited by S. Kalara (World Scientific, Singapore, 1992).
[7] P. Olesen, Phys. Lett. **160B**, 144 (1985).
[8] J. Polchinski and A. Strominger, Phys. Rev. Lett. **67**, 1681 (1991).
[9] F. Gliozzi, Acta Phys. Polon. B **23**, 1 (1992).
[10] D.C. Lewellen, Nucl. Phys. **B392**, 137 (1993).
[11] A. Polyakov, Nucl. Phys. **B268**, 406 (1986).
[12] Yu.A. Simonov, Phys. Lett. B **226**, 151 (1989).
[13] M. McGuigan and C.T. Thorn, Phys. Rev. Lett. **69**, 1312 (1992).
[14] W.-K. Tang, Phys. Rev. D **48**, 2019 (1993).
[15] M. Lüscher, Nucl. Phys. **B180**, 317 (1981).
[16] J.L. Cardy, Nucl. Phys. **B324**, 581 (1989).
[17] C. Itzykson, H. Saleur, and J.-B. Zuber, Europhys. Lett. **2**, 91 (1986).
[18] M.B. Green, Phys. Lett. B **266**, 325 (1991).
[19] M.B. Green, Phys. Lett. B **302**, 29 (1993).
[20] V.V. Nesterenko and Nguyen Suan Han, Int. J. Mod. Phys. A **3**, 2315 (1988).
[21] H. Arodź, A. Sitarz, and P. Węgrzyn, Acta Phys. Polon. B **22**, 495 (1991).
[22] P. Węgrzyn, Phys. Lett. B **269**, 311 (1991).
[23] B. Boisseau and P.S. Letelier, Phys. Rev. D **46**, 1721 (1992).
[24] A. Polyakov, Phys. Lett. **103B**, 207 (1981).
[25] P.O. Mazur and V.P. Nair, Nucl. Phys. **B284**, 146 (1986).
[26] R. Efraty, Phys. Lett. B **322**, 84 (1994).
[27] J. Pawełczyk, Phys. Lett. B **311**, 98 (1993).
[28] V.G.J. Rodgers, Mod. Phys. Lett. A **11**, 1001 (1992).
[29] G.D. Robertson, Phys. Lett. B **226**, 244 (1989).
[30] B. Barbashov and V.V. Nesterenko, *Introduction to the Relativistic String Theory* (World Scientific, Singapore, 1990).
[31] R. Omnes, Nucl. Phys. **B149**, 269 (1979).