Multistring solutions by soliton methods in de Sitter spacetime

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Exact solutions of the string equations of motion and constraints are systematically constructed in de Sitter spacetime using the dressing method of soliton theory. The string dynamics in de Sitter spacetime is integrable due to the associated linear system. We start from an exact string solution $q_{(0)}$ and the associated solution of the linear system $\Psi^{(0)}(\lambda)$, and we construct a new solution $\Psi(\lambda)$ differing from $\Psi^{(0)}(\lambda)$ by a rational matrix in λ with at least four poles $\lambda_0, 1/\lambda_0, \lambda_0^*, 1/\lambda_0^*$. The periodicity condition for closed strings restricts λ_0 to discrete values expressed in terms of Pythagorean numbers. Here we explicitly construct solutions depending on $(2+1)$-spacetime coordinates, two arbitrary complex numbers (the "polarization vector"), and two integers (n,m) which determine the string windings in the space. The solutions are depicted in the hyperboloid coordinates q and in comoving coordinates with the cosmic time T. Despite the fact that we have a single world sheet, our solutions describe *multiple* (here five) different and independent strings; the world sheet time τ turns out to be a multivalued function of T. (This has no analogue in flat spacetime.) One string is stable (its proper size tends to a constant for $T\to\infty$, and its comoving size contracts); the other strings are unstable (their proper sizes blow up for $T\to\infty$, while their comoving sizes tend to constants). These solutions (even the stable strings) do not oscillate in time. The interpretation of these solutions and their dynamics in terms of the sinh-Gordon model is particularly enlightening.

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I. INTRODUCTION

Since the propagation of strings in curved spacetimes started to be systematically investigated, a variety of new physical phenomena appeared [1,2]. These results are relevant both for fundamental (quantum) strings and for cosmic strings which behave essentially in a classical way.

String propagation has been investigated in nonlinear gravitational plane waves [3] and shock waves [4], black holes [5], conical spacetimes [6], and cosmological spacetimes [1,7].

Among the cosmological backgrounds, de Sitter spacetime occupies a special place. This is on one hand relevant for inflation and on the other hand string propagation turns out to be especially interesting there [1,7]. String instability, in the sense that the string's proper length grows indefinitely, is particularly present in de Sitter spacetime. The string dynamics in a de Sitter universe is described by a generalized sinh-Gordon model with a potential unbounded from below [14]. The sinh-Gordon function $\alpha(\sigma,\tau)$ having a clear physical meaning, $H^{-1}e^{\alpha(\sigma,\tau)/2}$, determines the string proper length. Moreover the classical string equations of motion (plus the string constraints) turn out to be integrable in a de Sitter universe [14,15]. More precisely, they are equivalent to a nonlinear σ model on the Grassmannian $SO(D,1)/O(D)$ with periodic boundary conditions (for closed strings). This σ model has an associated linear system [8] and, using it, one can show the presence of an infinite number of conserved quantities [13]. In addition, the string constraints imply a zero energy-momentum tensor, and these constraints are compatible with the integrability.

The so-called dressing method [8] in soliton theory allows one to construct solutions of nonlinear classically integrable models using the associated linear system. In the present paper we systematically construct string solutions in three-dimensional de Sitter spacetime. We start from a given exactly known solution of the string equations of motion and constraints in de Sitter spacetime [15] and then we "dress" it. The string solutions reported here indeed apply to cosmic strings in de Sitter spacetime as well. The dynamics of cosmic strings in expanding universes has been studied in the literature for the Friedmann-Robertson-Walker (FRW) cases (see, for example [9,10,11]). It must be noticed that the string behavior we found here in the de Sitter universe is essentially *different* from the standard FRW where the expansion factor $R(T)$ is a positive power of the cosmic time T. In such FRW universes, strings always oscillate in time; the comoving spatial string coordinates *contract* and the proper string size stays *constant* asymptotically for $T\to\infty$ [7,11]. In the cosmic string literature this is known as "string stretching." We called such behavior "stable" [7,15,11]. On the contrary, in de Sitter spacetime, as we show below, two types of asymptotic behaviors are present: (i) the proper string size and energy grow with the expansion factor ("unstable" behavior) or (ii) they tend to constant values ("stable" strings).

It is important at this point to ask: What important physics do we learn from these solutions? We think that the unstable string solutions in a de Sitter universe should provide essential clues about the feasibility of inflationary string scenarios [2,7]. These multistring solutions should play an important role in the back-reaction problem. Namely, they would become the source of the gravitational field itself through their string energy-momentum tensor. In addition, the string solutions in a de Sitter universe can be applied to study the physics of cosmic strings appearing in the inflationary stages of the Universe.

We apply here the dressing method as follows. We start from the exact ring-shaped string solution $q_{(0)}$ [15], and we find the explicit solution $\Psi^{(0)}(\lambda)$ of the associated linear system, where λ stands for the spectral parameter. Then we propose a new solution $\Psi(\lambda)$ that differs from $\Psi^{(0)}(\lambda)$ by a matrix rational in λ. Notice that $\Psi(\lambda=0)$ provides in general a new string solution. We then show that this rational matrix must have at least *four* poles $\lambda_0, 1/\lambda_0, \lambda_0^*, 1/\lambda_0^*$ as a consequence of the symmetries of the problem. The residues of these poles are shown to be one-dimensional projectors. We then prove that these projectors are formed by vectors which can also be expressed in terms of an arbitrary complex constant vector $|x_0\rangle$ and the complex parameter λ_0. This result holds for arbitrary starting solutions $q_{(0)}$.

Since we consider closed strings, we impose a 2π periodicity on the string variable σ. This restricts λ_0 to take discrete values that we achieved successfully to express in terms of Pythagorean numbers. In summary, our solutions depend on two arbitrary complex numbers contained in $|x_0\rangle$ and two integers n and m. The counting of degrees of freedom is analogous to 2+1 Minkowski spacetime except that left and right modes are here mixed up in a nonlinear and precise way.

The vector $|x_0\rangle$ show how indicates the polarization of the string. The integers (n,m) determine the string winding. They fix the way in which the string winds around the origin in the spatial dimensions (here S^2). Our starting solution $q_{(0)}(\sigma,\tau)$ is a stable string wound n^2+m^2 times around the origin in a de Sitter space.

The matrix multiplications involved in the computation of the final solution were done with the help of the computer program of symbolic calculation MATHEMATICA. The resulting solution $q(\sigma,\tau)=(q^0,q^1,q^2,q^3)$ is a complicated combination of trigonometric functions of σ and hyperbolic functions of τ. That is, these string solitonic solutions do not oscillate in time. This is a typical feature of string instability [5,7,15]. The new feature here is that strings (even stable solutions) do not oscillate either for $\tau\to 0$ or for $\tau\to\pm\infty$. Figures 3 and 4 depict spatial projections (q^1,q^2,q^3) of the solutions for two given polarizations $|x_0\rangle$ and different windings (m,n).

We plot in Figs. 5–11 the solutions for significative values of $|x_0\rangle$ and (m,n) in terms of the comoving coordinates (T,X^1,X^2):

$$T=\frac{1}{H}\ln(q^0+q^1),\quad X^1=\frac{1}{H}\frac{q^2}{q^0+q^1}\ ,\quad X^2=\frac{1}{H}\frac{q^3}{q^0+q^1}\ . \tag{1.1}$$

The first feature to point out is that our solitonic solutions describe *multiple* (here five or three) strings, as it can be seen from the fact that for a given time T we find several different values for τ. That is, τ is a *multivalued* function of T for any fixed σ (Figs. 5 and 6). Each branch of τ as a function of T corresponds to a different string. This is an entirely new feature for strings in curved spacetime, with no analog in flat spacetime where the time coordinate can always be chosen proportional to τ. In flat spacetime, multiple string solutions are described by multiple world sheets. Here, we have a *single* world sheet describing several independent and simultaneous strings as a consequence of the coupling with the spacetime geometry. Notice that we consider *free* strings. (Interactions among the strings as splitting or merging are not considered.) Five is the generic number of strings in our dressed solutions. The value five can be related to the fact that we are dressing a one-string solution $(q_{(0)})$ with *four* poles. Each pole adds here an unstable string.

In order to describe the real physical evolution, we eliminated numerically $\tau=\tau(\sigma,T)$ from the solution and expressed the spatial comoving coordinates X^1 and X^2 in terms of T and σ.

We plot $\tau(\sigma,T)$ as a function of σ for different fixed values of T in Figs. 7 and 8. It is a sinusoidal-type function. In addition to the customary closed string period 2π, another period appears which varies on τ. For small τ, $\tau=\tau(\sigma,T)$ has a convoluted shape while, for larger τ (here $\tau\le 5$), it becomes a regular sinusoid. These behaviors reflect very clearly in the evolution of the spatial coordinates and shape of the string.

The evolution of the five (and three) strings simultaneously described by our solution as a function of T, for positive T is shown in Figs. 9–11. One string is stable (the fifth one). The other four are unstable. For the stable string, (X^1,X^2) contracts in time precisely as e^{-HT}, thus keeping the proper amplitude $(e^{HT}X^1,e^{HT}X^2)$ and proper size constant. For this stable string $(X^1,X^2)\le 1/H$. ($1/H$=the horizon radius.) For the other (unstable) strings, (X^1,X^2) become very rapidly constant in time, the proper size expanding as the Universe itself like e^{HT}. For these strings $(X^1,X^2)\ge 1/H$. These exact solutions display remarkably the asymptotic string behavior found in Refs. [7,14].

In terms of the sinh-Gordon description, this means that for the strings outside the horizon the sinh-Gordon function $\alpha(\sigma,\tau)$ is the same as the cosmic time T up to a function of σ. More precisely,

$$\begin{aligned}\alpha(\sigma,T) \overset{T\gg 1/H}{=} &\ 2HT(\sigma,\tau)\\ &+\ln(2H^2\{[A^1(\sigma)']^2+[A^2(\sigma)']^2\})\\ &+O(e^{-2HT})\ .\end{aligned} \tag{1.2}$$

Here $A^1(\sigma)$ and $A^2(\sigma)$ are the X^1 and X^2 coordinates outside the horizon. For $T\to+\infty$ these strings are at the absolute *minimum* $\alpha=+\infty$ of the sinh-Gordon potential with infinite size. The string inside the horizon (stable string) corresponds to the *maximum* of the potential, $\alpha=0$. $\alpha=0$ is the only value in which the string can stay without being pushed down by the potential to $\alpha=\pm\infty$, and this also explains why only one stable string appears (it is not possible to put more than one string at the maximum of the potential without falling down). These features are *generically* exhibited by our one-soliton multistring solutions, independently of the particular initial state of the string [fixed by $|x^0\rangle$ and (n,m)]. For particular values of $|x^0\rangle$, the solution describes three strings, with symmetric shapes from $T=0$, for instance like a rosette or a circle with festoons (Figs. 9–11).

The string solutions presented here trivially embed on D-dimensional de Sitter spacetime ($D\geq 3$). It must be noticed that they exhibit the essential physics of strings in a D-dimensional de Sitter universe. Moreover, the construction method used here works in any number of dimensions.

This paper is organized as follows. In Sec. II we describe the string equations in de Sitter universe and its associated linear system. Section III deals with the dressing method in soliton theory, its application to this string problem, and the systematic construction of solutions. In Sec. IV we explicitly describe the starting background solution $q_{(0)}(\sigma,\tau)$ and the solution $\Psi^{(0)}(\lambda)$ of the associated linear system. In Sec. V, we analyze our multistring solutions and describe their physical properties.

II. THE STRING EQUATIONS AND THEIR ASSOCIATED LINEAR SYSTEM

We will consider a string propagating in D-dimensional de Sitter spacetime. In the conformal gauge, the string action is given by

$$S=\frac{1}{2\pi\alpha'}\int d\sigma\, d\tau[\langle \partial_\mu qJ\partial^\mu q\rangle+\lambda(\sigma,\tau)(\langle qJq\rangle-1)] \,. \tag{2.1}$$

Here $q\rangle$ is a $(D+1)$-dimensional real vector and

$$J=\mathrm{diag}(-1,1,\ldots,1) \,, \tag{2.2}$$

$\lambda(\sigma,\tau)$ is a Lagrange multiplier that enforces the constraint

$$\langle qJq\rangle=1 \,, \tag{2.3}$$

and (σ,τ) parametrizes the string world sheet, as usual. Extremizing the action Eq. (2.1), and eliminating the Lagrange multiplier, we find the equations of motion

$$q_{\xi\eta}\rangle+q\rangle\langle q_\xi Jq_\eta\rangle=0 \,, \tag{2.4}$$

and ξ,η are light cone coordinates in the world sheet:

$$\tau=\eta+\xi, \quad \sigma=\eta-\xi \,. \tag{2.5}$$

We use the notation

$$q_\xi\equiv\frac{\partial q}{\partial\xi}, \quad q_\eta\equiv\frac{\partial q}{\partial\eta} \,. \tag{2.6}$$

In addition, we have the string constraints (conformal conditions)

$$\langle q_\xi Jq_\xi\rangle=0, \quad \langle q_n Jq_\eta\rangle=0 \,. \tag{2.7}$$

The solution $q\rangle$ should be a periodic function of $\sigma=\eta-\xi$, with period 2π for closed strings.

We are going to find solutions of this equation by using the Riemann transform method [8,16]. The most important observation is that Eq. (2.4) can be rewritten in the form of a chiral field model on the Grassmanian $G_D=\mathrm{SO}(D,1)/\mathrm{O}(D)$. Indeed, any element $\mathbf{g}\in G_D$ can be parametrized with a real vector $q\rangle$ of the unit pseudolength

$$\mathbf{g}=1-2q\rangle\langle qJ, \quad \langle qJq\rangle=1 \,. \tag{2.8}$$

In terms of $\mathbf{g}$, the string equations (2.4)–(2.7) have the form

$$2\mathbf{g}_{\xi\eta}-\mathbf{g}_\xi\mathbf{g}\mathbf{g}_\eta-\mathbf{g}_\eta\mathbf{g}\mathbf{g}_\xi=0 \,, \tag{2.9}$$

and the conformal constraints are

$$\mathrm{tr}\mathbf{g}_\xi^2=0, \quad \mathrm{tr}\mathbf{g}_\eta^2=0 \,, \tag{2.10}$$

which are equivalent to Eqs. (2.7). The fact that $\mathbf{g}\in G_D$ implies that $\mathbf{g}$ is a real matrix with the properties

$$\mathbf{g}=J\mathbf{g}^tJ, \quad \mathbf{g}^2=I, \quad \mathrm{tr}\mathbf{g}=2, \quad \mathbf{g}\in\mathrm{SL}(D+1,R) \,. \tag{2.11}$$

These conditions are equivalent to the existence of the representation (2.8).

Equation (2.9) is the compatibility condition for the overdetermined linear system

$$\Psi_\xi=\frac{U}{1-\lambda}\Psi, \quad \Psi_\eta=\frac{V}{1+\lambda}\Psi \,, \tag{2.12}$$

where

$$U=\mathbf{g}_\xi\mathbf{g}, \quad V=\mathbf{g}_\eta\mathbf{g} \,. \tag{2.13}$$

Or in terms of the vector $q\rangle$,

$$U=2q_\xi\rangle\langle qJ-2q\rangle\langle q_\xi J \,,$$

$$V=2q_\eta\rangle\langle qJ-2q\rangle\langle q_\eta J \,.$$

The use of overdetermined linear systems to solve nonlinear partial differential equations associated with them goes back to Ref. [17]. (See Refs. [18] and [19] for further references.)

In order to fix the freedom in the definition of Ψ we shall identify

$$\Psi(\lambda=0)=\mathbf{g} \,. \tag{2.14}$$

This condition is compatible with the above equations since the matrix function Ψ at the point $\lambda=0$ satisfies the same equations as $\mathbf{g}$. Thus the problem of constructing exact solutions of the string equations is reduced to finding compatible solutions of the linear equations (2.12) such that $\mathbf{g}=\Psi(\lambda=0)$ satisfies the constraints Eqs. (2.10) and (2.11).

III. THE DRESSING METHOD IN SOLITON THEORY

A. The reduction group of the associated linear system

We will consider now the symmetry group (or the so-called "reduction group" [8,16]) enjoyed by the linear system of equations

$$\Psi_\xi=\frac{U}{1-\lambda}\Psi, \quad \Psi_\eta=\frac{V}{1+\lambda}\Psi , \tag{3.1}$$

when Eqs. (2.11) hold.

It follows from the condition $\langle qJq\rangle=1$ that the matrix $\mathbf{g}=I-2q\rangle\langle qJ$ anticommutes with U and V:

$$\mathbf{g}U+U\mathbf{g}=0, \quad \mathbf{g}V+V\mathbf{g}=0 .$$

This implies that the matrix function $\mathbf{g}\Psi(1/\lambda)$ satisfies the same equation as $\Psi(\lambda)$:

$$[g\Psi(1/\lambda)]_\xi=\frac{U}{1-\lambda}[g\Psi(1/\lambda)] ,$$
$$[g\Psi(1/\lambda)]_\eta=\frac{V}{1+\lambda}[g\Psi(1/\lambda)] . \tag{3.2}$$

Then, it can differ from $\Psi(\lambda)$ only on a matrix multiplier which does not depend on ξ,η:

$$g\Psi(1/\lambda)=\Psi(\lambda)\delta_1(\lambda) . \tag{3.3}$$

The vector $q\rangle$, the corresponding matrix $\mathbf{g}$, and the currents U,V are real. Therefore $\Psi^*(\lambda^*)$ is a solution of Eqs. (2.12) as well, and we have

$$\Psi^*(\lambda^*)=\Psi(\lambda)\delta_2(\lambda) . \tag{3.4}$$

In addition, by using Eq. (3.4) twice, we find

$$\delta_2(\lambda)\delta_2^*(\lambda^*)=I . \tag{3.5}$$

The fact that $\mathbf{g}\in SO(3,1)$ yields $JU^TJ=-U$, $JV^TJ=-V$ and implies that $[J\Psi^t(\lambda)J]^{-1}$ obeys the same equation (2.12) as $\Psi(\lambda)$:

$$[J\Psi^t(\lambda)J]^{-1}=\Psi(\lambda)\delta_3(\lambda) . \tag{3.6}$$

The transformations (3.3), (3.4), and (3.6) generate a finite group which is called the reduction group of the problem and which guarantees that the properties (2.11) hold for $\mathbf{g}=\Psi(\lambda=0)$.

B. Rational dressing

Suppose we know a particular solution $\mathbf{g}_{(0)}(\eta,\xi)$ of the string equations (2.4). We shall denote by $U_{(0)}(\eta,\xi)$, $V_{(0)}(\eta,\xi)$ its corresponding currents (2.13), and by $\Psi^{(0)}(\lambda,\eta,\xi)$ the corresponding compatible solution of the overdetermined system (2.12). We assume that $\Psi^{(0)}$ as well as $U_{(0)}$ and $V_{(0)}$ are explicitly known.

To construct a new solution $\mathbf{g}$ we assume that the corresponding Ψ function differs from $\Psi^{(0)}$ on a rational matrix multiplier $\Phi(\lambda,\eta,\xi)$:

$$\Psi(\lambda)=\Phi(\lambda)\Psi^{(0)}(\lambda) . \tag{3.7}$$

We assume that $\Phi(\lambda)$ is rational in λ, but, of course it might have a complex dependence in ξ,η. The dressing method consists of finding a matrix $\Phi(\lambda)$ such that $\Psi(\lambda)$ given by Eqs. (3.7) satisfies the linear system (3.1) and the symmetry conditions (3.3)–(3.6). Then, once $\Phi(\lambda)$ is known, the string solution $\mathbf{g}(\eta,\xi)$ follows from Eq. (2.14).

It follows from (2.14), (3.3), and (3.7) that Φ should obey the symmetries

$$\Phi(0)\Psi^{(0)}(0)\Phi(1/\lambda)=\Phi(\lambda)\Psi^{(0)}(0) , \tag{3.8}$$

$$\Phi^*(\lambda^*)=\Phi(\lambda) , \tag{3.9}$$

$$J\Phi^t(\lambda)J=\Phi^{-1}(\lambda) . \tag{3.10}$$

We assume that the constant matrices $\delta_1(\lambda)$, $\delta_2(\lambda)$, and $\delta_3(\lambda)$ coincide for the dressed and undressed solutions.

Suppose that the rational function $\Phi(\lambda)$ has a pole at the point λ_0. It follows from Eqs. (3.8) and (3.9) that it must have poles at the points $1/\lambda_0,\lambda_0^*,1/\lambda_0^*$ as well and, in addition, $\Phi(\infty)=I$. Thus, the simplest (generic) possible case is

$$\Phi(\lambda)=I+\frac{A}{\lambda-\lambda_0}+\frac{A^*}{\lambda-\lambda_0^*}+\frac{B}{\lambda-\lambda_0^{-1}}+\frac{B^*}{\lambda-\lambda_0^{*-1}} , \tag{3.11}$$

where A,B are matrix functions of (ξ,η) to be determined below. This simplest case will be called the *one-soliton solution* from now on. We choose this name since in the context of nonlinear integrable equations in an infinite space interval (the sine-Gordon equation, for instance), this minimal pole structure (the minimal number of poles compatible with the symmetry group) in λ generates the *one-soliton* solution (see for example [18]).

Here we have taken into account Eqs. (3.9) and (3.8). It follows from Eq. (3.10) that

$$\Phi^{-1}(\lambda)=I+\frac{JA^tJ}{\lambda-\lambda_0}+\frac{JA^\dagger J}{\lambda-\lambda_0^*}+\frac{JB^tJ}{\lambda-\lambda_0^{-1}}+\frac{JB^\dagger J}{\lambda-\lambda_0^{*-1}} , \tag{3.12}$$

where the dagger denotes Hermitian conjugation of a matrix. The condition $\Omega(\lambda)=\Phi(\lambda)\Phi^{-1}(\lambda)=I$ can be imposed in the following way: the right hand side $[\Omega(\lambda)]$ is a rational function of λ which takes the value I at the point $\lambda=\infty$, then $\Omega(\lambda)$ will be identically I if it does not have any singularity on the Riemann sphere of λ. Double poles would vanish, if and only if,

$$AJA^t=0, \quad BJB^t=0 . \tag{3.13}$$

Thus the matrices A,B are degenerated and we can write them as a sum of bivectors

$$A=\sum_i a_i\rangle\langle x_iJ, \quad B=\sum_i b_i\rangle\langle y_iJ . \tag{3.14}$$

The constraints (3.13) imply

$$\langle x_i|J|x_j\rangle=0 \quad \text{for all pairs } i,j , \tag{3.15}$$

which means that the vectors $x_i\rangle$ are null and mutually pseudo-orthogonal. Therefore, since pseudo-orthogonal null vectors are proportional, we have

$$x_i\rangle=c_ix\rangle$$

and without loss of generality, we take

$$A=a\rangle\langle xJ, \quad B=b\rangle\langle yJ . \tag{3.16}$$

Now the constraints (3.13) read

$$\langle xJx\rangle=0, \quad \langle yJy\rangle=0 . \tag{3.17}$$

In addition, by requiring the residues of $\Omega(\lambda)$ to vanish at the points λ_0, $1/\lambda_0$, λ_0^*, $1/\lambda_0^*$, we get

$$AJ+JA^t+AJ\left[\frac{B^t}{\lambda_0-\lambda_0^{-1}}+\frac{A^\dagger}{\lambda_0-\lambda_0^*}+\frac{B^\dagger}{\lambda_0-\lambda_0^{*-1}}\right]+\left[\frac{BJ}{\lambda_0-\lambda_0^{-1}}+\frac{A^*J}{\lambda_0-\lambda_0^*}+\frac{B^*J}{\lambda_0-\lambda_0^{*-1}}\right]A^t=0\ ,$$

$$BJ+JB^t+BJ\left[-\frac{A^t}{\lambda_0-\lambda_0^{-1}}+\frac{B^\dagger}{\lambda_0^{-1}-\lambda_0^{*-1}}+\frac{A^\dagger}{\lambda_0-\lambda_0^{*-1}}\right]+\left[-\frac{AJ}{\lambda_0-\lambda_0^{-1}}+\frac{B^*J}{\lambda_0^{-1}-\lambda_0^{*-1}}+\frac{A^*J}{\lambda_0-\lambda_0^{*-1}}\right]B^t=0\ ,$$

$$A^*J+JA^\dagger+A^*J\left[\frac{B^\dagger}{\lambda_0^*-\lambda_0^{*-1}}+\frac{A^t}{\lambda_0^*-\lambda_0}+\frac{B^t}{\lambda_0^*-\lambda_0^{-1}}\right]+\left[\frac{B^*J}{\lambda_0^*-\lambda_0^{*-1}}+\frac{AJ}{\lambda_0^*-\lambda_0}+\frac{BJ}{\lambda_0^*-\lambda_0^{-1}}\right]A^\dagger=0\ ,$$

$$B^*J+JB^\dagger+B^*J\left[-\frac{A^\dagger}{\lambda_0^*-\lambda_0^{*-1}}+\frac{B^t}{\lambda_0^{*-1}-\lambda_0^{-1}}+\frac{A^t}{\lambda_0^*-\lambda_0^{-1}}\right]+\left[-\frac{A^*J}{\lambda_0^*-\lambda_0^{*-1}}+\frac{BJ}{\lambda_0^{*-1}-\lambda_0^{-1}}+\frac{AJ}{\lambda_0^*-\lambda_0^{-1}}\right]B^\dagger=0\ .$$

Later on we shall demonstrate that the periodicity condition on σ can be satisfied only in the case where all poles $(\lambda_0,\lambda_0^*,\lambda_0^{-1},\lambda_0^{*-1})$ of $\Phi(\lambda)$ are purely imaginary [see Eqs. (4.9), (4.18), and (4.20)]. From now on, we shall denote $\lambda_0=i\kappa$, $\kappa\in R$. Substituting the bivectorial representation (3.16) in the above equations and by separating bivectors we get the system of vector equations with respect to $a\rangle,b\rangle$:

$$b\rangle\frac{i\langle xJy\rangle}{\kappa+\kappa^{-1}}+a^*\rangle\frac{i\langle x^*Jx\rangle}{2\kappa}+b^*\rangle\frac{i\langle y^*Jx\rangle}{\kappa-\kappa^{-1}}=x\rangle\ ,$$

$$b^*\rangle\frac{i\langle x^*Jy^*\rangle}{\kappa+\kappa^{-1}}+a\rangle\frac{i\langle x^*Jx\rangle}{2\kappa}+b\rangle\frac{i\langle x^*Jy\rangle}{\kappa-\kappa^{-1}}=-x^*\rangle\ ,$$

$$-a\rangle\frac{i\langle xJy\rangle}{\kappa+\kappa^{-1}}-b^*\rangle\frac{i\langle y^*Jy\rangle}{2\kappa^{-1}}+a^*\rangle\frac{i\langle x^*Jy\rangle}{\kappa-\kappa^{-1}}=y\rangle\ ,$$

$$-a^*\rangle\frac{i\langle x^*Jy^*\rangle}{\kappa+\kappa^{-1}}-b\rangle\frac{i\langle y^*Jy\rangle}{2\kappa^{-1}}+a\rangle\frac{i\langle y^*Jx\rangle}{\kappa-\kappa^{-1}}=-y^*\rangle\ .$$

By bivectorial separations we mean the following trick: suppose we have an equation of the form $p\rangle\langle X+X\rangle\langle p=0$ where $p\rangle,X\rangle$ are vectors and $p\rangle\neq 0$. Then the only solution of this equation is $X\rangle=0$ and the matrix equation is reduced to a vector one.

One can solve this system of linear equations and express the vectors $a\rangle,b\rangle$ in terms of $x\rangle,y\rangle$:

$$a\rangle=\frac{2i\kappa(\kappa^4-1)}{\delta}[(\kappa^4-1)x^*\rangle\langle y^*Jy\rangle+2(1+\kappa^2)y^*\rangle\langle x^*Jy\rangle+2(1-\kappa^2)y\rangle\langle x^*Jy^*\rangle]\ , \quad (3.18)$$

$$b\rangle=\frac{2i(\kappa^4-1)}{\kappa\delta}[(\kappa^4-1)y^*\rangle\langle x^*Jx\rangle+2\kappa^2(1+\kappa^2)x^*\rangle\langle y^*Jx\rangle+2\kappa^2(1-\kappa^2)x\rangle\langle x^*Jy^*\rangle]\ , \quad (3.19)$$

where δ is the scalar function

$$\delta=(1-\kappa^4)^2\langle x^*Jx\rangle\langle y^*Jy\rangle+4\kappa^2(1+\kappa^2)^2\langle x^*Jy\rangle\langle y^*Jx\rangle-4\kappa^2(1-\kappa^2)^2\langle x^*Jy^*\rangle\langle xJy\rangle\ . \quad (3.20)$$

At the moment we have satisfied the reduction constraints (3.9) and (3.10) completely, but the constraint Eq. (3.8) has not yet been imposed. One can prove without loss of generality, that Eq. (3.8) is verified, when the vectors $x\rangle,y\rangle$, satisfy

$$y\rangle=\Psi^{(0)}(0)x\rangle\ . \quad (3.21)$$

Assembling Eqs. (3.16) and Eqs. (3.18)–(3.21) all together one can find Φ [Eq. (3.11)] as a function of $x\rangle,\lambda,\lambda_0$:

$$\Phi=\Phi(\lambda,\lambda_0,x\rangle)\ . \quad (3.22)$$

Now, we are interested in the value of this function at $\lambda=0$, since it gives a new solution $\mathbf{g}=\Phi(0)\mathbf{g}_{(0)}$. One can check that

$$\mathbf{g}=\mathbf{g}_{(0)}\frac{4(1-\kappa^4)}{\delta}\{(1-\kappa^4)(F\mathbf{g}_{(0)}+\mathbf{g}_{(0)}F)\langle x^*Jx\rangle+2(1+\kappa^2)(\kappa^2F-\mathbf{g}_{(0)}F\mathbf{g}_{(0)})\langle x^*J\mathbf{g}_{(0)}x\rangle-2(1-\kappa^2)\mathrm{Re}[(\kappa^2H+\mathbf{g}_{(0)}H\mathbf{g}_{(0)})\langle x^*J\mathbf{g}_{(0)}x^*\rangle]\}\ , \quad (3.23)$$

where

$$F=\mathrm{Re}(x^*\rangle\langle xJ),\quad H=x\rangle\langle xJ\ .$$

Thus, we have parametrized the new solution $\mathbf{g}$ by a real number κ and a complex vector $x\rangle$ of zero pseudolength. Since $\mathbf{g}$ satisfies the conditions (2.11), it has the form of Eq. (2.8) and the corresponding vector $q\rangle$ can be found by projecting Eq. (2.8) on an arbitrary constant vector $p\rangle$. [For instance, (1,0,0,0).] We find, in this way,

$$q\rangle=\frac{p\rangle-\mathbf{g}p\rangle}{\sqrt{2(\langle pJp\rangle-\langle pJ\mathbf{g}p\rangle)}}\ . \quad (3.24)$$

Now, to construct a solution $q\rangle$ of the string equations (2.4) and (2.7), one needs only to determine the dependence of $x\rangle$ and κ on the variables ξ,η.

C. Evolution of $x\rangle$ and κ in ξ and η

Now the problem is to find the evolution of $x\rangle$ and κ in ξ and η. It follows from Eqs. (2.12) and (3.7) that

$$\Phi_\xi+\frac{1}{1-\lambda}\Phi U_{(0)}=\frac{1}{1-\lambda}U\Phi\ , \qquad (3.25)$$

$$\Phi_\eta+\frac{1}{1+\lambda}\Phi V_{(0)}=\frac{1}{1+\lambda}V\Phi\ , \qquad (3.26)$$

where U,V are still undetermined functions of η,ξ which do not depend on λ. Let us rewrite Eqs. (3.25) and (3.26) in the form

$$\Phi\left[-\partial_\xi+\frac{U_{(0)}}{1-\lambda}\right]\Phi^{-1}=\frac{U}{1-\lambda}\ , \qquad (3.27)$$

$$\Phi\left[-\partial_\eta+\frac{V_{(0)}}{1+\lambda}\right]\Phi^{-1}=\frac{V}{1+\lambda}\ . \qquad (3.28)$$

Consider the left-hand side (LHS) of (3.27). It is a rational function of λ with a pole at $\lambda=1$ and at $\lambda\in\{\lambda_0,\lambda_0^{-1},\lambda_0^*,\lambda_0^{*-1}\}$, but the RHS has only one pole at $\lambda=1$. Thus to fit Eq. (3.27), we have to set the residues at λ_0, λ_0^{-1}, λ_0^*, and λ_0^{*-1} equal to zero. In fact, it is sufficient to require the vanishing of the residue at λ_0 only. All other residues will vanish due to the action of the reduction group. The condition

$$\mathrm{res}|_{\lambda_0}\Phi\left[-\partial_\xi+\frac{U_{(0)}}{1-\lambda}\right]\Phi^{-1}=0 \qquad (3.29)$$

yields

$$A\left[\partial_\xi-\frac{U_{(0)}}{1-\lambda_0}\right]JA^t=0 \qquad (3.30)$$

and

$$A\left[\partial_\xi-\frac{U_{(0)}}{1-\lambda_0}\right]J\left[\frac{B^t}{\lambda_0-\lambda_0^{-1}}+\frac{A^\dagger}{\lambda_0-\lambda_0^*}+\frac{B^\dagger}{\lambda_0-\lambda_0^{*-1}}\right]+\left[\frac{B}{\lambda_0-\lambda_0^{-1}}+\frac{A^*}{\lambda_0-\lambda_0^*}+\frac{B^*}{\lambda_0-\lambda_0^{*-1}}\right]\left[\partial_\xi-\frac{U_{(0)}}{1-\lambda_0}\right]JA^t=0\ . \qquad (3.31)$$

Both equations will be satisfied if

$$\left[\partial_\xi-\frac{U_{(0)}}{1-\lambda_0}\right]x\rangle=0\ . \qquad (3.32)$$

Thus, the simultaneous solution of Eqs. (3.30) and (3.31) is

$$x\rangle=\Psi^{(0)}(\eta,\xi;\lambda_0)x_0\rangle\ , \qquad (3.33)$$

where $x_0\rangle$ is any complex constant vector of zero pseudolength, and λ_0 turns out not to depend on η,ξ.

Moreover, the solution (3.33) of Eq. (3.29) is also a solution of the equation

$$\mathrm{res}|_{\lambda_0}\Phi\left[-\partial_\eta+\frac{V_{(0)}}{1+\lambda}\right]\Phi^{-1}=0\ . \qquad (3.34)$$

Finally, the η,ξ dependence of the vector $x\rangle$ is given by Eq. (3.33). Together with Eq. (3.23) it gives the one-soliton solution.

The wave function $\Psi(\eta,\xi;\lambda)$ corresponding to the one-soliton solution [let us denote it by $\Psi_1(\eta,\xi;\lambda)$] can be regarded as a function of $\lambda,\lambda_0,\Psi^{(0)}(\eta,\xi,\lambda)$, and $x_0\rangle$ [see Eqs. (3.7) and (3.22)]:

$$\Psi_1(\eta,\xi;\lambda)=\Phi(\lambda,\lambda_0,J\Psi^{(0)}(\eta,\xi;\lambda_0)Jx_0\rangle)\Psi_0(\eta,\xi;\lambda)\ . \qquad (3.35)$$

The wave function corresponding to a n-soliton solution can be obtained recursively through the relation

$$\Psi_n(\eta,\xi;\lambda)=\Phi(\lambda,\lambda_{n-1},J\Psi_{n-1}(\eta,\xi;\lambda_{n-1})Jx_{n-1}\rangle)\times\Psi_{n-1}(\eta,\xi;\lambda)\ , \qquad (3.36)$$

and the corresponding solution of the chiral model $\mathbf{g}=\Psi_n(\eta,\xi;\lambda=0)$ will satisfy all the reduction conditions Eqs. (3.3), (3.4), and (3.6).

IV. THE CHOICE OF THE BACKGROUND SOLUTION

Let us now construct explicit solutions by applying the above procedure. To begin with we shall consider a three-dimensional de Sitter spacetime ($D=3$).

As a background starting solution we choose for simplicity the solution $q_{(0)}(\sigma,\tau)$ found in Ref. [15]. This solution corresponds to the trivial $\alpha=0$ solution of the sinh-Gordon equation and it is given by

$$q_{(0)}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sinh\tau\\ \cosh\tau\\ \cos\sigma\\ \sin\sigma\end{bmatrix},\quad q_{(0)\xi}=\frac{1}{\sqrt{2}}\begin{bmatrix}\cosh\tau\\ \sinh\tau\\ \sin\sigma\\ -\cos\sigma\end{bmatrix}, \qquad (4.1)$$

$$q_{(0)\eta}=\frac{1}{\sqrt{2}}\begin{bmatrix}\cosh\tau\\ \sinh\tau\\ -\sin\sigma\\ \cos\sigma\end{bmatrix},\quad b_{(0)}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sinh\tau\\ \cosh\tau\\ -\cos\sigma\\ -\sin\sigma\end{bmatrix}. \qquad (4.2)$$

For this solution, we have

$$q_{(0)\xi\xi}=q_{(0)\eta\eta}=b_{(0)},\quad q_{(0)\xi\eta}=q_{(0)}\ ,$$
$$b_{(0)\xi}=q_{(0)\eta},\quad b_{(0)\eta}=q_{(0)\xi}\ ,$$
$$\langle q_{(0)\xi}Jq_{(0)\eta}\rangle=-1,\quad \langle q_{(0)}Jq_{(0)}\rangle=1\ ,$$
$$\langle b_{(0)}Jb_{(0)}\rangle=1,\quad \text{other } \langle\cdot J\cdot\rangle=0\ ,$$
$$U_{(0)}=2q_{(0)\xi}\rangle\langle q_{(0)}J-2q_{(0)}\rangle\langle q_{(0)\xi}J\ ,$$
$$V_{(0)}=2q_{(0)\eta}\rangle\langle q_{(0)}J-2q_{(0)}\rangle\langle q_{(0)\eta}J\ ,$$

where J is given by Eq. (2.3). Let us define

$$Q(\xi,\eta)=(q_{(0)\xi},q_{(0)\eta},q_{(0)},b_{(0)})\ , \tag{4.3}$$

then we find by direct calculation that

$$Q^TJQ=\begin{bmatrix}0&-1&0&0\\-1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$
$$U_{(0)}Q=(0,2q_{(0)},2q_{(0)\xi},0)\ ,$$
$$V_{(0)}Q=(2q_{(0)},0,2q_{(0)\eta},0)\ ,$$

and

$$Q^TJU_{(0)}Q=\begin{bmatrix}0&0&0&0\\0&0&-2&0\\0&2&0&0\\0&0&0&0\end{bmatrix},$$
$$Q^TJV_{(0)}Q=\begin{bmatrix}0&0&-2&0\\0&0&0&0\\2&0&0&0\\0&0&0&0\end{bmatrix},$$
$$Q^TJQ_\xi=\begin{bmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{bmatrix},$$
$$Q^TJQ_\eta=\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix}.$$

We have to solve two compatible equations for Ψ:

$$\Psi^{(0)}_\xi=\frac{U_{(0)}}{1-\lambda}\Psi^{(0)},\quad \Psi^{(0)}_\eta=\frac{V_{(0)}}{1+\lambda}\Psi^{(0)}\ . \tag{4.4}$$

Let us make the gauge transformation

$$\Psi^{(0)}=Q(\xi,\eta)\Xi(\xi,\eta)\ , \tag{4.5}$$

then

$$Q^TJQ_\xi\Xi+Q^TJ\Xi_\xi=\frac{Q^TJU_{(0)}Q}{1-\lambda}\Xi\ ,$$
$$Q^TJQ_\eta\Xi+Q^TJQ\Xi_\eta=\frac{Q^TJV_{(0)}Q}{1+\lambda}\Xi\ , \tag{4.6}$$

and $\Xi(\xi,\eta)$ satisfies the following equations with *constant* coefficients:

$$\Xi_\xi=\begin{bmatrix}0&0&\mu^{-2}&0\\0&0&0&-1\\0&\mu^{-2}&0&0\\-1&0&0&0\end{bmatrix}\Xi(\xi,\eta)\ , \tag{4.7}$$

$$\Xi_\eta=\begin{bmatrix}0&0&0&-1\\0&0&\mu^2&0\\\mu^2&0&0&0\\0&-1&0&0\end{bmatrix}\Xi(\xi,\eta)\ . \tag{4.8}$$

The solution will be defined on the Riemann surface Γ which covers twice the complex plane λ:

$$\mu^2=\frac{1-\lambda}{1+\lambda}\ . \tag{4.9}$$

The points $\lambda=\pm1$ are the branching points of Γ.

The fundamental set of solutions of Eq. (4.7) and (4.8) is given by

$$\exp(\mu\eta+\mu^{-1}\xi)\begin{bmatrix}-\mu^{-1}\\-\mu\\-1\\1\end{bmatrix},\quad \exp(-\mu\eta-\mu^{-1}\xi)\begin{bmatrix}\mu^{-1}\\\mu\\-1\\1\end{bmatrix}, \tag{4.10}$$

$$\exp(i\mu\eta-i\mu^{-1}\xi)\begin{bmatrix}i\mu^{-1}\\-i\mu\\1\\1\end{bmatrix},\quad \exp(-i\mu\eta+i\mu^{-1}\xi)\begin{bmatrix}-i\mu^{-1}\\i\mu\\1\\1\end{bmatrix}. \tag{4.11}$$

It will be convenient to use the following linear combinations of the above solutions:

$$\Xi(\mu,\xi,\eta)=\Lambda(\mu)\Pi(\mu,\xi,\eta)\ , \tag{4.12}$$

where

$$\Lambda(\mu)=\mathrm{diag}(\mu^{-1},\mu,1,1)\ , \tag{4.13}$$

$$\Pi(\mu,\xi,\eta)=\frac{1}{\sqrt{2}}\begin{bmatrix}\cosh\gamma&-\sinh\gamma&\sin\theta&-\cos\theta\\\cosh\gamma&-\sinh\gamma&-\sin\theta&\cos\theta\\\sinh\gamma&-\cosh\gamma&-\cos\theta&-\sin\theta\\-\sinh\gamma&\cosh\gamma&-\cos\theta&-\sin\theta\end{bmatrix}, \tag{4.14}$$

and

$$\theta=\mu\eta-\frac{1}{\mu}\xi, \quad \gamma=\mu\eta+\frac{1}{\mu}\xi . \tag{4.15}$$

Thus, $\Psi^{(0)}(\mu,\xi,\eta)$ as expressed by

$$\Psi^{(0)}(\mu,\xi,\eta)=Q(\xi,\eta)\Lambda(\mu)\Pi(\mu,\xi,\eta) \tag{4.16}$$

is the solution of the system (4.4) satisfying the constraint (2.14) for the solution $q_{(0)}$. That is

$$\Psi^{(0)}(\mu=1,\xi,\eta)=1-2|q_{(0)}\rangle\langle q_{(0)}|J .$$

We want solutions periodic in σ with period 2π. We see from Eqs. (4.12)–(4.16) that we have hyperbolic functions on the argument

$$\frac{\sigma}{2}(\mu-\mu^{-1})+\frac{\tau}{2}(\mu+\mu^{-1}) , \tag{4.17}$$

and trigonometric functions with the argument

$$\frac{\sigma}{2}(\mu+\mu^{-1})+\frac{\tau}{2}(\mu-\mu^{-1}) .$$

The solution to the σ-periodicity condition requires to have

$$\mu=\exp[i\alpha] \tag{4.18}$$

with real α and $\cos\alpha$ and $\sin\alpha$ to be rational numbers. The general solution is given by the Pythagorean numbers:

$$\cos\alpha=\frac{m^2-n^2}{m^2+n^2}, \quad \sin\alpha=-\frac{2mn}{m^2+n^2}, \quad m,n=\text{integers} . \tag{4.19}$$

That is,

$$\mu=\frac{m+in}{m-in}, \quad m,n=\text{integers} . \tag{4.20}$$

We get in this way solutions with period $2\pi(m^2+n^2)$ in σ. Upon rescaling,

$$\sigma\to\sigma(m^2+n^2), \quad \tau\to\tau(m^2+n^2) , \tag{4.21}$$

we set the period to 2π. Notice that now the background solution $q_{(0)}(\sigma,\tau)$ will be wound (m^2+n^2) times around the origin in de Sitter space.

V. THE SOLITON-STRING SOLUTIONS AND THEIR PROPERTIES

We have now all the elements to obtain the explicit expression for the solution $|q(\eta,\xi)\rangle$ of the string Eqs. (2.4)–(2.7). The explicit expression for the solution $\Psi^{(0)}(\eta,\xi;\mu_0)$ given by Eq. (4.16) can be directly obtained by computing the indicated matrix multiplication; $Q(\eta,\xi)$ is given by Eqs. (4.1)–(4.3); $\Lambda(\mu)$ and $\Pi(\mu,\xi,\eta)$ are given by Eqs. (4.14). This was done with the help of the computer program of symbolic calculation MATHEMATICA.

By projecting $\Psi^{(0)}(\eta,\xi;\mu_0)$ thus obtained on a constant and complex null vector $|x_0\rangle$, we have directly the vector $|x\rangle$. The matrix $g(\xi,\eta)$ is obtained from Eq. (3.23) also using MATHEMATICA. Finally, the explicit solution $|q(\eta,\xi)\rangle$ is obtained by inserting $g(\xi,\eta)$ in Eq. (3.24). These string solutions depend on one complex parameter μ that depends on two integers n and m [see Eq. (4.20)], and one complex null vector $x_0\rangle$, that is, three complex independent numbers. Only two independent complex components remain in fact since $g(\xi,\eta)$ is homogeneous in $x_0\rangle$. As can be seen in Eq. (3.23), the change $x_0\rangle\to cx_0\rangle$ where c is a complex number, leaves the solution invariant. The dependence on $x_0\rangle$ is precisely like what happens for strings in D-dimensional Minkowski spacetime, in which the solution depends on $2(D-2)$ complex coefficients. They account for the $(D-2)$-transverse degrees of freedom and for the two helicity modes (right and left movers). Here, we are in three spacetime dimensions, and so we obtain two complex coefficients corresponding to the transverse degrees of freedom.

It can be noticed, that the linear system (3.1) satisfied by $\Psi(\eta,\xi,\lambda)$ is invariant under conformal transformations on ξ,η. Thus the dressing transformations do not generate conformal modes but only physical (transverse) modes.

The vector $x_0\rangle$ describes the polarization of the string; the integers M,n associated with the σ periodicity, label the string modes. In Minkowski spacetime, only one integer labels the right modes and another one labels the left modes. Here, we obtain two independent integers for each mode. Notice that our modes combine left and right movers in a nonlinear and precise way.

The resulting solution $q(\sigma,\tau)=(q^0,q^1,q^2,q^3)$ is a complicated combination of trigonometric functions of σ and hyperbolic functions of τ. From Eqs. (4.17)–(4.20), we see that we have trigonometric functions on the arguments,

$$\sigma\frac{2mn}{m^2+n^2} \quad \text{and} \quad \sigma\frac{m^2-n^2}{m^2+n^2} ,$$

and hyperbolic functions on the arguments,

$$\tau\frac{2mn}{m^2+n^2} \quad \text{and} \quad \tau\frac{m^2-n^2}{m^2+n^2} . \tag{5.1}$$

That is, these string-solitonic solutions, *do not oscillate in time.* This is a typical feature of string unstability [5,7] which is present for strings in inflationary type backgrounds, i.e., accelerated expanding like de Sitter, and in black holes. The new feature here is that the string does not oscillate in time, neither for $\tau\to 0$ nor for $\tau\to\pm\infty$. It can be noticed that in decelerated expanding backgrounds, as it is the case in the standard FRW expansion, string instability does not occur and the string behavior is oscillating in τ [7]. This was recently confirmed for all values of τ, in the FRW universe, where explicit string solutions has been found [11]. The nonoscillatory behavior in time can be understood from the fact that the string motion in de Sitter spacetime reduces to a sinh-Gordon equation with negative potential [14]. In $D=3$, this is precisely the standard sinh-Gordon equation, whose potential unbounded from below (see Fig. 1) is responsible for the instability. By defining

$$e^{\alpha(\sigma,\tau)}=-q_\xi\cdot q_\eta , \tag{5.2}$$

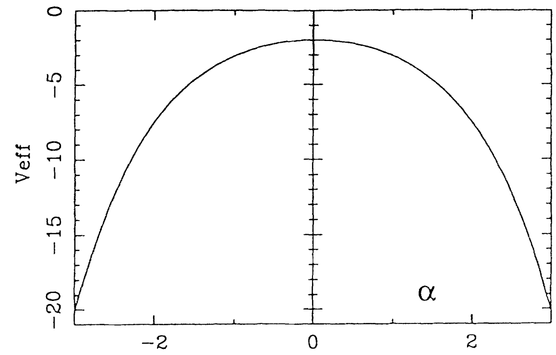

FIG. 1. Effective potential corresponding to the sinh-Gordon model.

the string equations (2.4) and string constraints (2.7) in de Sitter spacetime can be reduced to tine sinh-Gordon equation

$$\alpha_{\tau\tau}-\alpha_{\sigma\sigma}-e^{\alpha}+e^{-\alpha}=0\ . \tag{5.3}$$

Therefore, in order to find a solution in $D=3$ de Sitter spacetime, one can start from a σ-periodic solution of the sinh-Gordon equation (5.3) and insert it in the string equations (2.4):

$$[\partial_\tau^2-\partial_\sigma^2-e^{\alpha(\sigma,\tau)}]q(\sigma,\tau)=0\ . \tag{5.4}$$

Then, one must solve the *linear* equation (5.4) in $q(\sigma,\tau)$ and impose the constraints (2.3) and (2.7). This is actually an alternative method to the dressing method, to obtain string solutions in de Sitter spacetime. (See Fig. 2).

The function $e^{\alpha(\sigma,\tau)}$ has a clear physical interpretation, as it determines the proper string size. The invariant interval between two points on the string, computed with the spacetime metric, is given by

$$ds^2=\frac{1}{H^2}dq\cdot dq=\frac{1}{2H^2}e^{\alpha(\sigma,\tau)}(d\sigma^2-d\tau^2)\ . \tag{5.5}$$

The energy density of the sinh-Gordon model here,

$$\mathcal{H}=\frac{1}{2}\left[\left[\frac{\partial\alpha}{\partial\tau}\right]^2+\left[\frac{\partial\alpha}{\partial\sigma}\right]^2\right]-2\cosh\alpha(\sigma,\tau)\ , \tag{5.6}$$

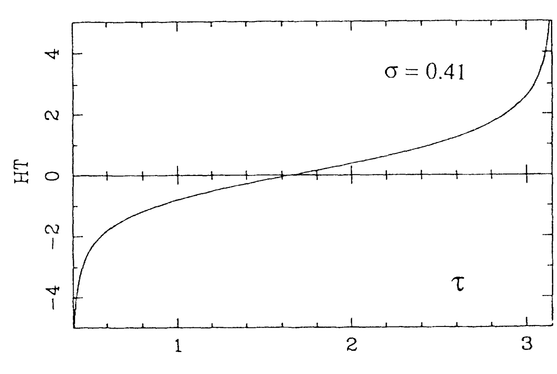

FIG. 2. Function $HT(\tau,\sigma)$, for fixed $\sigma=0.41$, for the $n=1$ string solution in $(1+1)$-dimensional de Sitter spacetime.

determines the potential

$$V_{\rm eff}=-2\cosh\alpha\ . \tag{5.7}$$

This potential has absolute minima at $\alpha=+\infty$ and $\alpha=-\infty$. As the time τ evolves, $\alpha(\sigma,\tau)$ generically approach one of these infinite minima. The first minimum corresponds to an infinitely large string whereas the second one describes a collapsed configuration. That is, the string in de Sitter spacetime will tend generically either to inflate (when $\alpha\to+\infty$) or to collapse to a point (when $\alpha\to-\infty$).

The background string solution $q_{(0)}(\sigma,\tau)$ given by Eq. (4.1) corresponds to the sinh-Gordon solution $\alpha=0$. This means a string with *finite constant* proper size (equal to $1/H$). In the sinh-Gordon model this corresponds to a particle at the maximum of the potential $V_{\rm eff}=-2$ and with zero velocity.

Let us recall that for a given time q_0, the de Sitter space is a sphere S^2 with radius $R=(1/H)\sqrt{1+q_0^2}$. For the background solution $q_{(0)}(\sigma,\tau)$ given by Eq. (4.1), we have $R(\tau)=(1/H)\sqrt{1+\frac{1}{2}\sinh^2\tau}$. As the de Sitter universe expands for $\tau\to\infty$, the string size $e^{\alpha(\sigma,\tau)}=1$ remains here constant. This solution is probably unstable under small perturbations.

It must be noticed that the integers (m,n) of the solitonic solutions (4.19) have the meaning of string winding. They label the different ways in which the string winds in the spatial compact dimensions (here S^2). Notice that our string solutions do not oscillate in time in spite of the fact that we are in a Lorentzian signature spacetime. (The dependence on τ is hyperbolic.)

In Figs. 3 and 4 we plot the one-soliton solutions $|q(\sigma,\tau)\rangle=(q^0,q^1,q^2,q^3)$ found here. They show the three-dimensional spatial projections (q^1,q^2,q^3) as a function of σ, for a given polarization vector $x^0\rangle$, different values of m,n, and two differents values of τ. Figures 3(a) and 3(b) show the same solution $[n=2,m=1,x^0\rangle=(1,-1,0.1,0.1i)]$ for two different values of τ. Figures 4(a) and 4(b) show the evolution for a higher winding number $(n=5)$, and polarization vector $x^0\rangle=(1,-1,1,i)$.

For comparison, let us recall, that in $D=2$ spacetime dimensions, in which the string motion reduces to the Liouville equation, the *exact general* solution is a string wound n times around the de Sitter space and evolving with it. The string covers n times de Sitter space which is here a circle S^1. This solution is given by [14]

$$q^0=-\cot n\tau,\quad q^1=\frac{\cos n\sigma}{\sin n\tau},\quad q^2=\frac{\sin n\sigma}{\sin n\tau}\ , \tag{5.8}$$

$$0<\sigma\le 2\pi,\quad 0<\tau\le\pi/n\ . \tag{5.9}$$

The invariant interval between two points of the string

$$ds^2=\frac{1}{H^2\sin^2 n\tau}(d\sigma^2-d\tau^2) \tag{5.10}$$

exhibits the typical feature of string instability: in the asymptotic regions $\tau\to 0^+$ and $\tau\to\pi/n$, the proper string length blows up. We also see that the string does not have "enough time" to oscillate in one expansion time of the Universe: the oscillation period of the string *coincides* with the expansion time of the Universe. When the string accomplishes one oscillation, the Universe has ended.

In order to analyze the exact solutions $|q(\sigma,\tau)\rangle$, it is convenient to use the coordinates (T,X^1,X^2) in this $(2+1)$-dimensional de Sitter spacetime:

$$q^0=\sinh HT+\frac{H^2}{2}\exp(HT)[(X^1)^2+(X^2)^2] , \quad (5.11)$$

$$q^1=\cosh HT-\frac{H^2}{2}\exp(HT)[(X^1)^2+(X^2)^2] , \quad (5.12)$$

$$q^2=H\exp(HT)X^1, \quad q^3=H\exp(HT)X^2 , \quad (5.13)$$

$$-\infty < T,X^1,X^2 < +\infty .$$

That is,

$$T=\frac{1}{H}\ln(q^0+q^1), \quad X^1=\frac{1}{H}\frac{q^2}{q^0+q^1}, \quad X^2=\frac{1}{H}\frac{q^3}{q^0+q^1} . \quad (5.14)$$

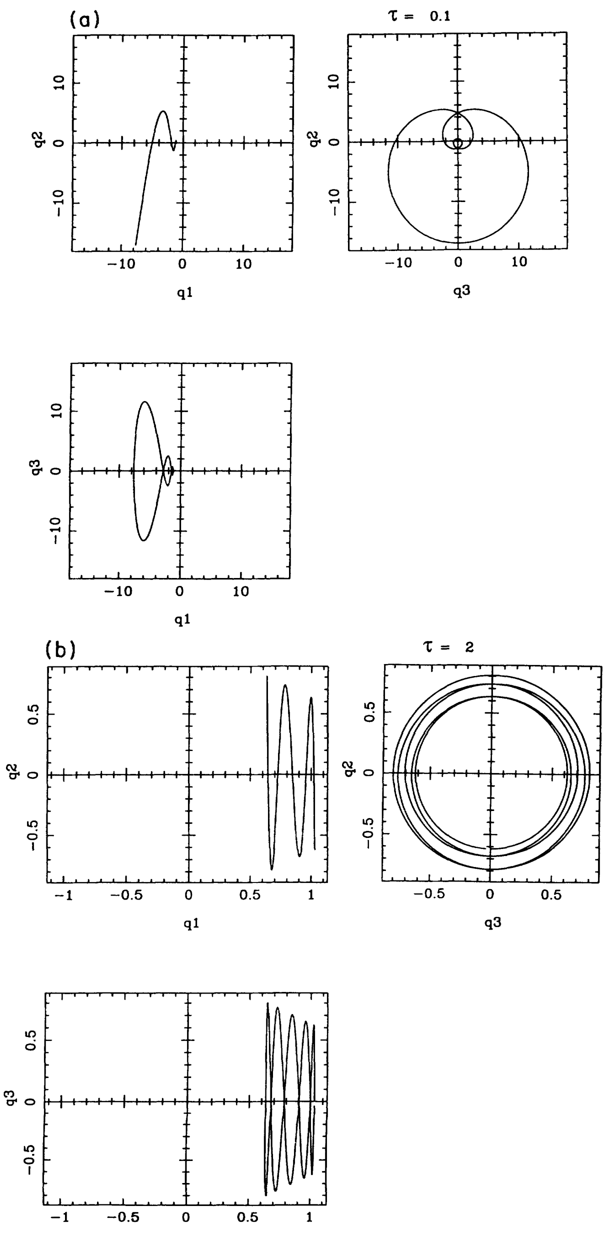

FIG. 3. Evolution of the string in (σ,τ) variables. The three projections (q^1,q^2), (q^1,q^3), and (q^2,q^3) are shown for $n=2$, $m=1$, and $\tau=0.1$ and 2 for $|x^0\rangle=(1,-1,0.1,0.1i)$.

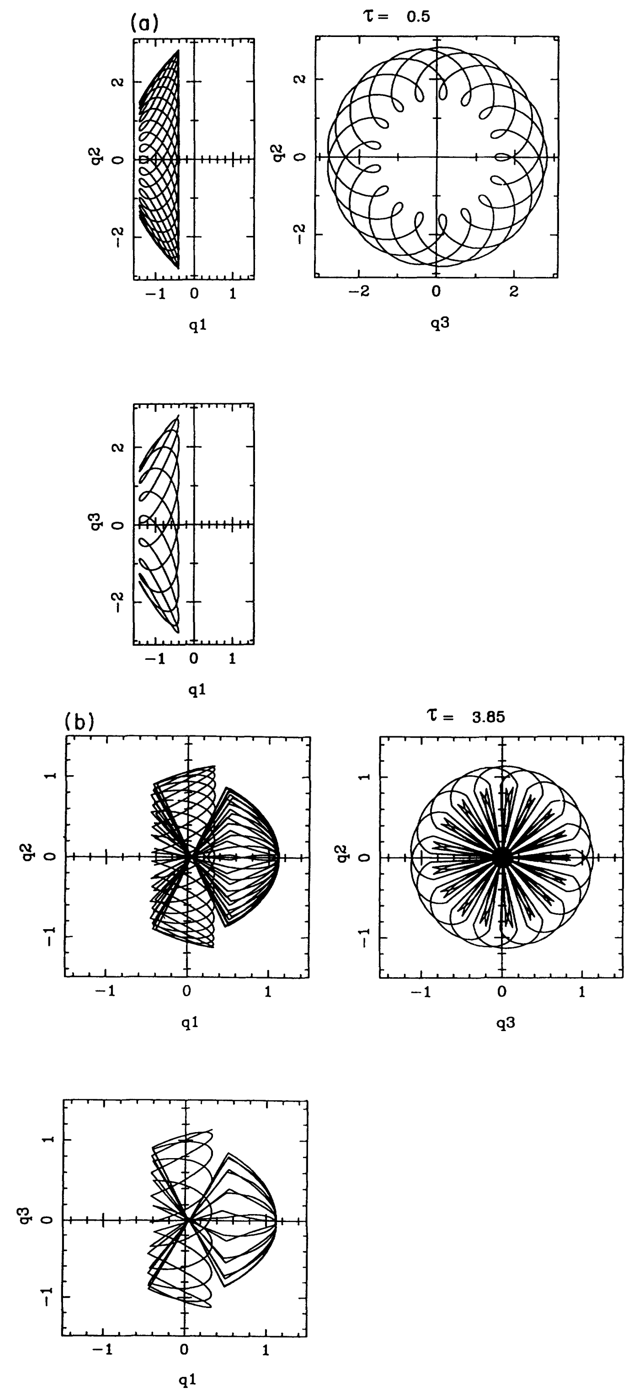

FIG. 4. Same as Fig. 3 for $n=5$, $\tau=0.5,3.85$ and $|x^0\rangle=(1,-1,i,1)$.

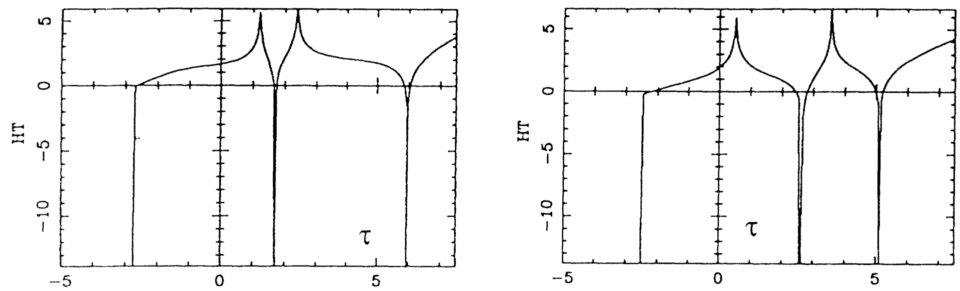

FIG. 5. Plot of the function $HT(\tau)$, for two values of σ, for $n=4, |x^0\rangle=(1+i, 0.6+0.4i, 0.3+0.5i, 0.77+0.79i)$. The function $\tau(T)$ is multivalued, revealing the presence of five strings.

The cosmic time T and the conformal time η are related by

$$\eta=-\frac{1}{H}e^{-HT}, \quad -\infty<\eta\leq 0 , \tag{5.15}$$

in terms of which the line element takes the form

$$ds^2=-dT^2+e^{2HT}[(dX^1)^2+(dX^2)^2]$$
$$=\frac{1}{H^2\eta^2}[-(d\eta)^2+(dX^1)^2+(dX^2)^2] . \tag{5.16}$$

We now analyze the properties and new features exhibited by these solutions.

First of all, let us analyze the cosmic time coordinate $T=T(\sigma,\tau)$. We have studied T as a function of τ for different fixed values of σ and vice versa. These functions have been obtained numerically for a wide family of solutions labeled by different values of the parameters n,m and $|x^0\rangle$. We report here only two significant cases, which show the *generic* features, irrespective of the particular values of these parameters.

Figure 5 shows T as a function of τ for the values of σ indicated in the picture. We depict T for $n=4$, $m=1$, and a generic $|x^0\rangle=(1+i, 0.6+0.4i, 0.3+0.5i, 0.77+0.79i)$.

In Fig. 6 we depict T as a function of τ for the solution with $n=4$, $m=1$, and $|x^0\rangle=(1,-1,i,1)$. We see that our solution in the generic case describes actually *five* strings, as it can be seen from the fact that for a given value of T we find five different values of τ. That is, τ is a *multivalued* function of T for any fixed σ. This is an entirely new feature for strings in curved spacetime. It has no analogy in flat spacetime where the time coordinate obeys $[\partial_\tau^2-\partial_\sigma^2]T=0$, and, therefore, using the conformal transformations

$$\sigma\pm\tau=f_\pm(\sigma'\pm\tau')$$

allows to choose the light-cone gauge, in which T (or a null-like combination of it) is proportional to τ. In curved spacetime, this is not possible in general. Only in some geometries like shock waves or gravitational plane waves, the light-cone choice is possible for all τ [2]. In asymptotically flat spacetimes this choice is only possible asymptotically [5]. When τ is a univalued function of the time coordinate, the solution of the string equations describes *only one* string. This is the case in geometries where the light-cone choice of gauge is possible for all τ. In spacetimes, as de Sitter, where τ is a multivalued function of the time coordinate, the solution of the string equations of motion and constraints describe a *multistring* configuration. That is, each branch of τ as a function of T corresponds to a different string.

This *multiple* number of strings arises as a consequence of the string dynamics in curved spacetimes, that is, from the *coupling* of the string with the spacetime geometry. Notice that here we have just *free* string equations of motion in curved spacetime. That is, interactions between the strings themselves, like splitting and merging, are not considered. We find that the geometry determines the simultaneous existence of several strings. They do not interact directly between them since they do not intersect. All the interaction is through the spacetime geometry. Notice that such phenomenon does not appear in $D=2$ [Eq. (5.9)], where time is a monotonic and periodic function of τ. This solution describes only one string in one period: $0<\tau\leq\pi/n$. For other periods we get identical copies of the same string. This is not the case of the $(2+1)$-dimensional solutions displayed in Figs. 5–11. They describe five or three different strings. Five is the generic number of strings in our dressed solutions. This value five can be related to the fact that we are dressing a one-string solution ($q_{(0)}$) with *four* poles. Each pole adds here an unstable string.

Figures 7 and 8 show the function $\tau=\tau(\sigma,T)$ as a func-

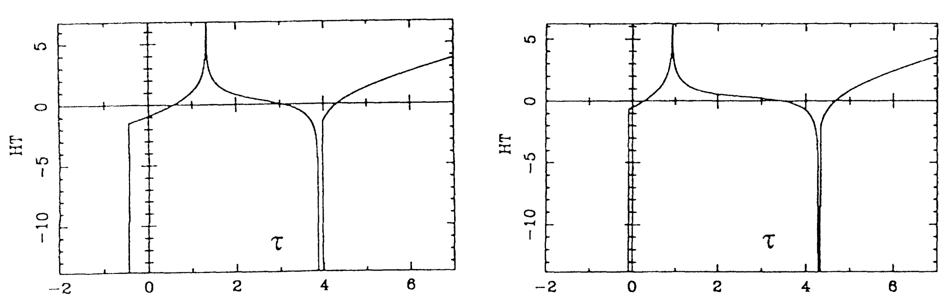

FIG. 6. Same as Fig. 5, for $n=4$, $|x^0\rangle=(1,-1,i,1)$. Because of a degeneracy, there are now only three strings.

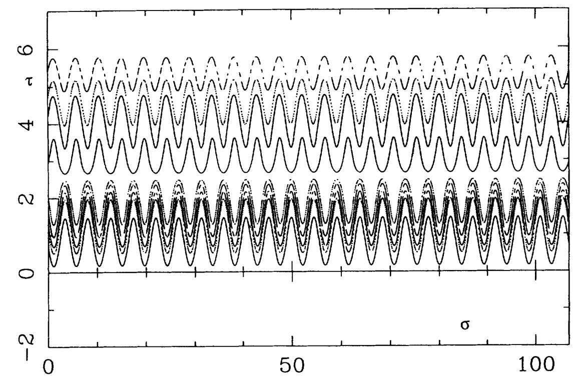

FIG. 7. $\tau=\tau(\sigma,T)$ for fixed T for $n=4$, $|x^0\rangle=(1,-1,i,1)$. Three values of HT are displayed, corresponding to $HT=0$ (full line), 1 (dots), and 2 (dashed line). For each HT, three curves are plotted, which correspond to the three strings. They are ordered with τ increasing.

tion of σ, the different values of T are indicated in the pictures, for the above solutions (i.e., polarization vectors $|x^0\rangle$ and windings (n,m) the same as above). The function $\tau=\tau(\sigma,T)$ being periodic in σ, it is plotted only for one period (2π) of σ. We see that *in addition* to the period 2π, *another* period in σ appears which depends on

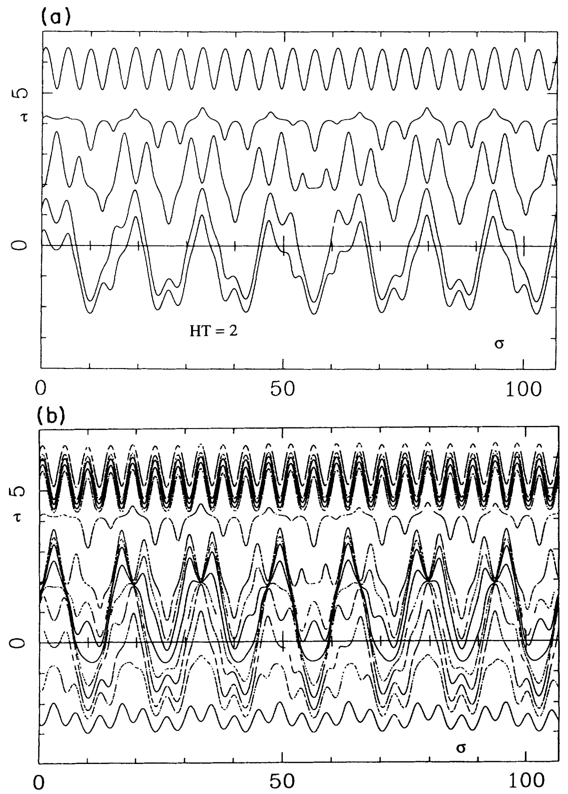

FIG. 8. Same as Fig. 7 for $n=4$, $x^0\rangle=(1+i,0.6+0.4i,0.3+0.5i,0.77+0.79i)$. (a) The five curves corresponding to the five strings at $HT=2$. (b) The five curves for three values of HT: $HT=0$ (full line), 1 (dots), and 2 (dashed line).

τ. $\tau=\tau(\sigma,T)$ is a sinusoidal-type function. It is more convoluted for small values of $|\tau|$ in the neighborhood of $\tau=0$ where several maxima and minima appear. As soon as the neighborhood of $\tau=0$ is left, $\tau=\tau(\sigma,T)$ becomes very fast a regular sinusoidal-type function of σ with a fixed period much smaller than 2π. [In all solutions studied here, $\tau=\tau(\sigma,T)$ reaches this asymptotic form for $\tau\sim5$.] The meaning of these small and large τ behaviors will become more clear in connection with the evolution of the spatial coordinates and shape of the string. The small τ behaviors are connected with the different (and complicated) ways in which the string winds at the beginning of its evolution, while the $\tau\to\infty$ uniform behavior is connected with the asymptotic configuration which is "frozen" in comoving coordinates. The large τ behavior turns to be τ-independent in comoving coordinates.

Let us analyze now the spatial coordinates $X^1(\sigma,\tau)$ and $X^2(\sigma,\tau)$ of this solution. Figures 9 and 10 show the time evolution of the *three* or *five* strings *simultaneously* described by this solution. In order to describe the real *physical* evolution, we eliminated $\tau=\tau(\sigma,T)$ from the solution and expressed $X^1(\sigma,\tau)=X^1(\sigma,T)$ and $X^2(\sigma,\tau)=X^2(\sigma,T)$ in terms of T. This was done numerically. Figure 10 shows the comoving coordinates (X^1,X^2) for different times HT. We see that for the fifth string, (X^1,X^2) collapse precisely as the inverse of the expansion factor e^{-HT}, while the other four strings keep (X^1,X^2) constant in time (in Fig. 9, it is the third string that collapses). That is, the first string keep its proper size constant while the proper size of the other four strings expand like e^{HT}. These exact solutions display remarkably the string behavior found asymptotically and approximately in Refs. [7]. In summary, when (X^1,X^2) are smaller or equal than $1/H$ (the horizon radius), they contract to a point keeping the proper amplitudes $(e^{HT}X^1,e^{HT}X^2)$ and proper size constant. When (X^1,X^2) are larger than $1/H$, they become very fast constant in time, the proper size expanding with the universe itself as e^{HT} (string instability).

In terms of the sinh-Gordon description [see Eqs. (5.2) and (5.3) and Fig. 1], this means that for strings outside the horizon, the sinh-Gordon function $\alpha(\sigma,\tau)$ for most of the history is the same as the cosmic time T up to a function of σ. We find, combining Eqs. (5.11)–(5.13) with Eq. (5.14),

$$\alpha(\sigma,\tau) \overset{T\gg1/H}{=} 2HT(\sigma,\tau) + \ln\{2H^2[(A^1(\sigma)')^2+(A^2(\sigma)')^2]\} + O(e^{-2HT}) . \quad (5.17)$$

Here $A^1(\sigma)$ and $A^2(\sigma)$ are the X^1 and X^2 coordinates outside the horizon. For $T\to\infty$ the string is at the absolute *minimum* $\alpha=+\infty$ of the sinh-Gordon potential and possess an infinite size.

The string inside the horizon corresponds to the *maximum* of the potential, $\alpha=0$. This is the stable string with contracting coordinates (X^1,X^2) and *constant* proper size, appearing in all the multistring solutions found here. The value $\alpha=0$ is the only one in which the string can stay without being pushed down by the potential to $\pm\infty$. This also explains why only one stable string appears: it is not possible to put more than one string at the maximum of the potential without falling down. The starting zero soliton solution $\alpha=0$ we have dressed is a particular and very simple stable string.

For degenerate choices of $|x^0\rangle$, the number of strings reduces to three (see Fig. 6). For large positive T two of the strings (strings 1 and 2) are of the unstable type and one (string 3) is of stable type. In addition, strings 1 and 2 become identical in the infinite T limit. In Fig. 9, we plot this solution for negative T. We see that string 2 is stable for $T\to-\infty$ (it has constant invariant size in such limit), whereas the invariant sizes of strings 1 and 3 collapse in this limit. In addition, there is an intermediate regime for $|T|\leq 3$ where the comoving size of the strings decreases by a factor of about 10.

The features above described are *generically* exhibited by our one-soliton multistring solutions independently of the particular initial state of the string. [Fixed by the values $|x^0\rangle$ and (n,m).]

It is interesting to see how the shape of the string becomes more symmetric for special values of $|x^0\rangle$. For instance, a rosette shape or a circle with many festoons are clearly shown by Figs. 9–11. They correspond to $|x^0\rangle=(1,-1,i,1)$ with $n=4,6$, respectively. These particularly symmetric vectors $|x^0\rangle$ yield also degenerate solutions, in the sense that they contain only three different strings instead of five, as it happens in the generic case.

We also see that for the symmetric initial conditions for the string state $|x^0\rangle$, the function $\tau=\tau(\sigma,T)$ becomes a perfectly symmetric periodic sinusoidal inside the period 2π, for *all* values of τ (including small $|\tau|$), and the additional very small period is practically the same for all $|\tau|$.

The number of string windings and festoons is related to the frequencies in Eq. (4.19) and expressed in terms of (n,m). The σ dependence is characterized by the fre-

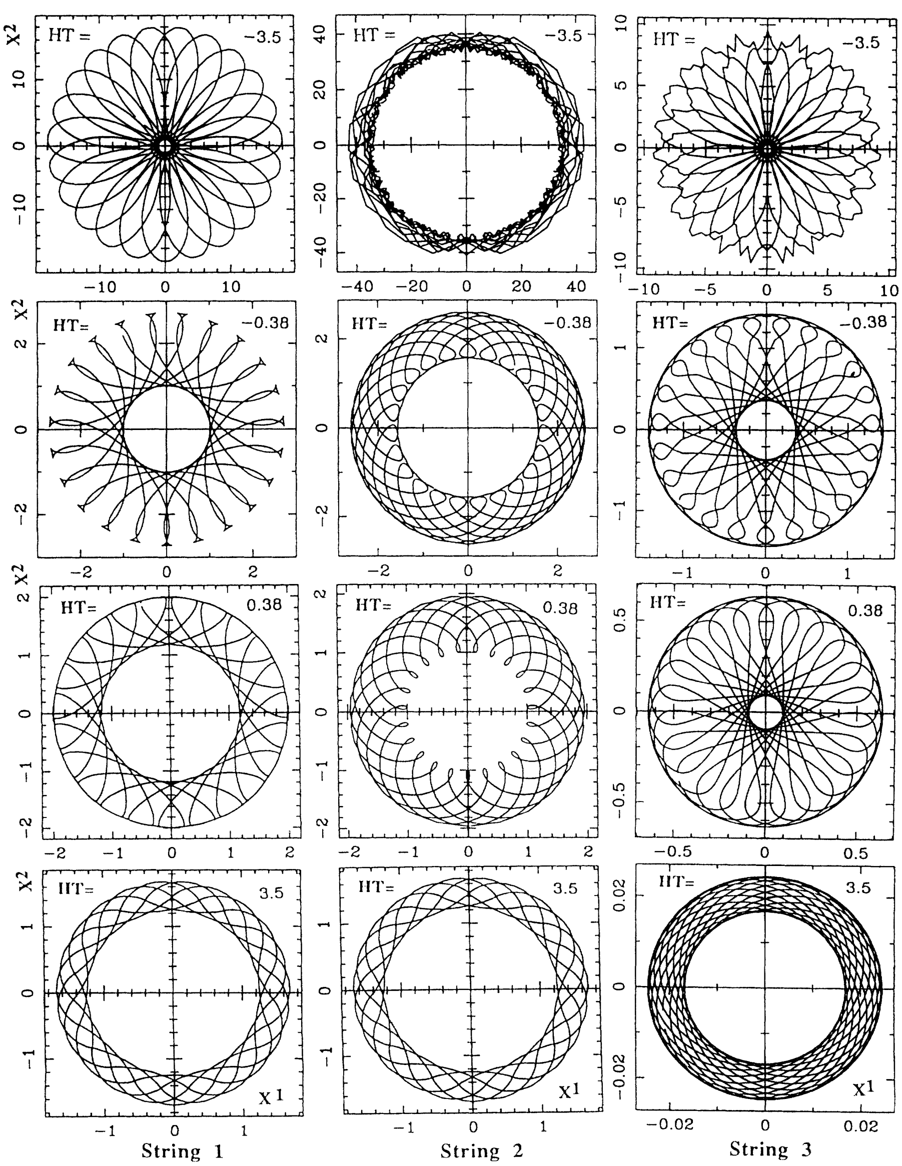

FIG. 9. Evolution as a function of cosmic time HT of the three strings, in the comoving coordinates (X^1,X^2), for $n=4$, $|x^0\rangle=(1,-1,i,1)$. The comoving size of string (1) stays constant for $HT<-3$, then decreases around $HT=0$, and stays constant again after $HT=1$. The invariant size of string (2) is constant for negative HT, then grows as the expansion factor for $HT>1$, and becomes identical to string (1). The string (3) has a constant comoving size for $HT<-3$, then collapses as e^{-HT} for positive HT.

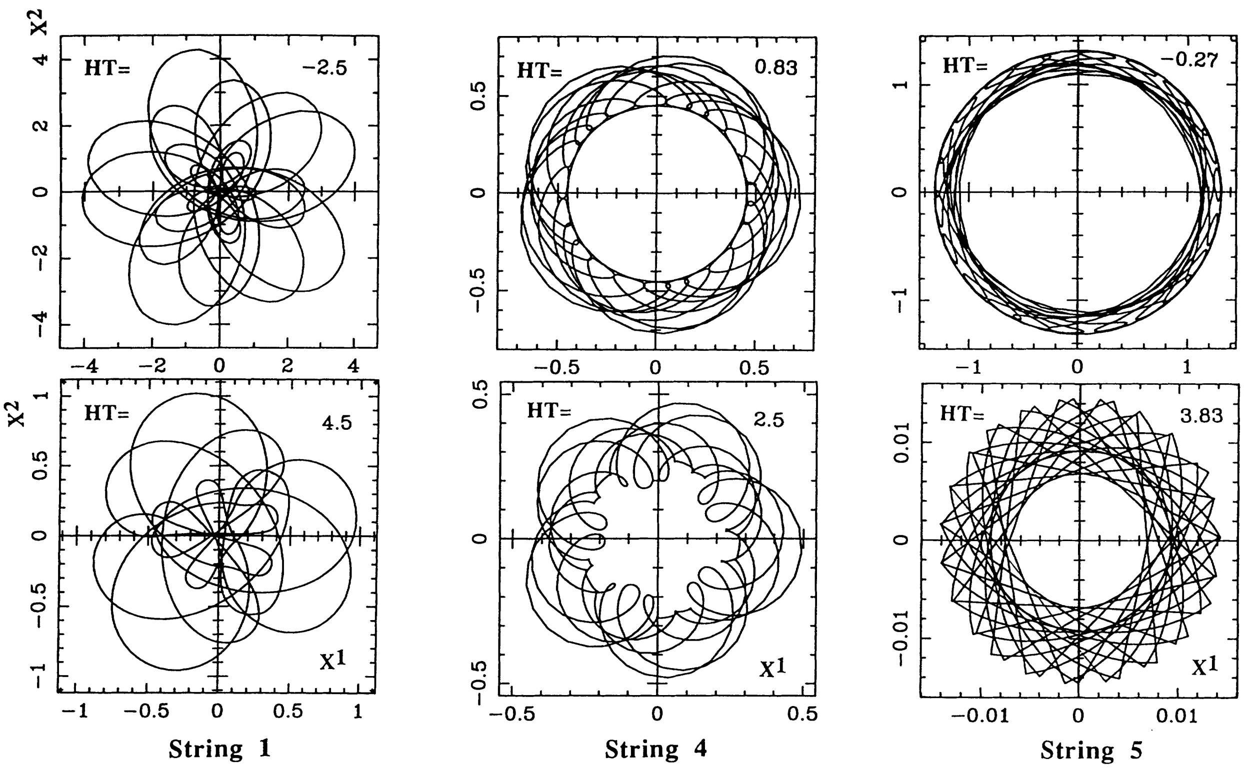

FIG. 10. Evolution of three of the five strings for $n=4$, $|x^0\rangle=(1+i,0.6+0.4i,0.3+0.5i,0.77+0.79i)$.

quencies [see Eq. (4.19)]

$$\Omega_1=\frac{2mn}{m^2+n^2},\quad \Omega_2=\frac{m^2-n^2}{m^2+n^2},$$

and the basic frequency

$$\Omega_0=\frac{1}{n^2+m^2}\quad \text{for } n^2+m^2 \text{ odd},$$

$$\Omega_0=\frac{2}{n^2+m^2}\quad \text{for } n^2+m^2 \text{ even}.$$

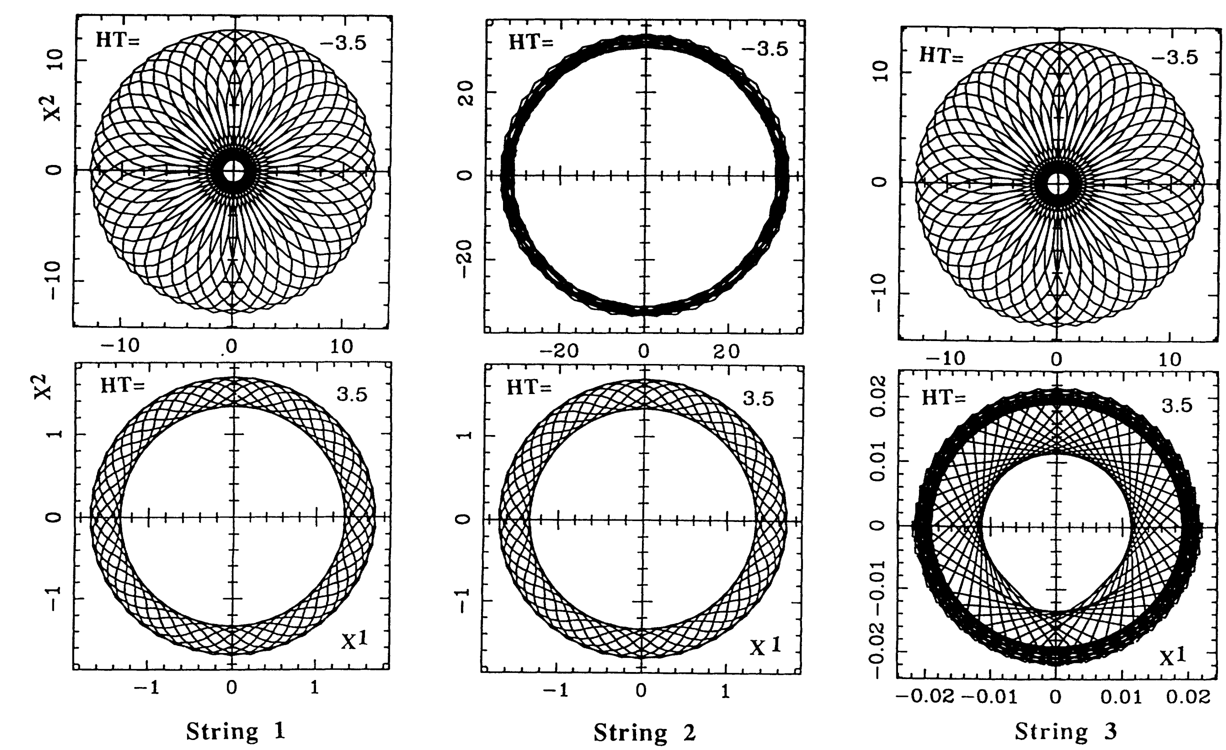

FIG. 11. Evolution of the three strings for the degenerate case $n=6$, $|x^0\rangle=(1,-1,i,1)$.

For $m=1$, the highest available frequency is the sum $\Omega_1+\Omega_2=(n^2+2n-1)/(n^2+1)$. This highest frequency determines the small period in $\tau=\tau(\sigma,T)$ as a function of σ for fixed large τ. That is,

$$2\pi\frac{n^2+1}{n^2+2n-1} .$$

In addition,

$$\frac{\Omega_1+\Omega_2}{\Omega_0}=n^2+2n-1 \quad \text{for odd } n ,$$

$$\frac{\Omega_1+\Omega_2}{\Omega_0}=(n^2+2n-1)/2 \quad \text{for even } n$$

gives the number of festoons in the strings at a given T (see Figs. 7–11).

Strings propagating in de Sitter spacetime enjoy as conserved quantities those associated with the O(3,1) rotations on the hyperboloid (2.3). They can be written as

$$L=\int_0^{2\pi}d\sigma\langle q\rangle\langle\dot q-\dot q\rangle\langle q\rangle J=\frac{1}{2}\int_0^{2\pi}d\sigma(U+V) .$$

In order to compute L it is convenient to relate U and V with $U_{(0)}$ and $V_{(0)}$ using Eqs. (3.27) and (3.28) and the asymptotic behavior of $\Phi(\lambda)$ for $\lambda\to\infty$ [see Eq. (3.11)]:

$$\Phi(\lambda)=1+\frac{C(\eta,\xi)}{\lambda}+O\left[\frac{1}{\lambda^2}\right] ,$$

where $C(\eta,\xi)$ is a matrix. We then find

$$U+V=U_{(0)}+V_{(0)}+2C(\eta,\xi)\sigma .$$

Since $C(\eta,\xi)$ is a periodic function of σ,

$$L=L_{(0)}$$

for all solutions considered here. We recall [15] that only L_{01} does not vanish for $q_{(0)}$, taking the value [see Eq. (4.21)]

$$L_{10}=-L_{01}=(n^2+m^2)\pi .$$

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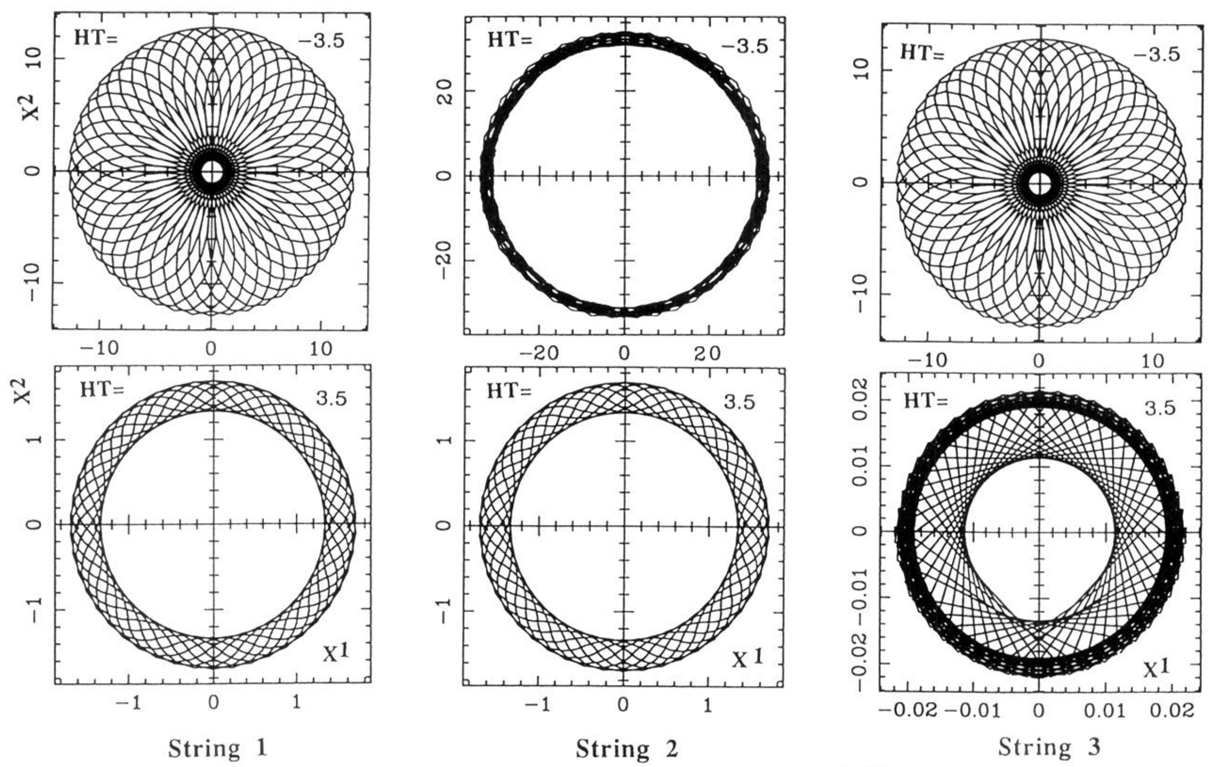

FIG. 11. Evolution of the three strings for the degenerate case $n=6$, $|x^0\rangle=(1,-1,i,1)$.