Nonperturbative isotropic multiparticle production in Yang-Mills theory

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We use singular Euclidean solutions to find multiparticle production cross sections in field theories. We investigate a family of time-dependent O(3) symmetrical solutions of the Yang-Mills equations, which govern the isotropic high-energy gauge boson production. At low energies our approach reproduces the instanton-induced cross sections. For higher energies we get new results. In particular, we show that the cross section for isotropic multiparticle production remains exponentially small in the running gauge coupling constant. The result applies both to baryon number violation in the electroweak theory and to QCD jet production. We find that the isotropic multigluon production cross section falls off approximately as a ninth power of energy but possibly might be observable.

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I. INTRODUCTION

The past few years have witnessed an increasing interest in the nonperturbative multiparticle production induced by classical solutions of the field equations. Typical examples are (i) baryon number violation in the electroweak theory (for reviews see [1–4]), (ii) multijet production in strong interactions [5, 6], and (iii) multipion production in heavy ion collisions [7–9].

While there seems to be a consensus in that classical field configurations are relevant in *many* → *many* processes (see, e.g., Refs. [10, 11]), it is not so clear in the case of the *two* → *many* ones. The difficulty with the latter processes is that an initial state with a few particles is a quantum rather than a classical one, even at asymptotically high energies. If one starts from a classical field with large energy at $t = +\infty$ and evolves it back in time according to the classical equations of motion which preserve the field energy, one would end up with another field configuration at $t = -\infty$ with the same energy. Meanwhile, at $t = -\infty$ the whole energy has to belong not to the classical field (which should be zero or, at best, contain only the wrong-frequency part not corresponding to any real particles) but to the few-particle quantum state. Therefore there cannot exist any *continuous* classical field configuration interpolating in time between $t = \pm\infty$.

A way to overcome this difficulty has been indicated by Khlebnikov [12], who, following a remarkable work of Iordanskii and Pitaevskii [13], has suggested the study of *singular* Euclidean solutions of field equations. Singular fields do not conserve energy across singularities; therefore, they are adequate to describe transitions between a quantum state with low or zero field energy and a (semi)classical one with high energy. Singular trajec-

tories in imaginary (Euclidean) time were introduced 60 years ago by Landau to calculate matrix elements between low- and high-energy states in quantum mechanics [14]. Even if a final high-energy state can be described semiclassically, the initial low-energy state needs not; therefore the Landau method gives an example how, nevertheless, the matrix elements can be treated semiclassically. It should be mentioned that an application of the Landau method to multiparticle production has been suggested by Voloshin [15]; however, only spatially constant fields have been considered in that work.

The energy of the initial state, i.e., the collision energy, is introduced as follows. Let the multiparticle production be initiated by, say, an annihilation of two high-energy particles with energies $E_{1,2}$ in the c.m. frame. To get the physical on-shell amplitude one has to calculate the two-point Green function in the singular background field, take its Fourier transform, and then apply the "leg amputation" procedure of Lehmann, Symanczik, and Zimmermann. The high-energy asymptotics of the Green function is apparently given by the Fourier transform of the singularity point [13], which is $\exp[-i(E_1 + E_2)t_{\rm sing}]$ in Minkowski space, or, if one passes to the Euclidean space with the usual substitution $-it \to t$, one gets $\exp(Et_{\rm sing})$, where $E = E_1 + E_2$ is the total c.m. energy of the process and $t_{\rm sing}$ is the time position of the field singularity in Euclidean space. We will see below that integrating over the positions of the singularity will result in the necessary conservation law: $E = E_{\rm field}$, where $E_{\rm field}$ is the energy of the produced multiparticle state.

Our aim is to calculate semiclassically the total cross section induced by two high-energy particles with a total energy E, i.e., the imaginary part of a forward scattering amplitude which, in its turn, can be expressed through the four-point Green function [we use $\phi(\mathbf{r}, t)$ to describe a generic field]:

$$\sigma(E^2 = (p_1 + p_2)^2) = (\text{flux factor}) \lim_{p_{1,2}^2 \to m^2} \text{Im} \int d^4 x_{1-4} e^{-ip_1(x_1 - x_3) - ip_2(x_2 - x_4)}$$
$$\times \langle (\partial_1^2 + m^2)\phi(x_1)(\partial_2^2 + m^2)\phi(x_2)(\partial_3^2 + m^2)\phi(x_3)(\partial_4^2 + m^2)\phi(x_4) \rangle. \tag{1.1}$$

The purpose of the paper is to demonstrate how to calculate the multiparticle production cross sections given generically by Eq. (1.1), using the technique of singular Euclidean solutions.

II. SINGULAR TRAJECTORIES IN QUANTUM MECHANICS

Before proceeding to the Yang-Mills case we describe a quantum-mechanical analogue of our approach. We consider a particle with coordinate $\phi(t)$ in a double-well potential, whose (Euclidean) action is given by

$$S = \frac{1}{g^2} \int dt \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{8} (\phi^2 - 1)^2 \right],$$ (2.1)

where $g^2 \ll 1$ is a dimensionless coupling constant.

A. Quantum-mechanical "cross section"

In the quantum-mechanical case "mass shell" means fixed energy for each "particle," equal to 1 in our notation. Therefore, to get an analogue of high-energy cross section one has to consider an "off-mass-shell" amplitude. Since we are only interested in the cross section computed to an exponential accuracy, it does not matter what specific operator $O(\phi)$ we use to go off shell. We choose $O(\phi)$ to be a low-power polynomial of ϕ. The analogue of Eq. (1.1) in quantum mechanics would be

$$\sigma(E) = \text{Im} \int dt_{1,2} e^{-iE(t_1 - t_2)} \left\langle O(\phi(t_1)) O(\phi(t_2)) \right\rangle.$$ (2.2)

One can decompose the Green function $\langle \rangle$ as a sum over intermediate states with wave functions $\Psi_n(\phi)$:

$$\left\langle O(\phi(t_1)) O(\phi(t_2)) \right\rangle = \sum_n \left| \int d\phi \Psi_0(\phi) O(\phi) \Psi_n^\star(\phi) \right|^2$$

$$\times \left[\theta(t_1 - t_2) e^{-iE_n(t_1 - t_2)} + \theta(t_2 - t_1) e^{-iE_n(t_2 - t_1)} \right].$$ (2.3)

Integrating over $t_{1,2}$ in Eq. (2.2) and taking the imaginary part we get

$$\sigma(E) = \pi \delta(E_n - E) V \left| \int d\phi \Psi_0(\phi) O(\phi) \Psi_E^\star(\phi) \right|^2,$$ (2.4)

where V is the time volume of the process [actually to be canceled by the omitted "flux factor" in Eq. (2.2)]. We see thus that the quantum-mechanical "cross section" comes to a square of a matrix element between the ground state Ψ_0 and a highly excited one Ψ_E.

According to the Landau method [14] this matrix element can be calculated to exponential accuracy as an exponent of the difference of two shortened actions, one with zero energy and the other with the given energy E:

$$\sigma(E) = \exp\left(2\left[- \int d\phi \sqrt{2U(\phi)} \right.\right.$$

$$\left.\left. + \int d\phi \sqrt{2U(\phi) - 2E} \right] \right),$$ (2.5)

where $U(\phi)$ is the potential energy of the field. Equation (2.5) is directly applicable to the simple case of a one-minimum potential; in this case one has to integrate, in the first term, from $\phi_{\min}$, corresponding to the minimum of the potential, to $\phi = \infty$, and in the second term from the energy-dependent turning point ϕ_1, where the integrand nullifies, to $\phi = \infty$. For the square of the amplitude one has to double the result in the exponent.

We arrive thus at singular Euclidean trajectories. One starts at (Euclidean) time $t = -\infty$ from the minimum of the potential, goes according to the equations of motion with zero energy to infinity (singularity) at some $t = -T/2$, and then, starting from the same singularity, goes with fixed energy E to the corresponding turning point ϕ_1 at $t = 0$. At the turning point $\dot{\phi} = 0$, and one can enter the Minkowski space to calculate the real-time evolution of the field. However, the Minkowski action is a pure phase; therefore, to find the cross section one needs not know this part of the trajectory. Instead, one has to square the amplitude, i.e., repeat the singular Euclidean trajectory in the opposite direction: starting from the above turning point ϕ_1 at $t = 0$, going with energy E to a singularity at $t = +T/2$, and ultimately returning to the minimum of the potential at $t \to +\infty$. The different branches of the trajectory are shown schematically in Fig. 1(a).

The Yang-Mills theory is, however, more like the double-well quantum mechanics, therefore we have to generalize Eq. (2.5) to the case of a two-minimum potential as in Eq. (2.1), allowing for the instanton transitions between the minima. To that end we first have to specify the initial and final states $\Psi_{0,E}$ entering Eq. (2.4). In a double-well potential all levels are known to be split into symmetric (s) and antisymmetric (a) states. For levels deep inside the wells the corresponding wave functions can be written as superpositions of states localized in the left and in the right wells:

$$\Psi_{s,a}(\phi) = \frac{\Psi(1 + \phi) \pm \Psi(1 - \phi)}{\sqrt{2}},$$

$$\Psi(1 \pm \phi) = \frac{\Psi_s(\phi) \pm \Psi_a(\phi)}{\sqrt{2}}.$$ (2.6)

To mimic in quantum mechanics the nonperturbative Chern-Simons changing transitions of the Yang-Mills theory we choose the initial state with low energy [Ψ_0 in the notation of Eq. (2.4)] to be localized near the left minimum and have approximately zero energy:

$$\Psi_0(\phi) = \frac{\Psi_s^{\approx 0}(\phi) + \Psi_a^{\approx 0}(\phi)}{\sqrt{2}}.$$ (2.7)

For the high-energy state Ψ_E we take a superposition which is predominantly localized in the right well and
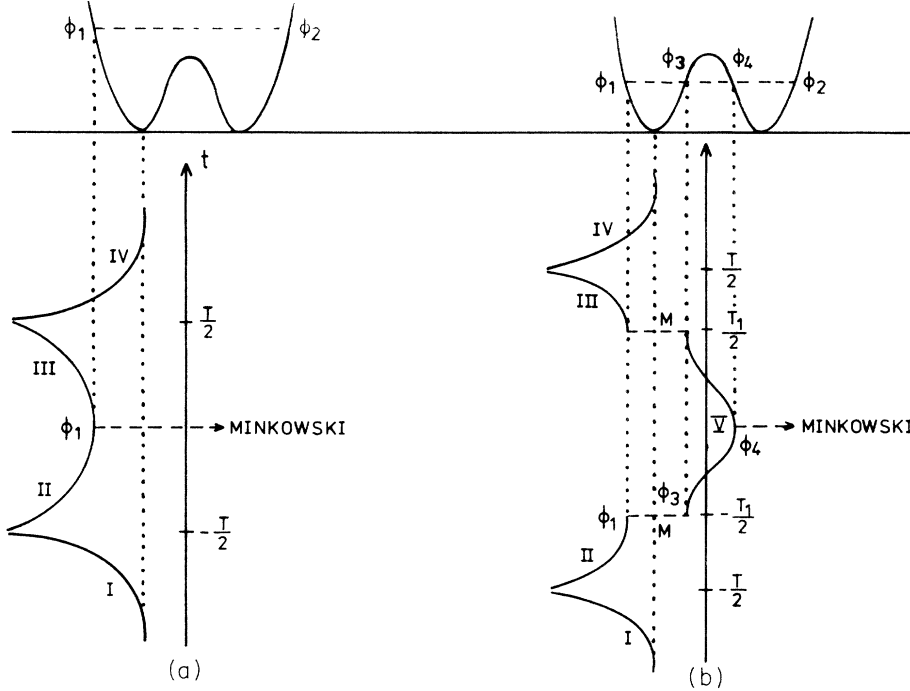
FIG. 1. Singular trajectories (a) below and (b) above the sphaleron.

has energy approximately equal to E:

$$\Psi_E(\phi) = \frac{\Psi_s^{\approx E}(\phi) - \Psi_a^{\approx E}(\phi)}{\sqrt{2}}. \tag{2.8}$$

Of course at energies E higher than the top of the potential barrier one cannot say anymore that the state (2.8) is localized near the right minimum, but it is probably the best one can do to define continuously in energy the transitions between the left and right wells. In the Yang-Mills theory a much more clear trigger of the transition with the change of the Chern-Simons number is given by the accompanying change of fermion chirality or baryon and/or lepton number violation. However, for energies less than the barrier top the state Ψ_E (2.8) is indeed localized near the right minimum. For energies higher than the barrier matrix elements for the transition to any state are exponentially small, so the concrete definition of the final state does not make a great difference.

With this definition of the initial and final states, Eq. (2.5) has to be slightly modified. For energies higher than the potential barrier the modification is cosmetic: the singular trajectory described above has to start in the specific minimum $\phi = -1$; it ends up in the same minimum [see Fig. 1(a)]. For energies lower than the barrier the Minkowskian part of the trajectory starting from the turning point ϕ_1 [see Fig. 1(b)] hits another turning point ϕ_3, where it again enters the forbidden zone and hence develops in imaginary (Euclidean) time. This branch (V) is a bounce resembling the kink plus antikink: at $t = 0$ (chosen for symmetry reasons) the trajectory reaches the turning point ϕ_4 belonging rather to the right well. From this point one can proceed in Minkowski time observing the final-state field in the right well. However,

the Minkowski action, being a pure phase, does not contribute to the cross section; in order to obtain it one has to repeat the trajectory in the opposite order and count the actions (with appropriate signs) along the branches I–V [see Fig. 1(b)]. We get, for the actions along the different branches,

$$S^{\mathrm{I}} = S^{\mathrm{IV}} = \frac{1}{2g^2} \int_{-\infty}^{-1} d\phi(\phi^2 - 1),$$

$$S^{\mathrm{II}} = S^{\mathrm{III}} = \frac{1}{2g^2} \int_{-\infty}^{\phi_1 = -\sqrt{1+\sqrt{\varepsilon}}} d\phi\sqrt{(\phi^2 - 1)^2 - \varepsilon},$$

$$S^{\mathrm{V}} = \frac{1}{g^2} \int_{\phi_3 = -\sqrt{1-\sqrt{\varepsilon}}}^{\phi_4 = \sqrt{1-\sqrt{\varepsilon}}} d\phi\sqrt{(\phi^2 - 1)^2 - \varepsilon}, \tag{2.9}$$

where we have introduced a dimensionless energy $\varepsilon = 8g^2E$. The final formula for the cross section is

$$\sigma(E) = \exp\left(-S^{\mathrm{I}} + S^{\mathrm{II}} - S^{\mathrm{V}} + S^{\mathrm{III}} - S^{\mathrm{IV}}\right). \tag{2.10}$$

At energies higher than the barrier ($\varepsilon > 1$) the branch V is absent and there is no contribution of S_{V} to $\sigma(E)$ so that we return to Eq. (2.5).

Let us comment on a few features of Eqs. (2.9), and (2.10). The actions $S^{\mathrm{I}} - S^{\mathrm{IV}}$ each diverge at $\phi = -\infty$; however, their sum is finite. To see that explicitly we can rewrite their sum as

$$S^{\mathrm{I-IV}} \equiv S^I + S^{\mathrm{IV}} - S^{\mathrm{II}} - S^{\mathrm{III}}$$

$$= \frac{1}{g^2}\left\{(\phi_1 + 1) - \frac{\phi_1^3 + 1}{3}\right.$$

$$\left. + \int_{-\infty}^{\phi_1} d\phi\left[(\phi^2 - 1) - \sqrt{(\phi^2 - 1)^2 - \varepsilon}\right]\right\} > 0,$$

$$(2.11)$$

which apparently is convergent. At $\varepsilon \to 0$ these branches cancel altogether and one is left with the piece S^{V}, which, in this limit, is an action on the infinitely separated instanton (kink) and anti-instanton (antikink), equal to $4/3g^2$. It gives the familiar Gamov suppression factor for tunneling at zero energy. We thus rewrite Eq. (2.10) as

$$\sigma(E) = \exp\left[-S^{\mathrm{I-IV}}(E) - S^{\mathrm{V}}(E)\right]. \qquad (2.12)$$

As the energy rises the behavior of the cross section is determined by the interplay of two opposite trends. The tunneling probability, represented by $\exp(-S^{\mathrm{V}})$, increases with energy whereas the overlap of the initial low-energy state with the high-energy one, represented by $\exp(-S^{\mathrm{I-IV}})$, decreases. This physical picture was suggested several years ago in Ref. [16]; now we are in a position to make it fully quantitative. At energies higher than the barrier there is no compensation for that trend from the side of S^{V}, so the cross section will definitely decrease, and we thus expect a maximum of the cross section at energies of the order of the top of the barrier, $\varepsilon = 1$.

B. Calculation of the cross section

The time dependence of the trajectories is given by

$$\frac{t - t_0}{2} = \int \frac{d\phi}{\sqrt{(\phi^2 - 1)^2 - \varepsilon}}. \qquad (2.13)$$

For $\varepsilon = 0$ (branches I and IV) we get trajectories going from -1 to $-\infty$ at $t = \pm T/2$:

$$\phi^{\mathrm{I}}(t) = \coth\frac{1}{2}\left(t + \frac{T}{2}\right),$$

$$\phi^{\mathrm{IV}}(t) = -\coth\frac{1}{2}\left(t - \frac{T}{2}\right) = \phi^{\mathrm{I}}(-t). \qquad (2.14)$$

At $\varepsilon \neq 0$ (branches II, III, and V) the trajectories are given by elliptic functions of the first and second kind.

Actually in order to calculate the cross section (2.10) as a function of energy it is not necessary to know the trajectories explicitly: the shortened actions contain less information. We first calculate the derivatives of shortened actions with respect to ε. At $\varepsilon < 1$ we have [**K**(k) is the complete elliptic integral of the first kind [17]]

$$g^2\frac{d\ln\sigma}{d\varepsilon} = -g^2\frac{dS^{\mathrm{I-IV}}}{d\varepsilon} - g^2\frac{dS^{\mathrm{V}}}{d\varepsilon}$$

$$= -\frac{1}{2}\frac{\mathbf{K}(k)}{\sqrt{1 + \sqrt{\varepsilon}}} + \frac{\mathbf{K}(k)}{\sqrt{1 + \sqrt{\varepsilon}}} \quad \left(k = \sqrt{\frac{1 - \sqrt{\varepsilon}}{1 + \sqrt{\varepsilon}}}\right)$$

$$= \frac{1}{8}\ln\frac{64}{\varepsilon} + \varepsilon\frac{3\ln\frac{64}{\varepsilon} - 10}{128} + O(\varepsilon^2). \qquad (2.15)$$

At $\varepsilon > 1$ we get

$$g^2\frac{d\ln\sigma}{d\varepsilon} = -g^2\frac{dS^{\mathrm{I-IV}}}{d\varepsilon} = -\frac{1}{2}\frac{\mathbf{K}(l)}{\sqrt{2\sqrt{\varepsilon}}} \quad \left(l = \sqrt{\frac{\sqrt{\varepsilon} - 1}{2\sqrt{\varepsilon}}}\right)$$

$$= -\frac{1}{8\varepsilon^{1/4}}B\left(\frac{1}{4}, \frac{1}{2}\right) + O(\varepsilon^{-3/4}),$$

$$B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}. \qquad (2.16)$$

Integrating these equations we get the cross section. At small energies we obtain a rising cross section

$$g^2\ln\sigma(\varepsilon) = -\frac{4}{3} + \frac{\varepsilon}{8}\left(\ln\frac{64}{\varepsilon} + 1\right)$$

$$+ \frac{\varepsilon^2}{256}\left(3\ln\frac{64}{e} - \frac{17}{2}\right) + O(\varepsilon^3), \qquad (2.17)$$

while at large energies we get a decreasing cross section

$$g^2\ln\sigma(\varepsilon) = -\frac{\varepsilon^{3/4}}{6}B\left(\frac{1}{4}, \frac{1}{2}\right) + O(\varepsilon^{1/4}). \qquad (2.18)$$

We notice that (2.18) is consistent with the exact bound for the matrix elements found in Ref. [18].

The maximum of the cross section is achieved at $\varepsilon = 1$, corresponding to the top of the barrier. The logarithm of the cross section there is exactly half that at $\varepsilon \to 0$, where the suppression is solely due to the double instanton action:

$$g^2\ln\sigma_{\max} = -\frac{2}{3}, \quad \varepsilon = 1. \qquad (2.19)$$

C. "Square root" suppression

The "square root" suppression of the cross sections at the maximum as compared to that at zero energies have been advocated by Zakharov [19] and Maggiore and Shifman [20] from unitarity considerations (this and other work on unitarity has been extensively reviewed in Ref. [4]). It is remarkable that we get the square root suppression in a different approach.[1] Probably it means that the correct formulas respect unitarity. Moreover, we

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[1]The same kind behavior has been found earlier by Voloshin [21] for the high-energy production accompanying the decay of a metastable vacuum.

can show that this square root suppression is of a more general nature. Indeed, let us compare $dS/d\varepsilon$ along the branches II, III, and V for $\varepsilon < 1$. According to a general theorem of the classical mechanics (see Appendix A) these derivatives are related to the periods of motion along these branches:

$$8g^2 \frac{dS^{\mathrm{I-IV}}}{d\varepsilon} = 4 \int_{-\infty}^{\phi_1} \frac{d\phi}{\sqrt{U(\phi) - \varepsilon}} = T_2,$$

$$-8g^2 \frac{dS^{\mathrm{V}}}{d\varepsilon} = 4 \int_{\phi_3}^{\phi_4} \frac{d\phi}{\sqrt{U(\phi) - \varepsilon}} = T_1,$$

$$U(\phi) = (\phi^2 - 1)^2, \quad T = T_1 + T_2. \tag{2.20}$$

Here T_1, T_2 are periods of motion along the branches V and II+III, respectively. We are going to prove that $T_1 = 2T_2$ for a wide class of potentials $U(\phi)$.

To that end let us consider the difference of integrals along two closed contours Γ_1 and Γ_2 in the complex ϕ plane (see Fig. 2):

$$2 \int_{\Gamma_1} \frac{d\phi}{\sqrt{U(\phi) - \varepsilon}} - 2 \int_{\Gamma_2} \frac{d\phi}{\sqrt{U(\phi) - \varepsilon}} = 0. \tag{2.21}$$

This difference is zero as far as the potential energy $U(\phi)$ has no singularities in the whole complex ϕ plane, so that the integrand has only cuts at the turning points ϕ_{1-4}. According to Eq. (2.20) the integral along the contour Γ_1 is equal to T_1 while that along Γ_2 is equal to twice T_2, as there are two equal contributions from two cuts to that integral. Therefore we indeed get the relation

$$T_1 = 2T_2, \quad -\frac{dS^{\mathrm{V}}}{d\varepsilon} = 2\frac{dS^{\mathrm{I-IV}}}{d\varepsilon}. \tag{2.22}$$

Integrating this equation in ε we arrive to the following important relation first noticed in Ref. [22]:

$$S^{\mathrm{V}}(\varepsilon) = S^{\mathrm{V}}(0) - 2S^{\mathrm{I-IV}}(\varepsilon). \tag{2.23}$$

Equations (2.22) and (2.23) are of a very general nature and are independent of the explicit form of the potential. [For the concrete potential considered above Eq. (2.22)
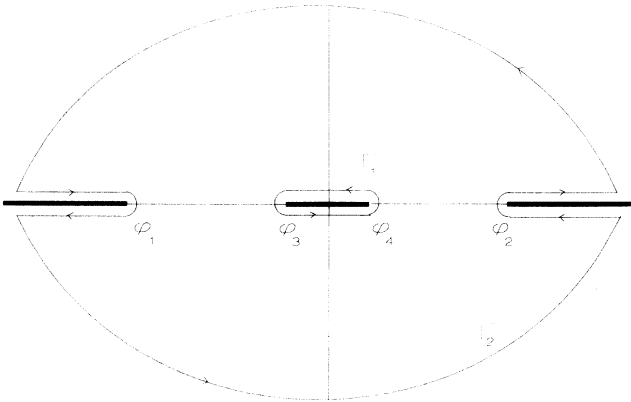
follows directly from Eq. 2.15)]. Moreover, we think that Eqs. (2.22) and (2.23) are valid also in field theories where tunneling may occur, Yang-Mills (YM) and Yang-Mills-Higgs (YMH) theories being examples. The analogue of Eq. (2.21) in field theory would be the following contour integral in the complex plane of an arbitrary parameter s parametrizing the solutions of equations of motion with a given field energy $\phi_E(x, s)$:

$$\int_{\Gamma} ds \frac{\sqrt{\int d^3x \, [d\phi_E(x, s)/ds]^2}}{\sqrt{\int d^3x \left\{ \frac{1}{2} (\nabla_i \phi_E)^2 + U[\phi] \right\} - E}}. \tag{2.24}$$

One can choose the time t as the parameter s, but that is not necessary since (2.24) is reparametrization invariant. One can try to use this freedom to introduce a parametrization $\phi_E(x, s)$ such as to ensure that both the numerator and the denominator have no singularities in the complex s plane, except the cuts due to the turning points of the denominator. (In the one-degree-of-freedom case such a parameter s is the "field" ϕ itself.) If that goal is achieved, one would prove the relations (2.22) and (2.23) in field theory. We have not proven them in the general case; however, we confirm these relations for the YM theory by a direct calculation in Sec. III, at least for small field energies.

At $\varepsilon = 1$ one reaches the top of the barrier where S_{V} vanishes. According to Eq. (2.23) it means that

$$S^{\mathrm{I-IV}}(1) = \frac{S^{\mathrm{V}}(0)}{2} = S^{\mathrm{inst}} = \frac{1}{g^2}\frac{2}{3}, \tag{2.25}$$

which reproduces Eq. (2.19) and gives the maximum of the cross section corresponding to the square root suppresion as compared to that at zero energy. Combining Eqs. (2.23) and (2.25) we can write

$$\ln \sigma(\varepsilon) = \begin{cases} -2S^{\mathrm{inst}} + S^{\mathrm{I-IV}}(\varepsilon), & \varepsilon < 1, \\ -S^{\mathrm{inst}}, & \varepsilon = 1, \quad S^{\mathrm{I-IV}}(\varepsilon) > 0, \\ -S^{\mathrm{I-IV}}(\varepsilon), & \varepsilon > 1, \end{cases} \tag{2.26}$$

This equation shows a jump of the derivative at $\varepsilon = 1$. It should be mentioned, though, that in a parametrically narrow strip near $\varepsilon = 1$ the semiclassical formulas are not valid anymore and one would expect a smooth match between the two regimes of Eq. (2.26).

D. Instanton interactions

Let us return to our concrete quantum-mechanical example. To make contact with the previous work on the instanton-induced cross sections at low energies we recall that a conventional way to present the results is via the instanton–anti-instanton interaction potential U_{int}. The cross section can be written as a Fourier transform of the time separation of the instanton and the anti-instanton [23–26]:

$$\sigma(E) = \int dT \exp\left[ET - U_{\mathrm{int}}(T) \right]. \tag{2.27}$$

FIG. 2.   Contour integration proving the relation between the periods, Eq. (2.22).

We define here U_{int} so that at large separations T it includes twice the free instanton action. The integral over T is performed by saddle point method, which is how one gets the energy dependence of the cross section. Comparing Eq. (2.27) with $\sigma(E)$ found above [Eq. (2.17)], we obtain the instanton interaction potential at large separations

$$U_{int}(T) = \frac{1}{g^2}\left[\frac{4}{3} - 8e^{-T} - 48Te^{-2T} + 136e^{-2T}\right.$$
$$\left. + O(e^{-3T})\right]. \tag{2.28}$$

The first term here is twice the instanton action and the second one reproduces correctly the well-known kink–anti-kink leading-order attraction (see, e.g., Ref. [27]). The third term is new: we can compare it only with the valley approach of Balitsky and Yung [28, 24, 29]. For the sum ansatz of a kink and antikink these authors get

$$U_{int}^{valley}(T) = \frac{1}{g^2}\left[\frac{4}{3} - 8e^{-T} + 24Te^{-2T} - 40e^{-2T}\right.$$
$$\left. + O(e^{-3T})\right], \tag{2.29}$$

which coincides with our result only in the leading order. To our mind, it means only that the sum of a kink and antikink is not too useful beyond the leading order. In an indirect way, however, our result for the next-to-leading kink-antikink interaction seems to be in accordance with another calculation, this time for the instanton–anti-instanton interaction in the YM theory. Indeed, as emphasized in Refs. [28, 24, 29], an instanton–anti-instanton configuration can be obtained from a kink–antikink one through a chain of conformal and gauge transformations. The separation between the YM instantons R and their sizes $\rho_{1,2}$ is related to the separation of a kink and antikink T as

$$e^{T/2} = \frac{R^2 + \rho_1^2 + \rho_2^2 + \sqrt{(R^2 + \rho_1^2 + \rho_2^2)^2 - 4\rho_1^2\rho_2^2}}{2\rho_1\rho_2}, \tag{2.30}$$

while the coupling g^2 should be replaced by the gauge coupling α according to the rule

$$\frac{1}{g^2}\frac{4}{3} \rightarrow \frac{4\pi}{\alpha}. \tag{2.31}$$

Therefore, the kink–antikink interaction (2.28) can be transformed into the instanton–anti-instanton interaction. We get, from Eqs. (2.28) and (2.30),

$$U_{int}^{YM}(R) = \frac{4\pi}{\alpha}\left[1 - 6\frac{\rho_1^2\rho_2^2}{R^4} + 12\frac{\rho_1^2\rho_2^2(\rho_1^2 + \rho_2^2)}{R^6}\right.$$
$$\left. - 72\frac{\rho_1^4\rho_2^4}{R^8}\ln\frac{R^2}{\rho^2} + O\left(\frac{\rho^8}{R^8}\right)\right]. \tag{2.32}$$

These terms are *exactly* those which are independently known today for the YM instanton interactions. It should be stressed that the last term has been computed from unitarity in a laborous two-loop calculation in the YM theory[2] [26]. It is remarkable that we reproduce the result of these difficult calculations in such a simple way.

III. SINGULAR YANG-MILLS FIELDS

We now turn to the YM theory and describe the analogues of the branches I–V (see Fig. 1), assuming that the cross section of the *two* → *many* processes are to the exponential accuracy given by Eq. (2.10). A derivation of the generalization of the Landau formula to field theory using the first order formalism can be found in the paper of Iordanskii and Pitaevskii [13]. We shall reproduce the well-known results for the instanton-induced cross sections at relatively low energies and get new results for high energies. In the case of QCD we shall get the cross section of isotropical multigluon production as function of energy: it decreases as approximately the ninth power of energy.

A. Exact singular solutions with zero energy

In contrast with the quantum mechanics with only one degree of freedom, in field theory there are infinitely many ways to get an infinite field energy. In principle one has to try any type of field singularities and the ultimate choice of the classical field should be made from the maximum of the cross section with a given trigger for the final state (if there is any). In this paper we restrict ourselves to O(3) symmetrical field configurations which at low energies reproduce the instanton-induced processes and lead to an isotropic final state. One might well doubt whether it is natural for high-energy collisions[3] and in a sense we prove that it is not, but at the moment we would like to follow the instanton-induced processes up to asymptotically high energies. For that reason we choose the singularity in such a way that at given times $\mp T/2$ the A_μ field has a powerlike singularity in r where r is the distance from the origin. We have nothing to say about non-O(3) symmetrical singular fields, such as light-cone singularities, etc. They would lead to another behavior of the cross section and a different final state.

For branches I and IV corresponding to $E = 0$, exact solutions of the needed type have already been constructed by Khlebnikov [12]: They are the usual Belavin-Polyakov-Schwarz-Tyupkin instanton (branch I) and anti-instanton (branch IV) in the singular Lorentz gauge with the scale parameter ρ^2 changed to $-\rho^2$:

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[2]Recently the result has been confirmed in Ref. [30] for the case of the maximum-attraction orientation, though there is still a difference between Refs. [30] and [26] for general orientations.

[3]We a grateful to C. Wetterich for a discussion of this point.

$$A_\mu^{(\mathrm{I})a}(\mathbf{r},t) = \frac{2\bar\eta_{\mu\nu}^a\left(t+\frac{T}{2}-\rho\right)_\nu \rho^2}{\left[\rho^2 - r^2 - \left(t+\frac{T}{2}-\rho\right)^2\right]\left[r^2 + \left(t+\frac{T}{2}-\rho\right)^2\right]}, \quad t < -\frac{T}{2} \quad (\text{branch I}),$$ (3.1)

$$A_\mu^{(\mathrm{IV})a}(\mathbf{r},t) = \frac{2\eta_{\mu\nu}^a\left(t-\frac{T}{2}+\rho\right)_\nu \rho^2}{\left[\rho^2 - r^2 - \left(t-\frac{T}{2}+\rho\right)^2\right]\left[r^2 + \left(t-\frac{T}{2}+\rho\right)^2\right]}, \quad t > \frac{T}{2} \quad (\text{branch IV}),$$ (3.2)

where $\eta, \bar\eta$ are 't Hooft symbols [31].

Equation (3.1) describes the evolution of the field with zero energy starting from zero field at $t = -\infty$ and getting to a singularity at $t = -T/2$. It is the analogue of Eq. (2.14) in the quantum-mechanical example. Equation (3.2) describes the conjugate process: it goes from a singularity at $t = T/2$ to zero field at $t = +\infty$. In case of the electroweak (EW) theory with Higgs bosons one can neglect the influence of the Higgs bosons as far as the characteristic scale of the fields is $\rho \ll m_W^{-1}$, where m_W is the W boson mass. This is the case in two limits we are now interested in: at $E \ll m_W/\alpha$, where we shall reproduce the well-known result, and at $E \gg m_W/\alpha$, see below. In the intermediate energy region one would have to solve the coupled YMH system; otherwise Eqs. (3.1) and (3.2) are exact. At $t = \mp T/2$ these solutions become singular at the origin $r = 0$:

$$A_\mu^{(\mathrm{I})a}\left(\mathbf{r},-\frac{T}{2}\right) = \frac{2\bar\eta_{\mu\nu}^a\left(\begin{array}{c}\mathbf{r}\\-\rho\end{array}\right)_\nu}{-r^2(r^2+\rho^2)},$$ (3.3)

$$A_\mu^{(\mathrm{V})a}\left(\mathbf{r},\frac{T}{2}\right) = \frac{2\eta_{\mu\nu}^a\left(\begin{array}{c}\mathbf{r}\\\rho\end{array}\right)_\nu}{-r^2(r^2+\rho^2)}.$$ (3.4)

We notice that (3.3) and (3.4) coincide for spatial components A_i but differ in sign for the time component A_4. This is a gauge artifact, however. Indeed, one can perform a time-dependent gauge transformation

$$A_\mu \to U^\dagger(\mathbf{r},t)A_\mu U(\mathbf{r},t) + iU^\dagger \partial_\mu U,$$ (3.5)

eliminating the A_4 components in both branches I and IV. (In fact, $A_4 = 0$ is the adequate gauge to use: all results are physically more transparent in it; we started with a Lorentz gauge to make formulas more compact.) It is easy to check that making a gauge transformation (3.5) we do not destroy the coincidence of the spatial components A_i at the singularity points. We cite the neccessary formulas in Appendix B. The time-dependent gauge transformations (3.5) are, however, defined up to time-independent transformations. In order not to intro-

duce new quantities, we use this gauge freedom to keep the A_i fields at $t = \pm T/2$ exactly in the form given by Eqs. (3.3) and (3.4):

$$A_i^{(\mathrm{I})\,a}\left(\mathbf{r},-\frac{T}{2}\right) = A_i^{(\mathrm{IV})\,a}\left(\mathbf{r},\frac{T}{2}\right)$$
$$= -\frac{2\rho^2}{r^2(r^2+\rho^2)}\left(\epsilon_{aij}r_j + \delta_{ai}\rho\right),$$
$$A_4^a = 0.$$ (3.6)

In general, in the $A_4 = 0$ gauge the spatial components $A_i(\mathbf{r},t)$ contain three O(3) symmetrical structures:

$$A_i^a(\mathbf{r},t) = \epsilon_{aij}n_j\frac{1-A(r,t)}{r} + (\delta_{ai} - n_a n_i)\frac{B(r,t)}{r}$$
$$+ n_a n_i\frac{C(r,t)}{r}, \quad \mathbf{n} = \frac{\mathbf{r}}{r}.$$ (3.7)

In Appendix B we quote the relation of these structures to the A_μ field in the Lorentz gauge and the YM action written in terms of the A,B,C functions.

In order to find the fields along branches II and III corresponding to a nonzero field energy E_{field}, one has to solve the YM equations (generally speaking, coupled to Higgs bosons) in the time interval $-T/2 < t < T/2$ with the singular boundary conditions (3.6). As in the quantum-mechanical example, the field energy E_{field} is directly related to the time interval T. However difficult technically, solving the YM equations with given boundary conditions is a well-formulated problem which can be solved numerically. We solve it analytically in two limiting cases: (i) $\rho \ll T$, corresponding to low energies $\alpha E\rho \ll 1$, and (ii) $\rho \gg T$, corresponding to high energies $\alpha E\rho \gg 1$. In the first case we reproduce the usual instanton results. The second case corresponds to asymptotically large energies, and the results here are new.

B. Reproducing instanton results

In the case $\rho \ll T$, i.e., low energies, we deal with a situation shown schematically in Fig. 1(b). Therefore,

one first has to find the singular solutions along branches II and III, and then go to the Minkowski space and find the analogue of the turning point ϕ_3. Then one has to construct again the Euclidean solution along branch V. Each time the boundary conditions for the solutions are determined at the preceding step, so that the procedure is straightforward. However, we have not solved the problem at arbitrary values of ρ/T, but only in the lowest order in ρ/T. The explicit construction of the YM fields along the branches I–V is rather lengthy and we relegate it to Appendix C. We find the following expressions for the *full* actions along branches I+IV [with the actions along branches II and III subtracted — we call it $S^{\mathrm{I-IV}}$; see Eq. 2.11) and along the branch V:

$$S^{\mathrm{I-IV}}_{\mathrm{full}} = \frac{4\pi}{\alpha}\left[6\left(\frac{\rho}{T_2}\right)^4 + O\left(\frac{\rho^6}{T_2^6}\right)\right],\qquad(3.8)$$

$$S^{\mathrm{V}}_{\mathrm{full}} = \frac{4\pi}{\alpha}\left[1 - 6\left(\frac{\rho}{T_2}\right)^4 - 6\left(\frac{\rho}{T_1 - T_2}\right)^4 + O\left(\frac{\rho^6}{T_2^6}\right)\right].\qquad(3.9)$$

In Eqs. (3.8) and (3.9) we have introduced the time $T_2 \equiv T - T_1$, which is the net time for branches II+III.

According to the general mechanical formula (see Appendix A), the energy of the field is the derivative of the full action in respect to the period of motion, which is T_2 for branches II+III and T_1 for branch V. Also, the field energy E_{field} should be the same on branches II, III, and V. From Eqs. (3.8) and (3.9) we have

$$\begin{aligned}E_{\mathrm{field}} &= -\frac{dS^{\mathrm{I-IV}}_{\mathrm{full}}}{dT_2} = \frac{4\pi}{\alpha}24\frac{\rho^4}{T_2^5}\\&= \frac{dS^{\mathrm{V}}_{\mathrm{full}}}{dT_1} = \frac{4\pi}{\alpha}24\frac{\rho^4}{(T_1 - T_2)^5}.\end{aligned}\qquad(3.10)$$

Hence we get a relation between the times along branches V and II+III:

$$T_1 = 2T_2 \quad\text{where}\quad T_2 = \rho\left(\frac{\alpha E\rho}{96\pi}\right)^{-1/5}.\qquad(3.11)$$

In Sec. II we obtained the same relation for the case of quantum mechanics and presented arguments that it could be of a more general nature. Now we have demonstrated the validity of this relation "experimentally" in the YM theory, at least for large periods, or, equivalently, for small energies E.

The corresponding shortened actions are (see Appendix A for the definitions)

$$S^{\mathrm{V}}_{\mathrm{short}} = S^{\mathrm{V}}_{\mathrm{full}} - E_{\mathrm{field}}T_1 = \frac{4\pi}{\alpha}\left[1 - 60\left(\frac{\alpha E\rho}{96\pi}\right)^{4/5}\right],\qquad(3.12)$$

$$S^{\mathrm{I-IV}}_{\mathrm{short}} = S^{\mathrm{I-IV}}_{\mathrm{full}} + E_{\mathrm{field}}T_2 = \frac{4\pi}{\alpha}30\left(\frac{\alpha E\rho}{96\pi}\right)^{4/5}.\qquad(3.13)$$

Adding these two actions in the exponent in accordance with Eq. (2.12) we get finally the cross section as a function of energy at small values of $\alpha E\rho$:

$$\sigma(E) = \exp\left\{\frac{4\pi}{\alpha}\left[-1 + 30\left(\frac{\alpha E\rho}{96\pi}\right)^{4/5}\right]\right\}.\qquad(3.14)$$

This formula coincides with the well-known result for small energies [23, 32] obtained for the instanton-induced total cross section in the saddle point approximation. The one-loop correction to Eq. (3.14) is also known [24, 25, 33, 34]: it adds, to the exponent of Eq. (3.14), a term

$$-\frac{4\pi}{\alpha}24\left(\frac{\alpha E\rho}{96\pi}\right)^{6/5}.\qquad(3.15)$$

The two-loop correction [26] adds

$$+\frac{4\pi}{\alpha}\frac{144}{5}\left(\frac{\alpha E\rho}{96\pi}\right)^{8/5}\left[\ln\frac{96\pi}{\alpha E\rho} + \mathrm{const}\right].\qquad(3.16)$$

We have not reproduced these terms directly by our new method (indirectly we got these terms from the quantum-mechanical calculation of Sec. II by using the conformal-symmetry arguments). We would like to stress that what looks like a loop expansion in the conventional approach appears to be equivalent to finding singular solutions of the *classical* equations of motion. We believe that it is a well-formulated program which may be performed for all values of the dimensional energy $\alpha E\rho$.

C. Singular solutions for high energies

Let us now consider the opposite case: $T \ll \rho$. It means that the time during which the field is developing is much less than the spatial spread of the field, see Eq. (3.6). During this short time the fields change only at distances $r \ll \rho$, but cannot change significantly at $r \sim \rho$. Therefore, the region $r > \rho$ cancels out in the difference between the shortenned actions. For that reason one can neglect r as compared to ρ in Eq. (3.6), so that the boundary conditions for the field simplifies. In terms of the A,B,C functions introduced in Eq. (3.7) the boundary condition becomes

$$A\left(r, \pm\frac{T}{2}\right) = 3,\quad B\left(r, \pm\frac{T}{2}\right) = C\left(r, \pm\frac{T}{2}\right) = -\frac{2\rho}{r}.\qquad(3.17)$$

We see that the structure $B + C$ is singular and thus much larger than the structures A and $B - C$; therefore we shall neglect the second two and work with one structure $B + C = D$. The equations of motion for $A, B - C$, and D will keep A and $B - C$ negligible, as compared to D, as far as r will remain much less than ρ, which is the

case of interest. Moreover, for the similar reasons one can neglect the spatial derivatives of $B + C$ and other less singular terms as compared to the terms $B^4, C^4/r^2$ in the action; see Appendix B, Eq. (B9). The resulting action written in terms of the large structure D becomes

$$S^{\text{II+III}} = \frac{3}{4\alpha} \int_{-T/2}^{T/2} dt \int_0^\infty dr \left(\frac{1}{2}\dot{D}^2 + \frac{D^4}{8r^2} \right). \tag{3.18}$$

The corresponding equation of motion is

$$-\ddot{D} + \frac{D^3}{2r^2} = 0 \tag{3.19}$$

with the boundary conditions

$$D\left(r, \pm\frac{T}{2}\right) = -\frac{4\rho}{r}. \tag{3.20}$$

The problem can be regarded as a simple one-degree-of-freedom mechanical one, with r viewed as a parameter of the trajectory. Equation (3.19) is integrable as there exists a conserved energy density:

$$e(r) = \frac{D^4}{8r^2} - \frac{1}{2}\dot{D}^2. \tag{3.21}$$

The time dependence of the field is determined by integrating Eq. (3.21):

$$t = \int_{-4\rho/r}^{D(r,t)} \frac{dD}{\sqrt{D^4/4r^2 - 2e(r)}}. \tag{3.22}$$

Introducing dimensionless quantities

$$x = -\frac{D}{[8e(r)r^2]^{1/4}}, \quad \beta = \frac{4\rho}{r[8e(r)r^2]^{1/4}}, \tag{3.23}$$

we rewrite Eq. (3.22) as

$$\frac{t}{2r}[8e(r)r^2]^{1/4} = \int_x^\beta \frac{dx}{\sqrt{x^4 - 1}}. \tag{3.24}$$

The half-period of the motion $T/2$ is found from Eq. (3.24) as an integral from the initial configuration

β up to the turning point being $x = 1$ in the new notation:

$$\frac{T}{4r}[8e(r)r^2]^{1/4} = \int_1^\beta \frac{dx}{\sqrt{x^4 - 1}}. \tag{3.25}$$

It can be written as an equation relating the period and the energy density:

$$\beta \int_1^\beta \frac{dx}{\sqrt{x^4 - 1}} = \frac{\rho T}{r^2}. \tag{3.26}$$

This equation has a solution for β and hence for $e(r)$ for all values of the right-hand side: small r corresponds to $\beta \to \infty$ while large r corresponds to $\beta \to 1$.

The shortened action density along the branches II+III is

$$s^{\text{II+III}}(r) = 2 \int_{-4\rho/r}^{D_{\min}} dD \sqrt{\frac{D^4}{4r^2} - 2e(r)}$$

$$= \frac{64\rho^3}{r^4} \frac{1}{\beta^3} \int_1^\beta dx \sqrt{x^4 - 1}. \tag{3.27}$$

Being integrated over r, this action is apparently divergent; however, one should not forget to subtract a similarly divergent piece corresponding to the zero-energy branches I and IV. To make the necessary subtraction we notice that zero energy corresponds to infinite period T and hence to $\beta \to \infty$; see Eq. (3.26). Putting $\beta = \infty$ in Eq. (3.27) we get the shortened action density for branches I and IV:

$$s^{\text{I+IV}}(r) = \frac{64\rho^3}{r^4} \frac{1}{3}, \tag{3.28}$$

which is indeed independent of T and should be subtracted from Eq. (3.27). Of course, one could get Eq. (3.28) directly by finding the action density along the solutions (3.1) and (3.2). To get the complete shortened action standing in the exponent for the cross section we integrate the difference of (3.27) and (3.28) over r. Using the relation (3.26) we integrate over β instead of r and obtain

$$S^{\text{I-IV}} = \frac{3}{4\alpha} \int_0^\infty dr(s^{\text{I+IV}} - s^{\text{II+III}}) = \frac{4\pi}{\alpha}\left(\frac{\rho}{T}\right)^{\frac{3}{2}} \frac{6}{\pi} \int_1^\infty \frac{d\beta}{\sqrt{\beta}} \left(\int_1^\beta \frac{dx}{\sqrt{x^4 - 1}}\right)^{1/2}$$

$$\times \left(\int_1^\beta \frac{dx}{\sqrt{x^4 - 1}} + \frac{\beta}{\sqrt{\beta^4 - 1}}\right) \left(\frac{1}{3} - \frac{1}{\beta} \int_1^\beta dx \sqrt{x^4 - 1}\right)$$

$$= \frac{4\pi}{\alpha} 5 \left(\frac{B(\frac{1}{4}, \frac{1}{2})}{8}\right)^4 \left(\frac{\rho}{T}\right)^{3/2} = \frac{4\pi}{\alpha} 0.9232 \left(\frac{\rho}{T}\right)^{3/2}. \tag{3.29}$$

We remind the reader that the above calculation is performed for the case $\rho \gg T$; hence Eq. (3.29) describes a rapid falloff of the cross section in this region.

Integrating the energy density $e(r)$ over r we get the field energy

$$E_{\text{field}} = \frac{3}{4\alpha} \int_0^\infty dr \, e(r) = \frac{4\pi}{\alpha T} \left(\frac{\rho}{T}\right)^{3/2} \frac{3}{\pi} \int_1^\infty \frac{d\beta}{\beta^{5/2}} \left(\int_1^\beta \frac{dx}{\sqrt{x^4-1}}\right)^{3/2} \left(\int_1^\beta \frac{dx}{\sqrt{x^4-1}} + \frac{\beta}{\sqrt{\beta^4-1}}\right)$$

$$= \frac{4\pi}{\alpha} \frac{3}{T} \left(\frac{\rho}{T}\right)^{3/2} \left(\frac{B(\frac{1}{4},\frac{1}{2})}{8}\right)^4 = \frac{4\pi}{\alpha} \frac{0.5539}{T} \left(\frac{\rho}{T}\right)^{3/2}. \tag{3.30}$$

One can immediately check that Eqs. (3.29) and (3.30) satisfy the general relation between the shortened action and the energy (see Appendix A):

$$\frac{dS_{\text{short}}^{\text{I-IV}}}{dT} = \frac{dE_{\text{field}}}{dT} T. \tag{3.31}$$

The energy E_{field} (3.30) actually has a double meaning. We have introduced it as the energy of a singular Euclidean field between the singularity points. It becomes purely potential energy at the turning point ($t = 0$) starting from where the field enters the classically allowed region and hence develops in real Minkowski time. Therefore, it is also the energy of the outgoing field or, equivalently, the energy of the *final* multiparticle state.

Simultaneously, it is the energy of the few-particle *initial* state. To see that we notice that the cross section is defined via the imaginary part of the four-point Green function of the initial particles; see Eq. (1.1). The high-energy asymptotics of a Green function in the background field is determined by the positions of the singularities of the field. In the case depicted in Fig. 1(a) there is one singularity at $t = -T/2$ for the amplitude and one singularity at $t = +T/2$ for the conjugated amplitude. Therefore, the asymptotics of the four-point Green function and hence of the cross section is

$$\exp\left[E_1(-T/2) + E_2(-T/2) - E_1 T/2\right.$$
$$\left. - E_2 T/2 - S_{\text{full}}(T)\right]$$
$$= \exp\left[-E_{\text{particle}} T - S_{\text{full}}(T)\right], \tag{3.32}$$

where $E_{\text{particle}} = E_1 + E_2$ is the energy of the initial two-particle state and S_{full} is the full or Lagrange action of the field:

$$S_{\text{full}} = S_{\text{full}}^{\text{I}} + S_{\text{full}}^{\text{IV}} - S_{\text{full}}^{\text{II}} - S_{\text{full}}^{\text{III}}. \tag{3.33}$$

On the other hand, the full actions are simply related to the shortened actions provided the fields satisfy the equations of motion (see Appendix A):

$$S_{\text{full}} = S_{\text{short}}^{\text{I}} + S_{\text{short}}^{\text{IV}} - S_{\text{short}}^{\text{II}} - S_{\text{short}}^{\text{III}} - E_{\text{field}} T$$
$$\equiv S^{\text{I-IV}} - E_{\text{field}} T \tag{3.34}$$

[the quantity $S^{\text{I-IV}}$ has been calculated in Eq. (3.29)]. Thus, we get from Eq. (3.32),

$$\exp\left[-E_{\text{particle}} T + S^{\text{I-IV}} + E_{\text{field}} T\right]. \tag{3.35}$$

Integrating over the separation between the singularities T we obtain the energy conservation law

$$E_{\text{particle}} = E_{\text{field}} \equiv E \tag{3.36}$$

whereas the cross section is given simply by

$$\sigma(E) = \exp\left[-S^{\text{I-IV}}(E)\right], \tag{3.37}$$

where the shortened action $S^{\text{I-IV}}$ should be expressed through the energy. It should be mentioned that in a case depicted in Fig. 1(b) there are additional field singularities at $t = \pm T_1/2$ and a similar analysis is slightly more complicated.

Expressing the time between the singularities T through the field energy (3.30) and substituting it into Eq. (3.29) we get, finally,

$$\sigma(E) = \exp\left[-\frac{4\pi}{\alpha} 1.316 \left(\frac{\alpha\rho E}{4\pi}\right)^{3/5}\right], \tag{3.38}$$

which is valid for $\alpha E \rho \gg 1$ and describes a decreasing cross section at large values of this parameter.

The result of this subsection can be formulated also in terms of the instanton–anti-instanton interaction potential U_{int}. We find, at small separations T, a strong repulsion:

$$U_{\text{int}}(T) = S_{\text{full}}^{\text{I-IV}}(T) = S^{\text{I-IV}} - ET = \frac{4\pi}{\alpha} 0.3693 \left(\frac{\rho}{T}\right)^{3/2},$$

$$T \ll \rho, \tag{3.39}$$

as contrasted with the usual attraction at large separations; see Eq. (2.32). We believe that it is a reasonable result if one defines U_{int} not as the action of an arbitrarily chosen Euclidean configuration such as the valley (we have shown in Sec. II that the valley leads to the wrong U_{int} already in the next-to-leading order), but through unitarity, as in Eq. (2.27). Physically, the effective repulsion of instantons at small separations appears here as a result of a rapidly decreasing overlap between low-energy and high-energy states. From the purely Euclidean viewpoint a closely situated instanton and anti-instanton resemble a vacuum state and therefore have to be effectively strongly repulsive if we wish to read off a nonperturbative contribution from that configuration. Equation (3.39) is good news for the QCD instanton-vacuum builders.

IV. ENERGY BEHAVIOR OF THE CROSS SECTION

We have established the behavior of the cross section for multi-gauge-boson production as a function of a dimensionless quantity $\alpha E \rho$, where ρ is the spatial size of the YM field, at small [Eq. (3.14)] and large [Eq. (3.38)] values of this parameter. Applications of this result are

different in the QCD case, where scale invariance is preserved at the classical level, and in the electroweak theory, where it is broken from the beginning. We discuss the two theories in turn.

A. QCD

In QCD the scale ρ is arbitrary and one has to integrate over it. Naturally, ρ will then be fixed by the maximum of the cross section as a function of the dimensionless parameter $\alpha E \rho$. We know the behavior of the $\ln \sigma(E)$ at small and large values of this parameter, and we plot it in Fig. 3, where we have added the next-to-leading term (3.15) at small $\alpha E \rho$. The low-energy and high-energy curves intersect and give the maximum at a point which is remarkably close to a point marked by a circle in Fig. 3, which is the position of the maximum that we would expect independently of the above calculations.

Indeed, in Sec. II we have presented general arguments in favor of the relation (2.23) and in Sec. III we have confirmed it at low energies for the YM case. If we assume Eq. (2.23) to be correct at all energies up to the sphaleron top of the barrier, we would get that, at the maximum corresponding to the top of the barrier, the cross section is exactly the square root of the cross section at zero energy. It means that the low- and high-energy branches of the cross section should intersect at

$$\ln \sigma_{\max} = -\frac{4\pi}{\alpha} \frac{1}{2}. \tag{4.1}$$

To what value of the dimensionless parameter $\alpha E \rho$ does this maximum correspond? Our arguments here are even more shaky; however, one could speculate that the maximum occurs at energy equal to the mass of the sphaleron. In the massless YM theory we are now dealing with, the sphaleron mass can be estimated as the potential energy of the instanton field exactly in the middle of the transition when the instanton passes the $N_{\text{CS}} = 1/2$ point. We have
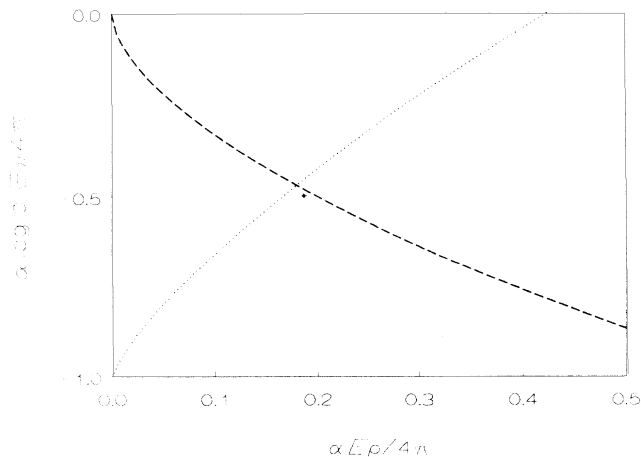

FIG. 3. The behavior of the natural log of the cross section as a function of $\alpha E \rho$ at small (dotted curve) and large (dashed curve) values of this parameter. The circle shows the expected maximum of the cross section.

$$M_{\text{sph}}(\rho) = \frac{1}{4g^2} 4\pi \int dr\, r^2 \left. \frac{96\rho^4}{(t^2 + r^2 + \rho^2)^4} \right|_{t=0} = \frac{4\pi}{\alpha} \frac{1}{\rho} \frac{3}{16}. \tag{4.2}$$

We find then that the maximum of the cross section should be achieved at the value of the dimensionless parameter

$$\left(\frac{\alpha E \rho}{4\pi} \right)_{\max} = \frac{3}{16} = 0.1875. \tag{4.3}$$

This is the point marked by a circle in Fig. 3, and we see that it is remarkably close to the intersection of the low- and high-energy curves. Higher order corrections to both curves should probably move the intersection point exactly to the position calculated here.

The fact that the spatial size of the classical field ρ scales down as $1/E$ at the maximum means that quantum corrections to the classical calculations of this paper should be controllable and small at $E \to \infty$. The one-loop quantum corrections, as usual, should make the gauge coupling run. Also, one would expect a large prefactor owing to the zero modes about the classical trajectory – something like $(2\pi/\alpha)^{4N_c}$, where N_c is the number of colors. The prefactor α starts to "run" only at the two-loop level. Combining renormalization-group arguments with the result (4.1), we expect the isotropical multigluon production cross section to be

$$\sigma_{\text{jet}}(E) = \frac{c}{E^2} \left(\frac{\Lambda}{E} \right)^b \left(b \ln \frac{E}{\Lambda} \right)^{4N_c + p} \times \left[1 + O \left(\frac{1}{\ln(E/\Lambda)} \right) \right], \tag{4.4}$$

where $b = 11N_c/3 - 2N_f/3 \approx 7$ is the one-loop Gell-Mann–low coefficient and Λ is $\Lambda_{\text{QCD}} \approx 200$ MeV. The coefficient c and the power p are determined by the second-loop Gell-Mann–low coefficient and by the Green functions of the initial particles of the process; that is where the nonuniversality of cross sections comes in. A calculation of these constants seems to be feasible.

Thus we arrive at a prediction that the total cross section for isotropical multigluon production in QCD should fall off as approximately the ninth power of their aggregate energy. However, a presumably large prefactor and the peculiarity of the events might help to make such processes observable; this subject deserves a more thorough study.

It is worth mentioning that the momentum distribution of the produced multigluon state can be found easily by solving the YM equations in Minkowski time starting from the "sphaleron" field (4.2). Since that field has only one parameter ρ related to energy through Eq. (4.3), the gluon momentum distrubution will have a scaling behavior with energy, modulo logarithmic corrections.

B. EW theory

In this case the size of the classical field ρ is determined not only by energy but also by the Higgs vacuum expec-

tation value v. At low energies one has to multiply the $\alpha E \rho$-dependent cross section (3.14) [with the addition of one-loop (3.15) and two-loop (3.16) corrections] by the factor

$$\exp(-2\pi^2 \rho^2 v^2). \tag{4.5}$$

Integrating over ρ, one gets the familiar expansion of the total baryon-number-violating (BNV) cross section in terms of the dimensionless energy measured in units of the sphaleron mass [26,30]

$$\sigma_{\text{tot}}^{B}(E) = \exp\left[\frac{4\pi}{\alpha} F(\varepsilon)\right], \quad \varepsilon = \frac{E}{\sqrt{6\pi} m_W / \alpha},$$

$$F(\varepsilon) = -1 + \frac{9}{8}\varepsilon^{4/3} - \frac{9}{16}\varepsilon^{6/3} - \frac{3}{16}\varepsilon^{8/3}\left[\ln \varepsilon + \text{const}\right]. \tag{4.6}$$

The growth of the function $F(\varepsilon)$ has lead in the past to hopes that the BNV processes may become unsuppressed at energies of the order of the sphaleron mass m_W/α, corresponding to $\varepsilon \sim 1$. However, unitarity arguments of Refs. [19, 20] indicated that the BNV cross section induced by instantons should not rise beyond the square root of the cross section at zero energies. We now arrive to the same conclusion from our singular solutions.

At very high energies ($E \gg m_W/\alpha$) the size of the field ρ is determined by the factor (3.38) rather than by the Higgs factor such as (4.5), meaning that $\rho \sim 1/\alpha E \ll 1/m_W$. Therefore one can neglect the Higgs influence and arrive at the conclusions of the preceding subsection, viz., that the cross section does not rise above the value of $\exp(-2\pi/\alpha) \approx 10^{-85}$.

At energies of the order of the sphaleron mass ($E \sim m_w/\alpha$) one cannot neglect the Higgs effects and has to find singular solutions of the YMH equations along the branches I–V. If our general arguments based on analyticity are correct, we would again expect that the maximum occurs at exactly the sphaleron energy and corresponds to the square root suppression.

The momentum distribution of the produced W and H bosons should follow from the Minkowski development of the sphaleron — the "fall" of the sphaleron. This distribution has been studied numerically in Refs. [35, 36]. We argue now that the same distribution would happen at high-energy collisions; however, its probability would be of the order of 10^{-85}.

V. MASSLESS $\lambda\phi^4$ THEORY

For future reference we would like to cite here the results for the massless $\lambda\phi^4$ theory whose Euclidean action is

$$S = \int d^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{\lambda}{4}\phi^4\right). \tag{5.1}$$

Instead of the singular instanton one has to start here with the singular Lipaton [37], which gives an O(3) symmetrical zero-energy field singular at $t = \mp T/2$ [12]:

$$\phi(\mathbf{r}, t) = 2\sqrt{\frac{2}{\lambda}} \frac{\rho}{r^2 + (t \pm T/2 \mp \rho)^2 - \rho^2}. \tag{5.2}$$

Since there is no tunneling and consequently no branch V in this theory, one gets a monotonically decreasing cross section as function of the dimensionless quantity $\lambda E\rho$. Probably it is in accordance with the theory not being asymptotically free. The calculations of Sec. III can be repeated without any serious change (in fact, they are much more simple). For small $\lambda E\rho$ the cross section has been actually calculated in Ref. [12] and we confirm it:

$$\ln \sigma_{tot} = -\frac{16\pi^2}{\lambda} \frac{3}{2^{1/3}} \left(\frac{E\rho\lambda}{16\pi^2}\right)^{2/3}, \quad \lambda E\rho \ll 1. \tag{5.3}$$

For large $\lambda E\rho$ the leading term can be obtained immediately from Eq. (3.38) by changing notations. Indeed, introducing a new variable $D(r, t)$,

$$\phi(r, t) = \frac{1}{\sqrt{2\lambda}} \frac{D(r, t)}{r}, \tag{5.4}$$

we get, for the action (5.1),

$$S = \frac{2\pi}{\lambda} \int_{-T/2}^{T/2} dt \int dr \left(\frac{\dot{D}^2}{2} + \frac{D^4}{8r^2}\right), \tag{5.5}$$

which coincides with (3.18) up to the overall factor. The boundary conditions for $D(r, t)$ as imposed by Eq. (5.2) also coincide with those of the YM case (3.20). Hence the behavior of the cross section can be immediately obtained from Eq. (3.38) by a suitable change of constants. We get

$$\ln \sigma_{\text{tot}} = -\frac{16\pi^2}{\lambda} 1.119 \left(\frac{\lambda E\rho}{16\pi^2}\right)^{3/5}, \quad \lambda E\rho \gg 1. \tag{5.6}$$

Equations (5.3) and (5.6) exhibit a monotonically decreasing cross section as function of $\lambda E\rho$; integrating over ρ, one falls into the trivial maximum at $\lambda E\rho = 0$ corresponding to zero particle production, which is naturally more probable than multiparticle production. It is a prerogative of field theories with nontrivial topology to have a nontrivial behavior of classical cross sections even in the massless limit.

VI. CONCLUSIONS

We have elaborated on the use of singular Euclidean solutions to describe semi-classical multiparticle production in field theories. We have shown that, by using instantonlike O(3) symmetrical singular solutions, one reproduces the usual formulas for the instanton-induced cross sections for low energies. However, the present approach allows us to solve the problem at all energies. We have found analytically a family of singular solutions labeled by ρ/T, which govern the high-energy behavior of multiparticle production both for the YM and $\lambda\phi^4$ theories. We have presented new arguments (not based on unitarity, but rather on analyticity) that the maximum of the instanton-induced cross sections in QCD and EW

theories is still exponentially small in the coupling constant, with the coefficient corresponding exactly to the square root suppression as compared to zero energy. An interpolation of the low- and high-energy branches of the cross section give numerically this square root point to a surprisingly good accuracy; see Fig 3. Simultaneously our result means that there is effectively a strong repulsion between instantons at small separations.

As a by-product of our study we get a nonperturbative formula for the isotropic multigluon production in QCD, Eq. (4.4). Presently unknown coefficients and powers of the logarithms in that formula seem to be calculable and deserve further study. It would be also extremely useful, if possible, to understand Eq. (4.4) in terms of some clever summation of Feynman graphs for a gluon branching process; we mean the approach initiated by Cornwall [38] and Goldberg [39].

As to the applications to the baryon number violating processes in the EW theory, our result is in line with other people, who, in the recent years, argued from various angles that the BNV instanton-induced cross section has to be small at all energies. However, we feel that it is not the end of the story, but rather its beginning. It should not be surprising that the *isotropic* multi-particle production is suppressed at high energies. One should expect that at high energies the events have at most the axial symmetry. Therefore, one should look for solutions with singularities of a different kind, not just O(3) symmetrical. The same applies to other cases where we expect that multiparticle production might be described semiclassically such as multijet production and low-x structure functions in QCD, multipion production in heavy-ion collisions, etc.

Our final remark concerns an alternative approach to semiclassical multi-particle production [40] in which the few initial high-energy particles are used as a (weak) source for classical fields. How can this approach be reconciled with our? We think that including the source explicitly, one has to look for anomalous solutions of field equations, which do not possess singularities however small the source, but which develop a singularity when the source is set identically to zero. We have observed such a scenario in quantum mechanics and a $d=2$ massive σ model, and believe that it is a general case. However, in a $d=4$ Yang-Mills theory the hopefully equivalent approach based on singular Euclidean solutions is seemingly more simple.

ACKNOWLEDGMENTS

APPENDIX A:
GENERAL RELATIONS FOR THE ACTION

Let S_{full} be the Lagrange action

$$S_{\text{full}} = \frac{1}{g^2} \int dt \left[\frac{1}{2} \dot{\phi}^2 + U(\phi) \right].$$ (A1)

Using energy conservation

$$\frac{1}{2} \dot{\phi}^2 - U(\phi) = -Eg^2,$$ (A2)

which is satisfied if the field $\phi(t)$ satisfies the equations of motion, we can present S_{full} as

$$S_{\text{full}} = \frac{1}{g^2} \int dt \dot{\phi}^2 + E \int dt$$
$$= \frac{1}{2g^2} \int d\phi \sqrt{U(\phi) - \varepsilon} + ET = S_{\text{short}} + ET,$$ (A3)

where T is the period of the motion. Equation (A3) introduces the so-called shortened action

$$S_{\text{short}} = \int p \, dq.$$ (A4)

In the main text we make intensive use of the general identities following from Eq. (A3):

$$\frac{dS_{\text{short}}}{dE} = -T, \quad \frac{dS_{\text{short}}}{dT} = -T \frac{dE}{dT}$$ (A5)

and

$$\frac{dS_{\text{full}}}{dT} = E.$$ (A6)

Branches I and IV correspond to zero energy, so that the full actions along these branches coincide with the shortened ones:

$$S_{\text{full}}^{\text{I}} = S_{\text{full}}^{\text{IV}} = S_{\text{short}}^{\text{I}} = S_{\text{short}}^{\text{IV}}.$$ (A7)

The relations (A5) and (A6) apply directly to branch V, in which case the period is denoted by T_1 [see Fig. 1(b)]:

$$\frac{dS_{\text{short}}^{\text{V}}}{dE} = -T_1, \quad \frac{dS_{\text{short}}^{\text{V}}}{dT_1} = -T_1 \frac{dE}{dT_1}, \quad \frac{dS_{\text{full}}^{\text{V}}}{dT_1} = E.$$ (A8)

The period of motion along branches II+III is denoted by T_2 (it becomes equal to the full time between the singularities T for energies above the barrier) and we have

$$S_{\text{full}}^{\text{II+III}} = S_{\text{short}}^{\text{II+III}} + ET_2.$$ (A9)

We usually combine this action with that along branches I and IV in order to cancel the divergencies. Denoting

$$S_{\text{full}}^{\text{I-IV}} \equiv S_{\text{full}}^{\text{I+IV}} - S_{\text{full}}^{\text{II+III}}, \quad S^{\text{I-IV}} \equiv S_{\text{short}}^{\text{I+IV}} - S_{\text{short}}^{\text{II+III}},$$ (A10)

we have

$$S_{\text{full}}^{\text{I-IV}} = S^{\text{I-IV}} - ET_2,$$ (A11)

so that

$$\frac{dS^{\mathrm{I-IV}}}{dE} = T_2, \quad \frac{dS^{\mathrm{I-IV}}}{dT_2} = T_2 \frac{dE}{dT_2}, \quad \frac{dS^{\mathrm{I-IV}}_{\mathrm{full}}}{dT_2} = -E.$$
$$\text{(A12)}$$

APPENDIX B: GAUGE TRANSFORMATION TO THE $A_4 = 0$ GAUGE

Let us consider an instantonlike field in the Lorentz gauge:

$$A^a_\mu(x) = 2\bar{\eta}^a_{\mu\nu} x_\nu \Phi(x^2) \tag{B1}$$

or

$$A_4 = A^a_4 \frac{\tau^a}{2} = (\mathbf{n}\tau) r \Phi(r^2 + t^2),$$

$$A_i = A^a_i \frac{\tau^a}{2} = \{-\tau_i t + i\left[(\mathbf{n}\tau)\tau_i - n_i\right] r\} \Phi(r^2 + t^2). \tag{B2}$$

We make a hedgehog gauge transformation

$$A'_\mu = U^\dagger(\mathbf{r}, t) A_\mu U(\mathbf{r}, t) + iU^\dagger \partial_\mu U,$$

$$U = \exp\left[i(\mathbf{n}\tau)P(r, t)\right] \tag{B3}$$

in order to eliminate the A'_4 component. The profile function of the gauge transformation is found from the equation

$$A_4 = iU\partial_4 U^\dagger = (\mathbf{n}\tau)\frac{\partial P}{\partial t} = (\mathbf{n}\tau) r \Phi(r^2 + t^2), \quad \text{(B4)}$$

which can be integrated once the function $\Phi(r^2 + t^2)$ is given.

With $P(r, t)$ known, the gauge-transformed spatial components A'_i take the O(3) symmetrical form

$$A'^a_i(\mathbf{r}, t) = \epsilon_{aij} n_j \frac{1 - A(r, t)}{r} + (\delta_{ai} - n_a n_i)\frac{B(r, t)}{r}$$
$$+ n_a n_i \frac{C(r, t)}{r}, \quad \mathbf{n} = \frac{\mathbf{r}}{r}, \tag{B5}$$

where

$$A(r, t) = \cos 2P - (\cos 2Pr + \sin 2Pt)2r\Phi, \tag{B6}$$

$$B(r, t) = -\sin 2P + (\sin 2Pr - \cos 2Pt)2r\Phi, \tag{B7}$$

$$C(r, t) = -2r\left[\frac{\partial P}{\partial r} + t\Phi\right]. \tag{B8}$$

If the 't Hooft symbol η instead of $\bar{\eta}$ is used in the definition of the field (B1), one has to change $t \to -t$ in all the above equations.

The YM action rewritten in terms of these structures is

$$S_{\mathrm{YM}} = \frac{1}{4g^2} \int d^4x \left(F^a_{\mu\nu}\right)^2 = \frac{1}{\alpha} \int dt \int dr \left[\dot{A}^2 + \dot{B}^2 + \frac{1}{2}\dot{C}^2\right.$$

$$\left. + A'^2 + B'^2 + \frac{(A^2 + B^2 - 1)^2}{2r^2} + \frac{2C(A'B - AB')}{r} + \frac{C^2(A^2 + B^2)}{r^2}\right]. \tag{B9}$$

The Chern-Simons number is

$$N_{\mathrm{CS}} = \frac{1}{2\pi} \int dr \left[A'B - B'A + \frac{C}{r}(A^2 + B^2 - 1)\right]. \tag{B10}$$

APPENDIX C: SINGULAR YM FIELDS AT LOW ENERGIES

1. Branches II and III

At low energies we do not expect large deviation on branches II and III from the zero-energy solutions (3.1) and (3.2), which are just singular instantons. To save the space we introduce the following notation. Let $A^{\mathrm{inst}}_\mu(t + T/2 - \rho)$ be a singular instanton in the Lorentz gauge defined by Eq. (3.1). Similarly, $A^{\mathrm{anti}}_\mu(t - T/2 + \rho)$ is shorthand notation for the singular anti-intstanton given by Eq. (3.2). The time arguments of the fields are what we wish to stress here. We next gauge transform the

fields to the $A_4 = 0$ gauge according to the formulas of Appendix B. We denote the resulting spatial components as $A_i(t + T/2 - \rho)$ and $A_i(-t + T/2 - \rho)$ (note that we have changed the sign of the argument for the case of the anti-instanton field, in agreement with the procedure of Appendix B).

We next notice that the net time for branches II and III is $T_2 \equiv T - T_1$ [see Fig. 1(b)]. The field along branch II has to stop at a turning point at time $-T_1/2$ and the field along branch III has to stop at time $T_1/2$. Since these are the same turning points, we can temporaly identify them with a single point $t = 0$; the singularities occur then at time $\pm T_2/2$.

To get the fields satisfying the boundary conditions at the singularity points we introduce the field

$$A^0_i = \begin{cases} A_i(t + T_2/2 + \rho), & t < 0 \\ A_i(-t + T_2/2 + \rho), & t > 0. \end{cases} \tag{C1}$$

The two branches match at $t = 0$; however, their time derivatives there have opposite signs. Equation (C1) satisfies the YM equation of motion everywhere except the point $t = 0$:

$$\frac{\delta S[A_i^0]}{\delta A_i} \approx 2\dot{A}_i(T_2/2 + \rho)\delta(t) = \delta(t)O(\rho^2/T_2^2). \quad (C2)$$

Therefore, we have to modify A^0 in the vicinity of the turning point at $t = 0$. We look for the solutions along the branches II and III in the form

$$A_i(t) = A_i^0(t) + B_i(t), \quad (C3)$$

where the additional field B_i satisfies the zero boundary conditions at the singularity points $\pm T_2/2$ and the equation of motion

$$0 = \frac{\delta S[A_i^0 + B_i]}{\delta A_i} = \frac{\delta S[A_i^0]}{\delta A_i} + [-\partial_t^2 + \mathcal{K}(A_i^0)]B + O(B^2), \quad (C4)$$

where $-\partial_t^2 + \mathcal{K}$ is the quadratic form of the YM action.

To the accuracy we are now interested in, the field B_i is of the order of ρ^2/T_2^2. Therefore, we can neglect the nonlinear terms in Eq. (C4) and the shifts by ρ in the arguments of the fields. Furthermore, in the region of $t \approx 0$ one can put $A^0 = 0$ in the quadratic form $\mathcal{K}$, since A^0 is also of the order of ρ^2/T_2^2 in this region. Solving Eq. (C4) at $t \approx 0$ and taking into account Eq. (C2) we get

$$B_i(t \approx 0) = A_i(|t| + T_2/2). \quad (C5)$$

The resulting field $A_i^0 + B_i$ is smooth at $t = 0$; it is shown schematically in Fig. 4(a). Equation (C5) gives the only solution of the equation of motion to the needed accuracy, which is even in t and decreases in the direction of the singularity points. One could, in principle, add to Eq. (C5) a solution of the homogeneous equation. Such solutions would be the zero modes of the singular
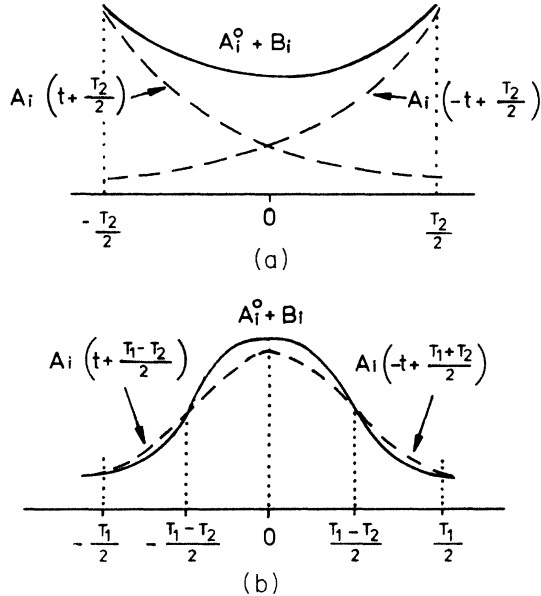

FIG. 4. A schematic view of (a) the singular solutions along branches II and III and (b) the bounce solution along branch V.

instanton; their addition would correspond to a shift or a distortion of the singularity points and therefore would not satisfy the necessary boundary conditions at $\pm T_2/2$. We thus obtain the field at the turning point $t = 0$:

$$\dot{A}_i^{\mathrm{II,III}}(\mathbf{r}, 0) = 0, \quad A_i^{\mathrm{II,III}}(\mathbf{r}, 0) = 2A_i(\mathbf{r}, T_2/2). \quad (C6)$$

Let us now calculate the full action along the branches II and III, subtracting those along the zero-energy branches I and IV:

$$S_{\mathrm{full}}^{\mathrm{I-IV}} = S^{\mathrm{I+IV}} - S_{\mathrm{full}}^{\mathrm{II+III}} = 2\int_{-\infty}^{-T_2/2} dt \int d^3x \mathcal{L}[A^I]$$
$$-\int_{-T_2/2}^{T_2/2} dt \int d^3x \mathcal{L}[A^{\mathrm{II,III}}]. \quad (C7)$$

Using equations of motion and gauge invariance this action can be calculated without undue difficulty. Let us add and subtract the action computed on A^0 (C1) in the time interval between $\mp T_2/2$. We write

$$S_{\mathrm{full}}^{\mathrm{I-IV}} = W_1 + W_2,$$
$$W_1 = 2\int_{-\infty}^{-T_2/2} dt \int d^3x \mathcal{L}[A_i(t + T_2/2)]$$
$$-2\int_{-T_2/2}^{0} dt \int d^3x \mathcal{L}[A_i(t + T_2/2)],$$
$$W_2 = -\int_{-T_2/2}^{T_2/2} dt \int d^3x \left\{\mathcal{L}[A^0 + B] - \mathcal{L}[A^0]\right\}. \quad (C8)$$

In W_1 we change the time variables and use the fact that the Lagrange density is even in time. We obtain

$$W_1 = 2\int_{T_2/2}^{\infty} dt \int d^3x \mathcal{L}[A_i(t)]. \quad (C9)$$

This integral can be calculated in two ways: (i) using the equation of motion for the A_i field and (ii) using the gauge invariance. The first method leads to the equation

$$W_1 = -\frac{1}{g^2}\int d^3x A_i^a(\mathbf{r}, T_2/2)\dot{A}_i^a(\mathbf{r}, T_2/2). \quad (C10)$$

Using the gauge invariance we can return to the Lorentz gauge and obtain

$$W_1 = \frac{1}{2g^2}\int_{T_2/2}^{\infty} dt \int d^3x \left(F_{\mu\nu}^a\right)^2$$
$$= \frac{4\pi}{\alpha}\left[\left(\frac{\rho}{T_2}\right)^4 + O\left(\frac{\rho^6}{T_2^6}\right)\right]. \quad (C11)$$

In W_2 we expand the integrand in B_i neglecting terms $O(B^3)$ and $O(B^4)$ and use the equations of motion for A_i^0 and B_i [Eqs. (C2) and (C4)]. We have

$$W_2 = -\frac{1}{g^2}\int_{-T_2/2}^{T_2/2} dt \int d^3x B_i^a(\mathbf{r}, t)\dot{A}_i^a(\mathbf{r}, T_2)\delta(t)$$
$$= -\frac{1}{g^2}\int d^3x A_i^a(\mathbf{r}, T_2/2)\dot{A}_i^a(\mathbf{r}, T_2/2) = W_1, \quad (C12)$$

where Eq. (C10) has been used in the last line. Adding

up the two pieces, we thus obtain the full action along branches II+III (with branches I and IV subtracted):

$$S_{\text{full}}^{\text{I−IV}} = \frac{4\pi}{\alpha}\left[6\left(\frac{\rho}{T_2}\right)^4 + O\left(\frac{\rho^6}{T_2^6}\right)\right].$$ (C13)

2. Branch V

We now go back to the turning point at $t = 0$ where we have already found the field, it is given by Eq. (C6). What we called $t = 0$ corresponds actually to two points: $t = \pm T_1/2$. At these points the field enters the allowed region and develops in Minkowski time. The solution of the Minkowski equations of motion is

$$A_i^{\text{Mink}} = A_i(T_2/2 + it) + A_i(T_2/2 - it).$$ (C14)

Indeed, this is a solution of Minkowski equations of motion for not too large t since A_i itself is a solution of Euclidean ones, and (C14) is evidently obeying boundary conditions (C6).

The field (C14) describes the outgoing gluon field which corresponds to multigluon production without changing of the Chern-Simons number. In order to change it we need one more tunneling transition corresponding to branch V of Fig. 1(b). Normally the Minkowski solution develops between the turning points ϕ_1 and ϕ_3 (as it was in our quantum-mechanical example) and determines the initial field configuration ϕ_3 for branch V. However, this is not the case in the lowest approximation we are dealing with. In this approximation the two turning points ϕ_3 and ϕ_1 coincide, so that branch V starts at the same field where branch II ends. It should be also mentioned that far from the centers in the leading approximation there is no difference between the field A_i of the singular and of the nonsingular instantons. We shall therefore denote the instanton field in the $A_4 = 0$ gauge by A_i as before.

Thus we have to find branch V as a solution of the Euclidean equations of motion with the boundary conditions given by Eq. (C6):

$$\dot{A}_i^{\text{V}}(\pm T_1/2) = 0, \quad A_i^{\text{V}}(\pm T_1/2) = 2A_i(T_2/2).$$ (C15)

Again we are looking for a solution in the form

$$A_i^{\text{V}} = A_i^0 + B_i,$$ (C16)

where this time A_i^0 is the field of the usual instanton in the $A_4 = 0$ gauge at $t < 0$ and that of an anti-instanton at $t > 0$; the centers of both are shifted by ΔT, to be found below. We have

$$A_i^0 = \begin{cases} A_i(t + T_1/2 - \Delta T), & t < 0 \\ A_i(-t + T_2/2 - \Delta T), & t > 0 \end{cases}$$ (C17)

[see Fig. 4(b)].

The shift in the positions of the instantons ΔT should be determined from the region near $-T_1/2$. In this region the field B_i obeys linearized equations of motion following from the quadratic form of the action in which one can neglect the instanton field being small in this region. The boundary conditions for B_i follow directly from Eqs.

(C15) and (C17):

$$B_i(t = -T_1/2) = 2A_i(T_2/2) - A_i(\Delta T),$$

$$\dot{B}_i(t = -T_1/2) = -\dot{A}_i(\Delta T).$$ (C18)

The field B_i should not increase in the direction of the center of instantons; then it does not include zero modes corresponding to the distortion of the instantons. We can satisfy this requirement only if $\Delta T = T_2/2$. Indeed, in this case the solution of equation of motion for B_i with the boundary conditions (C18) is

$$B_i = A_i\left[-t + (T_1 - T_2)/2\right],$$ (C19)

which clearly decreases for increasing t and hence does not contain the zero modes which would lead to the shift of the instanton position.

Let us turn now to the region near $t = 0$. Here we cannot apply immediately the method used above for branches II and III since the field A_i^0 is not small now. However, it is gauge equivalent to a small field $\tilde{A}_i^0$:

$$A_i^0(t) = U^\dagger(\mathbf{r})\tilde{A}_i^0 U(\mathbf{r}) + iU^\dagger\partial_i U,$$ (C20)

where $U(\mathbf{r})$ is a gauge transformation with the winding number equal to unity. One can apply the same considerations as above to the gauge transformed fields $\tilde{A}_i^0$ and $\tilde{B}_i = UB_iU^\dagger$. As a result we obtain the following expression for B_i near $t = 0$ [cf. Eq. (C5)]

$$B_i = U^\dagger(\mathbf{r})\tilde{A}_i\left(|t| + \frac{T_1 - T_2}{2}\right)U(\mathbf{r}).$$ (C21)

The solution of Minkowski equations of motion, which starts at the turning point at $t = 0$, is [cf. Eq. (C14)]

$$A_i^{\text{Mink}}(t) = U^\dagger(\mathbf{r})\left[\tilde{A}_i\left(it + \frac{T_1 - T_2}{2}\right) + \tilde{A}_i\left(-it + \frac{T_1 - T_2}{2}\right)\right]U(\mathbf{r}) + iU^\dagger\partial_i U.$$ (C22)

It describes gluon production around the minimum with the Chern-Simons number equal to one.

Now we can calculate the action along the branch V. The full action

$$S_{\text{full}}^{\text{V}} = \frac{2}{g^2}\int_{-T_1/2}^{0}\int d^3x \mathcal{L}[A_i^0 + B_i]$$ (C23)

can be expressed as a sum of twice the instanton action,

$$S^{\text{II}} = \frac{2}{g^2}\int_{-\infty}^{\infty}\int d^3x \mathcal{L}[A_i] = \frac{4\pi}{\alpha},$$ (C24)

and corrections of four types. The first is due to the fact that we have to calculate the action not along the whole t axis but in the time interval $(-\infty, -T_1/2)$:

$$\Delta S_1 = -\frac{2}{g^2} \int_{-\infty}^{T_1/2} \int d^3x \mathcal{L}\left[A_i \left(t + \frac{T_1 - T_2}{2} \right) \right]$$

$$= -\frac{4\pi}{\alpha} 3 \left(\frac{\rho}{T_2} \right)^4. \qquad (C25)$$

The second is a similar defect of the action, but coming from the time interval $(0, \infty)$:

$$\Delta S_2 = -\frac{2}{g^2} \int_0^{\infty} \int d^3x \mathcal{L}\left[A_i \left(t + \frac{T_1 - T_2}{2} \right) \right]$$

$$= -\frac{4\pi}{\alpha} 3 \left(\frac{\rho}{T_1 - T_2} \right)^4. \qquad (C26)$$

The third and fourth corrections are the contributions due to the B_i field:

$$\Delta S_3 + \Delta S_4$$

$$= \frac{1}{g^2} \int_{-T_1/2}^{T_1/2} \int d^3x \left\{ \mathcal{L}[A_i^0 + B_i] - \mathcal{L}[A_i^0] \right\}. \qquad (C27)$$

As above, we shall use here the equations of motion for both fields A_i^0 and B_i and get two contributions. One is the contribution of the boundaries at $t = \pm T_1/2$:

$$\Delta S_3 = \frac{2}{g^2} \int d^3x \left[B_i(t)\dot{A}_i \left(t + \frac{T_1 - T_2}{2} \right) \right.$$

$$\left. + \frac{1}{2} B_i(t)\dot{B}_i(t) \right]_{t=-T_1/2} = \Delta S_1. \qquad (C28)$$

The other is the contribution of the δ function at $t = 0$:

$$\Delta S_4 = \frac{1}{g^2} \int d^3x \left[B_i(t)\dot{A}_i \left(t + \frac{T_1 - T_2}{2} \right) \right.$$

$$\left. + \frac{1}{2} B_i(t)\dot{B}_i(t) \right]_{t=0} = \Delta S_2. \qquad (C29)$$

Adding up the four corrections we obtain finally the full action along branch V:

$$S_{\text{full}}^V = \frac{4\pi}{\alpha} \left[1 - 6 \left(\frac{\rho}{T_2} \right)^4 - 6 \left(\frac{\rho}{T_1 - T_2} \right)^4 \right]. \qquad (C30)$$

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