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Circular string instabilities in curved spacetime

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We investigate the connection between curved spacetime and the emergence of string instabilities, following the approach developed by Loustó and Sánchez for de Sitter and black hole spacetimes. We analyze the linearized equations determining the comoving physical (transverse) perturbations on circular strings embedded in Schwarzschild, Reissner-Nordström, and de Sitter backgrounds. In all three cases we find that the "radial" perturbations grow infinitely for $r \rightarrow 0$ (ring collapse), while the "angular" perturbations are bounded in this limit. For $r \rightarrow \infty$ we find that the perturbations in both physical directions (perpendicular to the string world sheet in four dimensions) blow up in the case of de Sitter space. This confirms results recently obtained by Loustó and Sánchez who considered perturbations around the string center of mass.

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I. INTRODUCTION

The classical equations of motion for a string in curved spacetime are generally nonintegrable due to their highly nonlinear nature, and even if the system for some specific background can be shown to be integrable, it may be a very hard task to actually write down the general solution in closed form. In many cases of interest it is on the other hand not so difficult to find special solutions. The standard way is to make an ansatz that somehow exploits possible symmetries of the background and somehow is also based on physical insight in the specific case under consideration. If properly chosen this ansatz may reduce the original system of coupled nonlinear partial differential equations to something simpler, and special solutions may be found by quadratures. So, for instance, if the background is axially symmetric one can look for circular strings, if the background is stationary one can look for stationary strings, etc.

A typical feature of nonlinear systems is the presence of regions of unstable and chaotic motion, i.e., perturbations around certain special solutions develop imaginary frequencies and grow infinitely. In a recent paper Loustó and Sánchez [1] consider such string instabilities in black hole and de Sitter spacetimes. Their starting point was a method of studying string solutions in curved spacetimes, originally developed by de Vega and Sánchez [2]. By considering first order string perturbations around the center of mass motion of the string (which is of course just a geodesic), Loustó and Sánchez fully analyzed the behavior of the solutions and found the regions of instability in the three cases of Schwarzschild, Reissner-Nordström, and de Sitter backgrounds.

In this paper we will consider a somewhat similar situation. However, we will take as the unperturbed string configuration a circular string in the same three backgrounds, and we will in fact follow the analysis of Ref. [1] very closely. We will use a method, developed by Frolov and the author [3] (see also [4,5]), to study covariantly the physical perturbations around circular strings embedded in the curved spacetimes mentioned above.

This program is mainly motivated by the recent interest in the dynamics of strings in curved backgrounds and the study of string instabilties in curved backgrounds: Basu, Guth, and Vilenkin [6,7] showed that circular cosmic strings may nucleate in the end of the de Sitter phase of the evolution of the Universe (see also [8,9]) and thereby avoid to be inflated away from our visible Universe. Later Vilenkin and Garriga [10] considered small perturbations around these nucleated strings both in the end of the de Sitter phase and after having entered the radiation-dominated aera. The evolution of circular cosmic strings in a radiation-dominated universe has also been considered in [11-13]. Furthermore, circular strings have recently been discussed in the context of a more systematic investigation of string dynamics in curved spacetimes [14–18], without considering perturbations around the rings, however. Finally we mention that superconducting charge-current carrying circular strings in black hole backgrounds have been considered in [19–21].

Our work is a natural continuation of the analysis of Loustó and Sánchez [1] and we hope it will give more insight into the connection between curved space-times and string instabilities. We will confirm that in some cases (for instance de Sitter spacetime at $r \rightarrow \infty$ [22,23]) the instabilities are really due to general features of the underlying curved background, while in other cases (for instance Reissner-Nordström black hole for $r \rightarrow 0$) they are just artifacts of the dynamics of the special unperturbed solution considered.

The paper is organized as follows. In Sec. II we will derive the equation of motion for the unperturbed circular string in the Schwarzschild, Reissner-Nordström, and de Sitter backgrounds. Then we use the general formalism of Ref. [3] to obtain the linearized equations determining the physical (transverse) perturbations. For sim-

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plicity we will take only four-dimensional spacetimes; it is of course trivial to include more "angular" space coordinates but concerning the perturbations they will all behave in the same way in our analysis. For both physical perturbations (one "radial" and one "angular") we get Schrödinger-like equations of the form

$$\frac{d^2 f}{d\tau^2}+V(r(\tau))f=0 ,$$

where f is the comoving perturbation, $r(\tau)$ is the radius of the unperturbed circular string, and the string time τ plays the role of the spatial coordinate.

In Sec. III we analyze these equations, taking r as a parameter. From the sign of $V(r)$ we find the regions where we expect that the perturbations develop imaginary frequencies and eventually grow infinitely, in the three backgrounds considered. While in Sec. III we only get indications of the emergence of string instabilities, in Sec. IV we consider the exact time evolution of the perturbations in the regions $r\to 0$ (ring collapse) and $r\to\infty$. This provides a connection between the less strictly obtained results of Sec. III and the question of bounded and/or unbounded perturbations.

The details of our results are presented in Secs. III and IV, and are summarized in Fig. 2. For $r\to 0$ the perturbations in the direction perpendicular to the string plane are bounded in all three backgrounds, while the perturbations in the plane of the string grow infinitely.

For $r\to\infty$ both physical perturbations are bounded in the case of Schwarzschild and Reissner-Nordström black holes (which is not surprising since these space-times are asymptotically flat), but unbounded in the case of de Sitter spacetime. Throughout the paper we use sign conventions of Misner-Thorne-Wheeler [24] and units where $G=1$, $c=1$ and the string tension $(2\pi\alpha')^{-1}=1$.

II. EQUATIONS FOR THE STRING PERTURBATIONS

The classical equations of motion for the bosonic string are in the conformal gauge given by

$$\ddot{x}^{\mu}-x''^{\mu}+\Gamma^{\mu}_{\rho\sigma}(\dot{x}^{\rho}\dot{x}^{\sigma}-x'^{\rho}x'^{\sigma})=0, \tag{2.1}$$

where an overdot and prime denote derivatives with respect to the string coordinates τ and σ, respectively. As usual these equations are supplemented with the two gauge constraints

$$g_{\mu\nu}\dot{x}^{\mu}x'^{\nu}=g_{\mu\nu}(\dot{x}^{\mu}\dot{x}^{\nu}+x'^{\mu}x'^{\nu})=0 . \tag{2.2}$$

In this paper we will consider perturbations around a circular string configuration embedded in Schwarzschild, Reissner-Nordström, and de Sitter spacetimes. In static coordinates these spacetimes are all special cases of the line element,

$$ds^2=-a(r)dt^2+\frac{dr^2}{a(r)}+r^2d\theta^2+r^2\sin^2\theta d\phi^2 , \tag{2.3}$$

so in the first place we will keep $a(r)$ as an arbitrary function. The components of the Christoffel symbol and of the Riemann tensor (that we will need later) corresponding to the metric (2.3) are listed in the Appendix.

Let us first consider the unperturbed circular string. It is obtained by the ansatz

$$t=t(\tau),\quad r=r(\tau),\quad \phi=\sigma,\quad \theta=\pi/2 , \tag{2.4}$$

describing a circular string in the equatorial plane with only one physical mode namely the radial $r(\tau)$. The equations of motion and the constraints (2.1) and (2.2) lead to

$$\begin{aligned}&\ddot{t}+\frac{a_{,r}}{a}\dot{t}\dot{r}=0 ,\\ &\ddot{r}-\frac{a_{,r}}{2a}\dot{r}^2+\frac{1}{2}aa_{,r}\dot{t}^2+ar=0 ,\\ &-a\dot{t}^2+\frac{\dot{r}^2}{a}+r^2=0 .\end{aligned} \tag{2.5}$$

This system of equations can alternatively be formulated as a Hamiltonian system:

$$\mathcal{H}=\frac{1}{2}aP_r^2-\frac{1}{2a}P_t^2+\frac{r^2}{2}\equiv 0 . \tag{2.6}$$

We can then eliminate the cyclic coordinate t:

$$P_t=-a\dot{t}=\text{const}\equiv -E,\quad \text{i.e.,}\ \dot{t}=\frac{E}{a} , \tag{2.7}$$

where E is the "energy" of the string. The radial coordinate r is then determined by

$$\dot{r}^2-E^2+ar^2=0 , \tag{2.8}$$

that is solved by

$$\tau-\tau_0=\pm\int_{r_0}^{r}\frac{dx}{\sqrt{E^2-x^2a(x)}} . \tag{2.9}$$

By inverting this relation for $r(\tau)$ we then obtain $t(\tau)$ by integration of equation (2.7). Note also that the line element (2.3) is now

$$ds^2=r^2(d\sigma^2-d\tau^2) ; \tag{2.10}$$

i.e., the invariant string size is given by $r(\tau)$.

We will now consider the perturbations in the two physical directions normal to the string world sheet. From the two tangent vectors

$$\dot{x}^{\mu}=(\dot{t},\dot{r},0,0),\quad x'^{\mu}=(0,0,0,1) , \tag{2.11}$$

we find the two normal vectors

$$n^{\mu}_{\perp}=\left[0,0,\frac{1}{r},0\right],\quad n^{\mu}_{\parallel}=\left[\frac{\dot{r}}{ar},\frac{\dot{t}a}{r},0,0\right] , \tag{2.12}$$

satisfying the equations

$$g_{\mu\nu}n^{\mu}_{R}x^{\nu}_{,A}=0,\quad g_{\mu\nu}n^{\mu}_{R}n^{\nu}_{S}=\delta_{RS} . \tag{2.13}$$

Here (R,S) takes the values "$\parallel$" and "$\perp$" and $A=(\tau,\sigma)$. Obviously $n^{\mu}_{\perp}$ is perpendicular to the string plane (that is equal to the equatorial plane) while $n^{\mu}_{\parallel}$ is in the string plane. The general physical perturbation can then be expressed as

$$\delta x^{\mu}=n^{\mu}_{\parallel}\delta x_{\parallel}+n^{\mu}_{\perp}\delta x_{\perp} , \tag{2.14}$$

where $\delta x_\parallel$ and $\delta x_\perp$ are the comoving perturbations, i.e., the perturbations as seen by an observer traveling with the unperturbed circular string. In the following we will call these perturbations for the angular perturbation ($\delta x_\perp$) and the radial perturbation ($\delta x_\parallel$), respectively.

According to the general covariant analysis of physical perturbations propagating along strings in curved spacetimes, carried out by Frolov and the author [3], the perturbations are determined by the matrix equation

$$\Box\delta x_R+2\mu_{RS}{}^A(\delta x^S)_{,A}+(\nabla_A\mu_{RS}{}^A)\delta x^S -\mu_{RT}{}^A\mu_S{}^T{}_A\delta x^S+\frac{2}{G^C{}_C}\Omega_R{}^{AB}\Omega_{S,AB}\delta x^S -h^{AB}x^\mu_{,A}x^\nu_{,B}R_{\mu\rho\sigma\nu}n^\rho_R n^\sigma_S\delta x^S=0 \ . \tag{2.15}$$

Here h_{AB} and G_{AB} are the intrinsic and induced metric, respectively, while $\Omega_{R,AB}$ and $\mu_{RS,A}$ are the second fundamental form and normal fundamental form [25], respectively:

$$\Omega_{R,AB}=g_{\mu\nu}n^\mu_R x^\rho_{,A}\nabla_\rho x^\nu_{,B} \ , \tag{2.16}$$

$$\mu_{RS,A}=g_{\mu\nu}n^\mu_R x^\rho_{,A}\nabla_\rho n^\nu_S \ , \tag{2.17}$$

where ∇_ρ is the spacetime covariant derivative. $\Box$ and ∇_A are the world-sheet d'Alambertian and covariant derivative, respectively:

$$\Box=\frac{1}{\sqrt{-h}}\partial_A(\sqrt{-h}\,h^{AB}\partial_B) \ , \tag{2.18}$$

$$\nabla_A\mu_{RS}{}^A=\partial_A\mu_{RS}{}^A+\Gamma^A_{BA}\mu_{RS}{}^B, \text{ etc} \ . \tag{2.19}$$

Finally $R_{\mu\rho\sigma\nu}$ represents the spacetime Riemann tensor.

This extremely complicated system of two coupled linear second-order partial differential equations fortunately simplifies enormously for the special cases considered here. One can show that all components of the normal fundamental form vanish, while the only nonvanishing components of the second fundamental form are

$$\Omega_{\parallel\tau\tau}=\Omega_{\parallel\sigma\sigma}=-E \ . \tag{2.20}$$

The nonvanishing components of the relevant projections of the Riemann tensor become

$$h^{AB}x^\mu_{,A}x^\nu_{,B}R_{\mu\rho\sigma\nu}n^\rho_\perp n^\sigma_\perp=a-1+\frac{r}{2}a_{,r} \ , \tag{2.21}$$

$$h^{AB}x^\mu_{,A}x^\nu_{,B}R_{\mu\rho\sigma\nu}n^\rho_\parallel n^\sigma_\parallel=\frac{r}{2}(ra_{,rr}+a_{,r}) \ . \tag{2.22}$$

Finally the d'Alambertian reduces to (conformal gauge) $\Box=\partial^2_\sigma-\partial^2_\tau$ and $G^A{}_A=h^{AB}G_{AB}=2r^2$. The original system (2.15) now decouples and leads to the two equations

$$(\partial^2_\sigma-\partial^2_\tau)\delta x_\perp-\left[a-1+\frac{r}{2}a_{,r}\right]\delta x_\perp=0 \ ,$$
$$(\partial^2_\sigma-\partial^2_\tau)\delta x_\parallel+\left[2\frac{E^2}{r^2}-\frac{r^2}{2}a_{,rr}-\frac{r}{2}a_{,r}\right]\delta x_\parallel=0 \ . \tag{2.23}$$

These equations can further be reduced to ordinary differential equations by Fourier transforming the comoving perturbations:

$$\delta x_\perp=\sum_{n=-\infty}^{n=+\infty}{}^{\sim} C_{n\perp}(\tau)e^{-in\sigma} \ ,$$
$$\delta x_\parallel=\sum_{n=-\infty}^{n=+\infty}{}^{\sim} C_{n\parallel}(\tau)e^{-in\sigma} \ , \tag{2.24}$$

where $C_{n\perp}=C^*_{-n\perp}$, $C_{n\parallel}=C^*_{-n\parallel}$ and the tilde denotes a summation for $|n|\neq 0,1$ only. The zero modes and the $|n|=1$ modes are excluded from the summations since they do not correspond to "real" perturbations on a circular string [10]. They describe spacetime translations and rotations that do not change the shape of the string. They therefore correspond to simply "jumping" from one unperturbed circular string to another unperturbed circular string. We are then left with the two equations ($|n|\geq 2$):

$$\ddot{C}_{n\perp}+\left[n^2+a-1+\frac{r}{2}a_{,r}\right]C_{n\perp}=0 \ ,$$
$$\ddot{C}_{n\parallel}+\left[n^2+\frac{r^2}{2}a_{,rr}+\frac{r}{2}a_{,r}-2\frac{E^2}{r^2}\right]C_{n\parallel}=0 \ . \tag{2.25}$$

Until now we have kept the function a in the line element (2.3) as an arbitrary function of r. As announced in the abstract we will, however, only consider the three cases of Schwarzschild, Reissner-Nordström, and de Sitter backgrounds. These spacetimes are essentially the scalar curvature flat cases of (2.3) since, from the Appendix, the condition $R=\text{const}\equiv K$ is

$$R=\frac{2}{r^2}(1-a)-4\frac{a_{,r}}{r}-a_{,rr}=K \ , \tag{2.26}$$

that is integrated to

$$a(r)=1+\frac{\alpha}{r}+\frac{\beta}{r^2}-\frac{K}{12}r^2 \ , \tag{2.27}$$

where α and β are constants. This expression covers the cosmologically and gravitationally interesting cases of de Sitter ($\alpha=\beta=0$, $K=12H^2$), Schwarzschild ($\beta=K=0$, $\alpha=-2M$) as well as Reissner-Nordström ($K=0$, $\alpha=-2M$, $\beta=Q^2$) spacetimes. In these cases (2.25) leads to

$$\ddot{C}_{n\perp}+\left[n^2+\frac{\alpha}{2r}-\frac{K}{6}r^2\right]C_{n\perp}=0 \ ,$$
$$\ddot{C}_{n\parallel}+\left[n^2+\frac{\alpha}{2r}+2\frac{\beta}{r^2}-2\frac{E^2}{r^2}-\frac{K}{6}r^2\right]C_{n\parallel}=0 \ , \tag{2.28}$$

and $r(\tau)$ is determined by (2.9):

$$\tau-\tau_0=\pm\int_{r_0}^{r}\frac{dx}{[(K/12)x^4-x^2-\alpha x+(E^2-\beta)]^{1/2}} \ . \tag{2.29}$$

For $K=0$ (the black hole cases) $r(\tau)$ is then a trigonometric function, while for $K\neq 0$ (de Sitter case) it is generally, but not always, elliptic.

III. ANALYSIS OF STRING PERTURBATIONS

In this section we analyze the equations for the perturbations (2.28) taking r as the parameter. Both equations are Schrödinger-like equations of the form $\ddot{f}+V(r)f=0$, and we will then say that the solutions are oscillatory in time τ if $V(r)$ is positive but nonoscillatory in time τ, developing imaginary frequencies, if $V(r)$ is negative. We are using the words oscillatory and nonoscillatory in a weak (and sloppy) sense, that should not be confused with the more strict use of the words in the mathematical literature, where (say) oscillatory behavior usually means that a function has infinitely many zeroes. Note also that there is no simple one-to-one correspondence between oscillatory (nonoscillatory) and boundedness (unboundedness) of the solutions. This can for instance be seen from the simple example

$$\ddot{f}+\frac{1}{4\tau^2}f=0\ .$$

In this case we would say that the solutions are oscillatory on the positive half axis, but the general solution

$$f(\tau)=A\sqrt{\tau}+B\sqrt{\tau}\ln\tau\ ,$$

is in fact neither oscillatory in the strict mathematical sense nor is bounded. In Sec. IV we will see, however, that in most of the cases considered here nonoscillatory behavior of the perturbations, developing imaginary frequencies, will actually lead to unbounded solutions, indicating that the underlying unperturbed string configuration is unstable, in agreement with the string instability characterization given by Loustó and Sánchez [1].

It should be stressed also that we are still only talking about the evolution of the comoving string perturbations. The transformation to the perturbations as seen by an observer at rest is a highly nontrivial problem, that to our knowledge has only been done in the case of flat Minkowski space [10]. In this paper we consider curved spacetimes where everything is of course somewhat more complicated.

Let us now consider the perturbations in the three cases.

A. Schwarzschild black hole

In this case the unperturbed string solution is (2.29):

$$r(\tau)=M+\sqrt{M^2+E^2}\cos\tau\ ; \tag{3.1}$$

i.e., the string has its maximal radius $r_{\text{max}}=M+\sqrt{M^2+E^2}$ at $\tau=0$. It then contracts through the horizon $r_{\text{hor}}=2M$ for

$$\tau=\arccos(M/\sqrt{M^2+E^2})\in\ ]0,\pi/2[,$$

and eventually falls into the singularity $r=0$ for

$$\tau=\arccos(-M/\sqrt{(M^2+E^2)})\in\ ]\pi/2,\pi[\ .$$

Mathematically speaking it of course continues oscillating but for our purposes we only consider the process of collapse from the maximal radius; this somehow resembles the radial infall of a point particle. The collapse into the Schwarzschild singularity of course takes infinite coordinate time. This can be explicitly seen by integrating (2.7) outside the horizon:

$$t=E\tau+2M\ln\left|\frac{\tan(\tau/2)+\delta}{\tan(\tau/2)-\delta}\right|\ , \tag{3.2}$$

where $\delta\equiv(\sqrt{M^2+E^2}-M)/E$. It follows that $t(0)=0$ and $t[\arccos(M/\sqrt{M^2+E^2})]=\infty$. We note in passing that the expression (3.2) is remarkably simple as compared to the corresponding relation for the radial infall of a point particle; see for instance Ref. [24]. From the physical point of view we are mostly interested in the circular string and its perturbations outside the horizon, but considering our whole analysis as a stability analysis of some special solutions to some nonlinear differential equations, there is no reason not to continue the solutions all the way through the horizon into the singularity at $r=0$.

The equations determining the modes of $\delta x_\perp$ and $\delta x_\parallel$, respectively, are

$$\begin{aligned}\ddot{C}_{n\perp}+\left[n^2-\frac{M}{r}\right]C_{n\perp}&=0\ ,\\ \ddot{C}_{n\parallel}+\left[n^2-\frac{M}{r}-2\frac{E^2}{r^2}\right]C_{n\parallel}&=0\ .\end{aligned} \tag{3.3}$$

According to the discussion at the beginning of Sec. III $\delta x_\perp$ is oscillatory in time for $r>M/4$, i.e., the first mode ($|n|=2$) becomes nonoscillatory at $r=M/4$. The higher modes become nonoscillatory for smaller and smaller r and for $r=0$ $\delta x_\perp$ is extremely nonoscillatory (nonoscillatory for all modes). $\delta x_\parallel$ is oscillatory in time for $r>(M+\sqrt{M^2+32E^2})/8$ and the picture is then similar to $\delta x_\perp$ for smaller and smaller r. Note that $(M+\sqrt{M^2+32E^2})/8<r_{\text{max}}=M+\sqrt{M^2+E^2}$ so that both $\delta x_\perp$ and $\delta x_\parallel$ are oscillatory when the string is near its maximal size. $\delta x_\perp$ is also oscillatory at the horizon $r=2M$ while $\delta x_\parallel$ is oscillatory (nonoscillatory) at the horizon for $E^2<7M^2$ ($E^2>7M^2$).

B. Reissner-Nordström black hole

In this case the unperturbed string solution is (2.29),

$$r(\tau)=M+\sqrt{M^2+E^2-Q^2}\cos\tau;\quad M^2\geq Q^2\ , \tag{3.4}$$

and we now have to distinguish between two different cases namely $E^2\geq Q^2$ and $E^2<Q^2$.

(i) $E^2\geq Q^2$. The string has its maximal radius $r_{\text{max}}=M+\sqrt{M^2+E^2-Q^2}$ for $\tau=0$. It first contracts through the horizon $r_+=M+\sqrt{M^2-Q^2}$ for $\tau=\arccos(\sqrt{M^2-Q^2}/\sqrt{M^2+E^2-Q^2})\in\ ]0,\pi/2[$ and then through the inner horizon $r_-=M-\sqrt{M^2-Q^2}$ for $\tau=\arccos(-\sqrt{M^2-Q^2}/\sqrt{M^2+E^2-Q^2})\in\ ]\pi/2,\pi[$. It eventually falls into the Reissner-Nordström singularity $r=0$ for $\tau=\arccos(-M/\sqrt{M^2+E^2-Q^2})\in\ ]\pi/2,\pi]$.

(ii) $E^2<Q^2$. The dynamics of the unperturbed string is similar to case (i) all the way from r_{max} through the two

horizons, but in this case the string reaches a *minimal* size $r_{\rm min}=M-\sqrt{M^2+E^2-Q^2}>0$ for $\tau=\pi$.

The equations determining the modes of $\delta x_\perp$ and $\delta x_\parallel$, respectively, are

$$\ddot{C}_{n\perp}+\left[n^2-\frac{M}{r}\right]C_{n\perp}=0\ ,$$

$$\ddot{C}_{n\parallel}+\left[n^2-\frac{M}{r}+\frac{2}{r^2}(Q^2-E^2)\right]C_{n\parallel}=0\ . \tag{3.5}$$

Formally the $C_{n\perp}$ equation is identical to the $C_{n\perp}$ equation in the Schwarzschild case, but one should remember that in this case the charge of course is present through the expression for $r(\tau)$.

We now find that $\delta x_\perp$ is oscillatory for $r>M/4$, i.e., the first mode ($|n|=2$) becomes nonoscillatory at $r=M/4$. The higher modes become nonoscillatory for smaller r and for $r\to 0$ $\delta x_\perp$ is extremely nonoscillatory. At first sight this seems to be similar to the Schwarzschild case, but we should take into account that the dynamics of the unperturbed circular string is very different here: string (i) is oscillatory at the maximal radius and near the horizon. Near the inner horizon it is oscillatory (nonoscillatory) for $Q^2>7M^2/16$ ($Q^2<7M^2/16$). For string (ii) the discussion is similar from the maximal radius through the two horizons. However, string (ii) has a minimal radius and it follows that if $Q^2-E^2>7M^2/16$ then it is actually *always* oscillatory (since $M/4<r_{\rm min}$).

For the radial perturbations it turns out that the situation is quite complicated. We find that the first mode ($|n|=2$) is oscillatory outside the interval $]M/8-[M^2/64-(Q^2-E^2)/2]^{1/2},\ M/8+[M^2/64-(Q^2-E^2)/2]^{1/2}[$. The higher modes are oscillatory outside smaller and smaller subintervals. It follows that $\delta x_\parallel$ is always oscillatory near the maximal radius, and for $Q^2-E^2\geq M^2/32$ [this is only possible for string (ii)] it is oscillatory for all r. The exact location of the nonoscillatory interval as compared to the two horizons and to $r_{\rm min}$ (for $E^2<Q^2$) is very complicated since we have two parameters (E^2 and Q^2) to play with, so almost all situations are possible. The result is most easily visualized by Fig. 1 accompanied by the following comments.

In region (i1) we have $E^2>Q^2$ and $\delta x_\parallel$ is nonoscillatory at the (outer) horizon and all the way towards the singularity $r=0$.

In region (i2) we have $E^2>Q^2$ and $\delta x_\parallel$ is oscillatory at the horizon but becomes nonoscillatory before the inner horizon, from which it is nonoscillatory all the way towards the singularity $r=0$.

In region (i3) we have $E^2>Q^2$ and $\delta x_\parallel$ is oscillatory at both horizons but becomes nonoscillatory for $r\to 0$.

In (ii1) we have $-M^2/32<E^2-Q^2<0$. In this narrow band (represented by a thick line in Fig. 1) $\delta x_\parallel$ is oscillatory at the horizon and near the minimal radius, but there is a nonoscillatory region between the horizon and $r_{\rm min}$.

In (ii2) we have $E^2-Q^2\leq -M^2/32$ and the string is oscillatory for all r, so that is a very interesting region.

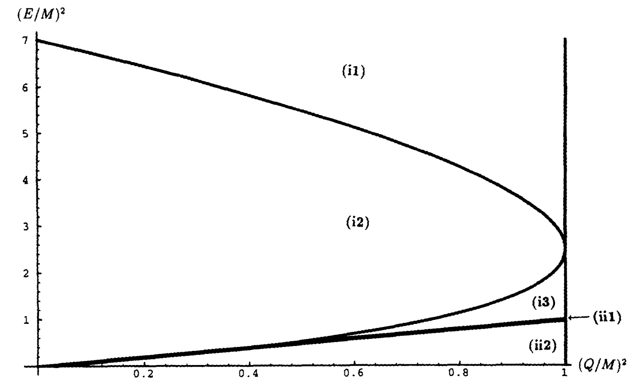

FIG. 1. The location of the critical radius where the nonoscillatory behavior sets in for the $|n|=2$ mode in the case of Reissner-Nordström black hole. The details of this figure are explained at the end of Sec. III B. Note that we only consider $Q^2\leq M^2$.

C. de Sitter spacetime

In this case $r(\tau)$ is in general given by a Weierstrass elliptic function. The detailed dynamics of unperturbed circular strings has been discussed elsewhere [6,8,15,16,18] so we shall not go into it here. We will consider only the following three types of solutions, whose existence is clear from (2.8) when $a=1-H^2r^2$.

(i) For $4H^2E^2\leq 1$ there is a solution starting with a maximal radius $r^2_{\rm max}=(1-\sqrt{1-4H^2E^2})/2H^2$ and then collapsing to $r=0$. It is always inside the horizon $r_{\rm hor}=1/H$.

(ii) Still for $4H^2E^2\leq 1$ there is another solution starting with a minimal radius $r^2_{\rm min}=(1+\sqrt{1-4H^2E^2})/2H^2$ and then expanding through the horizon towards infinity.

(iii) For $4H^2E^2>1$ there is a solution starting at $r=0$ and then expanding through the horizon towards infinity.

The equations for the modes of $\delta x_\perp$ and $\delta x_\parallel$, respectively, are

$$\ddot{C}_{n\perp}+(n^2-2H^2r^2)C_{n\perp}=0\ ,$$

$$\ddot{C}_{n\parallel}+\left[n^2-2H^2r^2-2\frac{E^2}{r^2}\right]C_{n\parallel}=0\ . \tag{3.6}$$

Let us consider the three configurations described above one by one.

(i) $\delta x_\perp$ is oscillatory for $r^2<2/H^2$; i.e., it is always oscillatory since $r^2_{\rm max}<2/H^2$. On the other hand we find that $\delta x_\parallel$ is oscillatory in the interval $r^2\in\,](1-\sqrt{1-E^2H^2})/H^2,\ (1+\sqrt{1-E^2H^2})/H^2[$; i.e., the first mode ($|n|=2$) becomes nonoscillatory at the boundaries of this interval. The higher modes are oscillatory in larger and larger intervals containing the above interval. Note that $(1+\sqrt{1-E^2H^2})/H^2>r^2_{\rm max}>(1-\sqrt{1-E^2H^2})/H^2$ so that $\delta x_\parallel$ is always oscillatory when the string has its maximal size. On the other hand $\delta x_\parallel$ is extremely nonoscillatory for $r\to 0$.

(ii) The critical string radii are the same as in case (i), but the dynamics of the unperturbed string is completely different. We have that $r^2_{\rm min}<2/H^2$ so that $\delta x_\perp$ is oscillatory at $r_{\rm min}$ and at the horizon, but it is nonoscillatory from $r^2=2/H^2$ towards infinity. For $\delta x_\parallel$ the picture is more or less the same. We find that $(1+\sqrt{1-E^2H^2})/H^2 > r^2_{\rm hor} > r^2_{\rm min} > (1-\sqrt{1-E^2H^2})/H^2$ so that $\delta x_\parallel$ is oscillatory at $r_{\rm min}$ and at the horizon, but it is nonoscillatory from $r^2=(1+\sqrt{1-E^2H^2})/H^2$ towards infinity. Note that the nonoscillatory behavior of $\delta x_\parallel$ sets in a little before $\delta x_\perp$, but in both cases it is outside the horizon, and of course for higher and higher modes the nonoscillatory behavior sets in for larger and larger r.

(iii) Again the critical radii are the same as in the other two cases. $\delta x_\perp$ is oscillatory from $r=0$ through the horizon to $r^2=2/H^2$ where the first ($|n|=2$) nonoscillatory behavior sets in. It is then nonoscillatory all the way towards infinity. The higher modes become nonoscillatory for larger r. Finally $\delta x_\parallel$ is oscillatory in the interval $r^2 \in](1-\sqrt{1-E^2H^2})/H^2,\ (1+\sqrt{1-E^2H^2})/H^2[$ surrounding the horizon, but it is extremely nonoscillatory for both $r\to 0$ and $r\to\infty$.

In de Sitter space there is also a stationary circular string solution [3,15,16]. From (2.8) follows that it is described by $4E^2H^2=1$, $2H^2r^2=1$. The stability of this solution was already considered in [3] in a different gauge, so let us restate the result here. The equations for the modes of $\delta x_\perp$ and $\delta x_\parallel$, respectively, are (3.6):

$$\ddot{C}_{n\perp}+(n^2-1)C_{n\perp}=0\ ,\qquad \ddot{C}_{n\parallel}+(n^2-2)C_{n\parallel}=0\ . \tag{3.7}$$

So, as already stated in [3], we get instabilities for the $|n|=1$ modes (and for the zero modes). However, as explained after Eq. (2.24), we do not consider these modes as "real" perturbations [10] since they just correspond to "jumping" from one unperturbed string to another one, without changing shape. It may be a matter of taste whether or not to include the $|n|=1$ modes, but in any case we get the somewhat surprising result that the stationary circular string solution in de Sitter spacetime is actually stable against "real" perturbations where $|n|\geq 2$.

This concludes our investigations of oscillatory and nonoscillatory (in the weak sense considered here) behavior of the comoving perturbations around a circular string in the three backgrounds of Schwarzschild, Reissner-Nordström, and de Sitter spacetimes. In the next section we will relate some of these results to the question of bounded or unbounded comoving perturbations.

IV. TIME-EVOLUTION AND ASYMPTOTIC BEHAVIOR

In this section we address the question of bounded or unbounded comoving perturbations by considering the time evolution of some of the solutions found in Sec. III; i.e., we now take τ as the parameter. It is clear from the general equations determining the perturbations (2.28) that we can only expect unbounded behavior of the solutions in the 2 regions $r\to 0$ and $r\to\infty$, so we will restrict ourselves by considering the solutions in these regions only. It is also clear that in the two cases of black holes we need only consider $r\to 0$ since for $r\to\infty$ (that can of course only be obtained for $E^2\to\infty$, i.e., infinite string energy) we have Minkowski space, where the perturbations are obviously bounded; they are just ordinary plane waves. For de Sitter spacetime, on the other hand, we can get unbounded perturbations for both $r\to 0$ and $r\to\infty$.

In all cases it will turn out that the behavior of the perturbations in the asymptotic regions corresponds to different cases of the motion of a particle in the potential $\alpha(\tau-\tau_0)^{-\beta}$ [1,26,27], in the sense that they are described by a stationary Schrödinger equation with τ playing the role of the spatial parameter. It is an elementary observation that if $\alpha<0$ and $\beta\geq 2$ there are singular solutions for $\tau\to\tau_0$. Therefore, as soon as we have obtained the potential with the two parameters α and β we can conclude whether the perturbations blow up, indicating that the underlying circular string is unstable. For completeness we will, however, give the full solutions in the asymptotic regions, demonstrating explicitly if and how the perturbations blow up.

A. Schwarzschild black hole

In this case we have $r(\tau)=M+\sqrt{M^2+E^2}\cos\tau$ with $r\to 0$ corresponding to $\tau\to\tau_0\equiv\arccos(-M/\sqrt{M^2+E^2})$ from below (cf. Sec. III A). For $r\to 0$ we then have approximately

$$r(\tau)\approx |E|(\tau_0-\tau)\ , \tag{4.1}$$

and the two equations determining the perturbations (3.3) become approximately

$$\ddot{C}_{n\perp}-\frac{M}{|E|(\tau_0-\tau)}C_{n\perp}=0\ ,\qquad \ddot{C}_{n\parallel}-\frac{2}{(\tau_0-\tau)^2}C_{n\parallel}=0\ . \tag{4.2}$$

Let us first consider the perturbations in the angular direction (the $C_{n\perp}$'s). Keeping in mind that $\tau_0-\tau$ is positive in the relevant range of τ we find the two real independent solutions in terms of Bessel functions [28]:

$$f=\sqrt{\tau_0-\tau}\,J_1(2\sqrt{M/|E|}\sqrt{\tau_0-\tau})\ ,\qquad g=\sqrt{\tau_0-\tau}\,N_1(2\sqrt{M/|E|}\sqrt{\tau_0-\tau})\ . \tag{4.3}$$

The most interesting feature of these solutions is that they are actually bounded [28]:

$$f\to\sqrt{M/|E|}(\tau_0-\tau),\quad g\to -\frac{1}{\pi}\sqrt{|E|/M}\ ,\quad \tau\to\tau_0\ . \tag{4.4}$$

This therefore provides an example where the solutions were classified as nonoscillatory (according to Sec. III A), but where the actual time evolution demonstrates that the solutions are bounded, and they are in fact oscillatory in the strict mathematical sense of having infinitely many zeros. This is, however, an exceptional case; in the other cases under consideration here we will find that nonoscillatory behavior at $r\to 0$ or $r\to\infty$ leads to unbounded solutions.

For the perturbations in the radial direction (the $C_{n\parallel}$'s) we find the complete solution

$$C_{n\parallel}(\tau)=\alpha_n(\tau_0-\tau)^2+\frac{\beta_n}{\tau_0-\tau}\ , \tag{4.5}$$

where α_n and β_n are arbitrary constants. This solution is indeed unbounded for $\tau\rightarrow\tau_0$.

B. Reissner-Nordström black hole

Here we only consider the case where $E^2\geq Q^2$ to ensure that we have solutions collapsing to $r=0$. Then the unperturbed string is determined by $r(\tau)=M+\sqrt{M^2+E^2-Q^2}\cos\tau$ and $r\rightarrow 0$ corresponds to $\tau\rightarrow\tau_0\equiv\arccos(-M/\sqrt{M^2+E^2-Q^2})$ from below (cf. Sec. III B). The approximate solution for $r\rightarrow 0$ is then

$$r(\tau)\approx\sqrt{E^2-Q^2}(\tau_0-\tau),\quad E^2>Q^2 \tag{4.6}$$

and

$$r(\tau)\approx\frac{M}{2}(\tau_0-\tau)^2,\quad E^2=Q^2\ . \tag{4.7}$$

The two equations determining the perturbations (3.5) become

$$\ddot{C}_{n\perp}-\frac{M}{\sqrt{E^2-Q^2}(\tau_0-\tau)}C_{n\perp}=0,\quad E^2>Q^2\ , \tag{4.8}$$

$$\ddot{C}_{n\parallel}-\frac{2}{(\tau_0-\tau)^2}C_{n\parallel}=0,\quad E^2>Q^2\ , \tag{4.9}$$

and

$$\ddot{C}_{nR}-\frac{2}{(\tau_0-\tau)^2}C_{nR}=0,\quad E^2=Q^2\ . \tag{4.10}$$

Equation (4.8) is solved by (4.3) with $|E|$ replaced by $\sqrt{E^2-Q^2}$, and the solution is bounded. Equations (4.9) and (4.10) are solved in the form (4.5), so the solutions blow up for $\tau\rightarrow\tau_0$. Notice that for $E^2=Q^2$ the perturbations in the two physical directions are determined by exactly the same equation.

C. de Sitter spacetime

Finally, we come to the de Sitter case, and we first consider the asymptotic region $r\rightarrow\infty$, i.e., we consider the strings (ii) and (iii) of Sec. III C. The asymptotic behavior of the unperturbed string is most easily found directly from the equation of motion (2.8). For $a=1-H^2r^2$ and $r\rightarrow\infty$ we find

$$\dot{r}^2-H^2r^4\approx 0\ , \tag{4.11}$$

so that, for an expanding solution ($\dot{r}\geq 0$),

$$r(\tau)\approx\frac{1}{H(\tau_0-\tau)}\ , \tag{4.12}$$

for some constant τ_0, and $r\rightarrow\infty$ corresponds to $\tau\rightarrow\tau_0$ from below. The two equations determining the perturbations (3.6) become

$$\ddot{C}_{n\perp}-\frac{2}{(\tau_0-\tau)^2}C_{n\perp}=0\ ,\qquad \ddot{C}_{n\parallel}-\frac{2}{(\tau_0-\tau)^2}C_{n\parallel}=0\ . \tag{4.13}$$

It follows that they both blow up in exactly the same way as $C_{n\parallel}(\tau)$ blows up at $r\rightarrow 0$ in the Schwarzschild case; compare with (4.2) and (4.5).

In the region $r\rightarrow 0$ (considering now the unperturbed strings (i) and (ii) of Sec. III C) we find from (2.8) the asymptotic behavior $\dot{r}^2\approx E^2$. For a collapsing string ($\dot{r}\leq 0$) this leads to

$$r(\tau)\approx|E|(\tau_0-\tau)\ , \tag{4.14}$$

for some constant τ_0, and $r\rightarrow 0$ corresponds to $\tau-\tau_0$ from below. The two equations determining the perturbations (3.6) in this limit become

$$\ddot{C}_{n\perp}+n^2C_{n\perp}=0\ ,\qquad \ddot{C}_{n\parallel}-\frac{2}{(\tau_0-\tau)^2}C_{n\parallel}=0\ . \tag{4.15}$$

Obviously $C_{n\perp}(\tau)$ is finite while $C_{n\parallel}(\tau)$ blows up in the same way as for $r\rightarrow\infty$.

V. CONCLUSION

In conclusion we have studied the comoving perturbations around circular strings embedded in the curved spacetimes of Schwarzschild, Reissner-Nordström, and de Sitter backgrounds. The results of our analysis are summarized in Fig. 2, the details are presented in Secs. III and IV. The main conclusions were already drawn in the Introduction but let us say a few more words here. Our results for the perturbations on the circular strings in de Sitter spacetime in the asymptotic region $r\rightarrow\infty$ confirm the results of Loustó and Sánchez [1] and also the results of Gasperini, Sánchez, and Veneziano [22,23] for highly unstable strings. For the black holes we get the same results as Loustó and Sánchez in the Schwarzschild case, while in the Reissner-Nordström case the results are different for the angular perturbations. This is, however, a "normal" situation; some special solutions of a nonlinear differential equation are stable and some are unstable. In this sense the de Sitter spacetime at $r\rightarrow\infty$ provides an exceptional case.

Region	Mode	Schwarzschild	Reissner-Nordström	de Sitter
$r\rightarrow 0$	$\delta x_\perp$	bounded	bounded	bounded
$r\rightarrow 0$	$\delta x_\parallel$	unbounded	unbounded	unbounded
$r\rightarrow\infty$	$\delta x_\perp$	bounded	bounded	unbounded
$r\rightarrow\infty$	$\delta x_\parallel$	bounded	bounded	unbounded

FIG. 2. This diagram summarizes the results obtained in this paper. $\delta x_\perp$ and $\delta x_\parallel$ corresponds to the comoving angular and radial perturbations, respectively.

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APPENDIX

In this Appendix we give the explicit expressions for the nonvanishing components of the Christoffel symbol and Riemann tensor corresponding to the line element (2.3).

The metric is

$$g_{tt}=-a,\quad g_{rr}=\frac{1}{a},\quad g_{\theta\theta}=r^2,\quad g_{\phi\phi}=r^2\sin^2\theta,\quad a=a(r)\ . \tag{A1}$$

The Christoffel symbol is

$$\Gamma^t_{tr}=\frac{a_{,r}}{2a},\quad \Gamma^r_{rr}=\frac{-a_{,r}}{2a},\quad \Gamma^r_{tt}=\frac{1}{2}aa_{,r}\ ,$$
$$\Gamma^r_{\theta\theta}=-ar,\quad \Gamma^r_{\phi\phi}=-ar\sin^2\theta,\quad \Gamma^\phi_{\phi r}=\frac{1}{r}\ , \tag{A2}$$
$$\Gamma^\phi_{\phi\theta}=\cot\theta,\quad \Gamma^\theta_{\theta r}=\frac{1}{r},\quad \Gamma^\theta_{\phi\phi}=-\sin\theta\cos\theta\ .$$

The Riemann tensor is

$$R_{rtrt}=\frac{1}{2}a_{,rr},\quad R_{r\theta r\theta}=\frac{-r}{2a}a_{,r},\quad R_{r\phi r\phi}=\frac{-r}{2a}a_{,r}\sin^2\theta\ ,$$
$$R_{t\theta t\theta}=\frac{r}{2}aa_{,r},\quad R_{t\phi t\phi}=\frac{r}{2}aa_{,r}\sin^2\theta, \tag{A3}$$
$$R_{\theta\phi\theta\phi}=r^2(1-a)\sin^2\theta\ .$$

The Ricci tensor is

$$R_{tt}=-a^2R_{rr}=a\left[\frac{a_{,rr}}{2}+\frac{a_{,r}}{r}\right],$$
$$R_{\phi\phi}=\sin^2\theta R_{\theta\theta}=(1-a-ra_{,r})\sin^2\theta\ . \tag{A4}$$

Finally the scalar curvature is

$$R=-a_{,rr}+\frac{2}{r^2}(1-a)-\frac{4}{r}a_{,r}\ . \tag{A5}$$

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