Reduced Hamiltonian descriptions

Henry E. Kandrup*
Department of Astronomy and Department of Physics and Institute for Fundamental Theory, University of Florida, Gainesville, Florida 32611
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Suppose that the evolution of some large, isolated system is Hamiltonian, but that one is interested only in the evolution of a smaller piece of the system, i.e., a subsystem. Alternatively, suppose that one is interested in the evolution of two halves of a single Hamiltonian system, but that one is not interested in the details of their mutual interaction. It is well known that, in either case, if the degrees of freedom of the two pieces are not completely decoupled, an exact reduced description of the individual components cannot be Hamiltonian. It is, however, shown here that, if one only allows for the *average* effects of each piece on the other, neglecting detailed correlations between the two components in a generalized self-consistent field approximation, one is always led to an approximate reduced description which *is* Hamiltonian.

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I. INTRODUCTION

It is, perhaps, reasonable to expect that the "fundamental" description of an isolated, or nearly isolated, system, e.g., the stars within a galaxy, or the Universe as a whole, is Hamiltonian. However, for any of a number of reasons, one may not be interested in all of this isolated system. One might, for example, be concerned only with some small piece of the system, the "subsystem," interacting with the remainder of the system, which serves as an environment or bath." Alternatively, one might be concerned with the two separate halves of some interacting system, but not with the details of their mutual interaction.

Concrete examples of interest in astrophysics and cosmology might include the following: (1) the evolution of a single dilaton mode during the epoch of inflation, which is coupled somehow to the rest of the Universe, which plays the role of a bath; (2) the coupling of some mode of a quantum field to the rest of the Universe, and the problem of decoherence; (3) the motion of a single test star within a galaxy comprised of many other field stars; (4) the coupling of some galaxy of a surrounding environment, e.g., the other galaxies within some rich cluster; (5) the interaction of charged particles via an electromagnetic field, where one is interested in the distribution functions for the particles and for the electromagnetic field, but not in correlations between the particles and the field.

It is well known that, if one considers a reduced description for a subsystem, or for the two halves of a single interacting system, that description will no longer be Hamiltonian [1]. This is important because a Hamiltonian evolution, which incorporates conservation of phase (i.e., Liouville's theorem), is much more constrained than a non-Hamiltonian evolution, which will in general be dissipative (or antidissipative) [2]. Whether or not a Hamiltonian description is desirable may depend on the physical setting which one envisions. However, it is of some interest to determine quite generally whether there exists some reasonable, controlled approximation in which the exact reduced description can be approximated as Hamiltonian.

The objective of this paper is to answer that question in the affirmative. Specifically, it will be shown quite generally that, if one implements a generalized self-consistent field approximation (S.C.F.A.), allowing only for the "average" interactions between the pieces of the system, one can always extract a reduced description that is Hamiltonian. In other words, *every mean field description of a Hamiltonian system is itself Hamiltonian.*

That this is the case will be demonstrated by an explicit construction. The starting point is the formulation of a general Hamiltonian appropriate for two interacting systems, written in terms of the basic physical degrees of freedom. These degrees of freedom can be more or less anything, from individual particles to collective degrees of freedom to the modes of a field. For specificity, in what follows they will be denoted "particles." For simplicity, it will also be assumed that the degrees of freedom are countable, but this is not essential, at least formally.

Given such a Hamiltonian, the analysis will be formulated in terms of a statistical description, the fundamental object being taken as a classical many-particle distribution function, which is assumed to satisfy an appropriate many-particle Liouville equation [3]. However, one could work equally well in terms of a quantum wave function that satisfies a quantum Liouville equation [4] or a many-particle Wigner function which satisfies a generalized transport equation [5]. In each case, the basic conclusion would be the same.

The content of the self-consistent field approximation, which leads to a Hamiltonian description, is simply that

*Electronic address: kandrup@astro.ufl.edu

the many-particle distribution function for the two pieces of the composite system may be approximated as factorizing into a product of reduced distribution functions for the two pieces. It is therefore clear that the construction can also be generalized trivially to the case of more than two interacting systems. For example, the standard Vlasov-Poisson system of plasma physics or galactic dynamics [6] corresponds to a special case where one considers each of N particles as an individual system and treats the interaction between the particles in a self-consistent field approximation.

II. CONSTRUCTION

Let $\{q_A,p_A\}$ denote canonical coordinates for the "subsystem" and $\{x_i,v_i\}$ canonical coordinates for the "bath." The fundamental object in a statistical description is then an $(N+n)$-particle distribution function μ, so defined that

$$\mu(q_1,p_1,\ldots,x_1,v_1,\ldots,t)dq_1dp_1\cdots dq_Ndp_Ndx_1 \\ dv_1\cdots dx_ndv_n\equiv\mu(\{q_A,p_A\},\{x_i,v_i\})d\Gamma_q d\Gamma_x \quad (1)$$

represents the joint probability of finding the subsystem particles in elements dp_Adq_A around specified points (q_A,p_A) and the bath particles in elements dx_idv_i around points (x_i,v_i). The quantities $d\Gamma_q$ and $d\Gamma_x$ denote, respectively, the natural phase space volume elements for the system and bath; i.e.,

$$d\Gamma_q=\prod_{A=1}^{N}dq_Adp_A \quad \text{and} \quad d\Gamma_x=\prod_{i=1}^{N}dx_idv_i\ . \quad (2)$$

The evolution of μ is assumed generated from a Hamiltonian

$$H=H_S(\{q_A,p_A\},t)+H_B(\{x_i,v_i\},t) \\ +H_I(\{q_A,p_A\},\{x_I,v_i\},t)\ , \quad (3)$$

so that the $(N+n)$-particle Liouville equation takes the form

$$\frac{\partial\mu}{\partial t}+\sum_{A=1}^{N}\left[\frac{\partial H}{\partial p_A}\frac{\partial\mu}{\partial q_A}-\frac{\partial H}{\partial q_A}\frac{\partial\mu}{\partial p_A}\right] \\ +\sum_{i=1}^{n}\left[\frac{\partial H}{\partial v_i}\frac{\partial\mu}{\partial x_i}-\frac{\partial H}{\partial x_i}\frac{\partial\mu}{\partial v_i}\right]=\frac{\partial\mu}{\partial t}-[H,\mu]=0\ , \quad (4)$$

where

$$[a,b]=\sum_{i=1}^{N}\left[\frac{\partial a}{\partial q_A}\frac{\partial b}{\partial p_A}-\frac{\partial a}{\partial p_A}\frac{\partial b}{\partial q_A}\right] \\ +\sum_{i=1}^{N}\left[\frac{\partial a}{\partial x_i}\frac{\partial b}{\partial v_i}-\frac{\partial a}{\partial v_i}\frac{\partial b}{\partial x_i}\right] \quad (5)$$

denotes Poisson brackets.

Although μ is the fundamental object in the statistical description, it is assumed that one is actually more interest in the evolution of f and g, the reduced distribution functions for the subsystem and bath, which are defined respectively by integrating over the degrees of freedom of the bath and the subsystem:

$$f(\{q_A,p_A\},t)\equiv\int d\Gamma_x\mu(\{q_A,p_A\},\{x_i,v_i\},t) \quad (6)$$

and

$$g(\{x_i,v_i\},t)\equiv\int d\Gamma_q\mu(\{q_A,p_A\},\{x_i,v_i\},t)\ . \quad (7)$$

It is straightforward to derive equations for $\partial f/\partial t$ and $\partial g/\partial t$. Thus, for example, by integrating Eq. (4) over the degrees of freedom of the bath, one finds that

$$\frac{\partial f}{\partial t}+\sum_{A=1}^{N}\left[\frac{\partial H_S}{\partial p_A}\frac{\partial f}{\partial q_a}-\frac{\partial H_S}{\partial q_A}\frac{\partial f}{\partial p_A} \right. \\ \left. +\int d\Gamma_a\left[\frac{\partial H_I}{\partial p_A}\frac{\partial\mu}{\partial q_A}-\frac{\partial H_I}{\partial q_A}\frac{\partial\mu}{\partial p_A}\right]\right]=0\ , \quad (8)$$

and, similarly, an integration of the degrees of freedom of the subsystem yields

$$\frac{\partial g}{\partial t}+\sum_{i=1}^{n}\left[\frac{\partial H_B}{\partial v_i}\frac{\partial g}{\partial x_i}-\frac{\partial H_B}{\partial x_i}\frac{\partial g}{\partial v_i} \right. \\ \left. +\int d\Gamma_q\left[\frac{\partial H_I}{\partial v_i}\frac{\partial\mu}{\partial x_i}-\frac{\partial H_I}{\partial x_i}\frac{\partial\mu}{\partial v_i}\right]\right]=0\ . \quad (9)$$

In general, Eqs. (8) and (9) do not yield simply a coupled system for the reduced distributions f and g, because the full many-particle μ does not factorize exactly into a product fg. Rather, there will exist nontrivial correlations between the subsystem and the bath. To make this explicit, it is convenient to write μ in the form

$$\mu(\{q_A,p_A\},\{x_i,v_i\},t) \\ \equiv f(\{q_A,p_A\},t)g(\{x_i,v_i\},t)+\mu_c\ . \quad (10)$$

The physical content of this decomposition is immediate: If there exist no correlations between the subsystem and bath, the joint probability for the subsystem to have specified $\{q_A,p_A\}$ and for the bath to have specified $\{x_i,v_i\}$ will be given as the product fg of the individual probabilities f and g. If, however, correlations are present, the joint probability will not factorize exactly, and one can define μ_c as representing the statistical effects of these correlations on the joint probabilities.

If H_I vanishes identically, it is mathematically consistent to assume a complete absence of correlations: If no correlations are present initially, none will ever be generated by the dynamics. However, if H_I is nonzero, even a state without initial correlations will immediately develop correlations because of the coupling between subsystem and bath.

By writing μ in the form (10), Eqs. (8) and (9) can be rewritten in a fashion which exhibits explicitly the effects of the correlations. Specifically, one has

$$\frac{\partial f(\{q_A,p_A\},t)}{\partial t}+\sum_{A=1}^{N}\left[\frac{\partial H_S}{\partial p_A}\frac{\partial f}{\partial q_A}-\frac{\partial H_A}{\partial q_A}\frac{\partial f}{\partial p_A}+\frac{\partial\langle H_I(q,p)\rangle}{\partial p_A}\frac{\partial f}{\partial q_A}-\frac{\partial\langle H_I(q,p)\rangle}{\partial q_A}\frac{\partial f}{\partial p_A}\right]$$
$$=-\sum_{A=1}^{N}\int d\Gamma_x\left[\frac{\partial H_I}{\partial p_A}\frac{\partial\mu_c}{\partial q_A}-\frac{\partial H_I}{\partial q_A}\frac{\partial\mu_c}{\partial p_a}\right]=0 \quad (11)$$

and

$$\frac{\partial g(\{x_i,v_i\},t)}{\partial t}+\sum_{i=1}^{n}\left[\frac{\partial H_B}{\partial v_i}\frac{\partial g}{\partial x_i}-\frac{\partial H_B}{\partial x_i}\frac{\partial g}{\partial v_i}+\frac{\partial\langle H_I(x,v)\rangle}{\partial v_i}\frac{\partial g}{\partial x_i}-\frac{\partial\langle H_I(x,v)\rangle}{\partial x_i}\frac{\partial g}{\partial v_i}\right]$$
$$=-\sum_{i=1}^{n}\int d\Gamma_q\left[\frac{\partial H_I}{\partial v_i}\frac{\partial\mu_c}{\partial x_i}-\frac{\partial H_I}{\partial x_i}\frac{\partial\mu_c}{\partial v_i}\right]=0\ , \quad (12)$$

where

$$\langle H_I(q,p)\rangle=\int d\Gamma_x g(\{x_i,v_i\},t)\times H_I(\{x_i,v_i\},\{q_A,p_A\},t) \quad (13)$$

and

$$\langle H_I(x,v)\rangle=\int d\Gamma_q f(\{q_A,p_A\},t)\times H_I(\{x_i,v_i\},\{q_A,p_A\},t)\ . \quad (14)$$

Note that the quantities $\langle H_I(q,p)\rangle$ and $\langle H_I(x,v)\rangle$ represent, respectively, the "average" interaction Hamiltonians responsible for the evolution of the subsystem and the bath.

If the Hamiltonian H is sufficiently simple, corresponding, e.g., to a coupled set of oscillators, one can proceed immediately to solve for μ_c as a functional of f and g to formulate a close, albeit nonlocal, system of equations for f and g [7]. If the Hamiltonian is more complicated, this is not as easily done. However, there still exist various ways in which one can proceed to derive nonlocal equations for f and g, including the introduction of time-dependent project operators. As discussed more carefully elsewhere [8], the idea is quite simple.

Suppose that one has a projection operator $P(t)$, to chosen that (i) P projects out from μ the uncorrelated product fg and (ii) the operations of projection and time evolution commute when acting on μ, i.e.,

$$P\mu=fg \quad \text{and} \quad P\frac{\partial\mu}{\partial t}=\frac{\partial}{\partial t}(P\mu)\ . \quad (15)$$

The single Liouville equation $\partial\mu/\partial t+L\mu=0$ is then equivalent to the coupled system

$$\frac{\partial fg}{\partial t}+PLfg=-PL\mu_c \quad (16)$$

and

$$\frac{\partial\mu_c}{\partial t}+(1-P)L\mu_c=-(1-P)Lfg\ . \quad (17)$$

However, it is straightforward to solve Eq. (17) formally to yield μ_c as a functional of fg at retarded times $<t$, and then substitute that solution back into Eq. (16). The net results is an equation of the form

$$\frac{\partial fg}{\partial t}+PLfg=-P(t)L(t)\mathcal{G}(t,0)\mu_c(0)-P(t)L(t)\int_0^t d\tau\,\mathcal{G}(t,\tau)[1-P(\tau)]\times L(\tau)f(\tau)g(\tau)\ , \quad (18)$$

where, in terms of a time ordering T,

$$\mathcal{G}(t_1,t_2)\equiv T\exp\left\{-\int_{t_1}^{t_2}dt_3[1-P(t_3)]L(t_3)\right\}\ . \quad (19)$$

One possible choice for $P(t)$ is the simple integral operation

$$P(t)=f(\{q_A,p_A\},t)\int d\Gamma_q+g(\{x_i,v_i\},t)\int d\Gamma_x-f(\{q_A,p_A\},t)g(\{x_iv_i\},t)\int d\Gamma_q\int d\Gamma_x\ . \quad (20)$$

If one integrates Eq. (18) over the degrees of freedom of the bath, and indulges in some straightforward, albeit tedious, manipulations, he or she concludes that

$$\frac{\partial f(\{q_A,p_A\},t)}{\partial t}+\sum_{A=1}^{N}\left[\frac{\partial H_S}{\partial p_A}\frac{\partial f}{\partial q_A}-\frac{\partial H_S}{\partial q_A}\frac{\partial f}{\partial p_A}+\frac{\partial\langle H_I(q,p)\rangle}{\partial p_A}\frac{\partial f}{\partial q_A}-\frac{\partial\langle H_I(q,p)\rangle}{\partial q_A}\frac{\partial f}{\partial p_A}\right]$$
$$=-\int d\Gamma_x[1-P(t)]L(t)\mathcal{G}(t,\tau)\mu_c(0)-\int_0^t d\tau\int d\Gamma_x\Delta(t)\mathcal{G}(t,\tau)\Delta(\tau)f(\tau)g(\tau)\ , \quad (21)$$

where

$$\Delta=\sum_A\frac{\partial}{\partial q_A}[H_I-\langle H_I(q,p)\rangle]\frac{\partial}{\partial p_A}+\sum_i\frac{\partial}{\partial x_i}[H_I-\langle H_I(x,v)\rangle]\frac{\partial}{\partial v_i}\ . \quad (22)$$

An analogous integration over the degrees of freedom of the subsystem then yields

$$\frac{\partial g(\{x_i,v_i\},t)}{\partial t}+\sum_{i=1}^{n}\left[\frac{\partial H_B}{\partial v_i}\frac{\partial g}{\partial x_i}-\frac{\partial H_B}{\partial x_i}\frac{\partial g}{\partial v_i}+\frac{\partial\langle H_I(x,v)\rangle}{\partial v_i}\frac{\partial g}{\partial x_i}-\frac{\partial\langle H_I(x,v)\rangle}{\partial x_i}\frac{\partial g}{\partial v_i}\right]$$
$$=-\int d\Gamma_q[1-P(t)]L(t)\mathcal{G}(t,\tau)\mu_c(0)-\int_0^t d\tau\int d\Gamma_q\Delta(t)\mathcal{G}(t,\tau)\Delta(\tau)f(\tau)g(\tau)\ , \quad (23)$$

Equations (21) and (23) demonstrate explicitly that μ_c can be eliminated from the evolution equations for f and g, but only at the expense of allowing for a nonlocal description. They also show that, aside from the propagation of any nontrivial initial conditions $\mu_c(0)$, the effects of correlations are quadratic in Δ, which involves the difference between the true interaction Hamiltonian H_I and the average values $\langle H_I(q,p)\rangle$ and $\langle H_I(x,v)\rangle$. It thus follows that, to the extent that no significant initial correlations are present and that Δ can be treated small, i.e., that the differences between the true and the average Hamiltonians are small, the right hand sides of Eqs. (21) and (23) can be neglected, so that one recovers a coupled system of equations involving only f and g. The self-consistent field approximation corresponds precisely to assuming that $\Delta\approx\mu_c(0)\approx 0$.

There is no guarantee that this approximation is justified. However, there are certain cases in which one may be able to argue that the correlations are weak, so that μ_c and Δ can be ignored. For example, if one assumes initial conditions without detailed correlations, i.e., an assumption of initial molecular chaos (Boltzmann's *Stoßzahlansatz* [9]), one may be able to argue that, at least for sufficiently short time scales, correlations will be relatively unimportant. The idea here is that the "average" effects of the bath on the subsystem, and vice versa, are presumably collective in origin, whereas the correlational effects involve noncollective processes. The crucial point, then, is that the natural time scale associated with the collective processes will typically be much shorter than the time scale for noncollective processes. This is, for example, the motivation for the Vlasov approximation in a plasma or for a self-gravitating system of stars [10].

What remains to be shown now is that, if one assumes $\mu_c\equiv 0$, he or she is led to an approximate reduced description which is necessarily Hamiltonian. There are a number of different ways in which, for some dynamical system, one can effect a Hamiltonian formulation. Perhaps the most general way, and the way which will be used here, is to proceed at a formal algebraic level, which manifests the symplectic character of the evolution. This will be done through an explicit construction, which entails the identification of a cosymplectic structure (i.e., a generalization of the ordinary Poisson brackets) and a Hamiltonian function [11].

The idea is in fact straightforward. Suppose that the fundamental dynamical variables are the distribution functions f and g, defined in an appropriate infinite-dimensional phase space. The object then is to identify Lie brackets $\{\mathcal{A},\mathcal{B}\}$, acting on pairs of functionals $\mathcal{A}[f,g]$ and $\mathcal{B}[f,g]$, and a Hamiltonian $\mathcal{H}[f,g]$, so chosen that the evolution equations for f and g reduce respectively to

$$\frac{\partial f}{\partial t}+\{\mathcal{H},f\}=0 \quad (24)$$

and

$$\frac{\partial g}{\partial t}+\{\mathcal{H},g\}=0\ . \quad (25)$$

The Lie brackets can be taken to be of the form

$$\{\mathcal{A},\mathcal{B}\}=\int d\Gamma_q f\left[\frac{\delta\mathcal{A}}{\delta f},\frac{\delta\mathcal{B}}{\delta f}\right]+\int d\Gamma_x g\left[\frac{\delta\mathcal{A}}{\delta g},\frac{\delta\mathcal{B}}{\delta g}\right]\ , \quad (26)$$

where $\delta/\delta f$ and $\delta/\delta g$ denote functional derivatives and $[a,b]$ denotes ordinary Poisson brackets defined in the $(N+n)$-particle phase space, as given by Eq. (5). This is nothing other than the Morrison [12] brackets for the Vlasov-Poisson system, as formulated for a two-component system with distribution functions f and g. Equation (26) is closely related to the more familiar brackets

$$\{\mathcal{M},\mathcal{N}\}=\sum_{i,j,k=1}^{3}\epsilon_{imk}\omega_i\frac{\partial M}{\partial\omega_j}\frac{\partial N}{\partial\omega_k} \quad (27)$$

which generate the Lie algebra associated with the three-dimensional rotation group, and thus play an important role in the Hamiltonian dynamics of mechanical systems. It is straightforward to verify that the bracket operation (26) is antisymmetric and that it satisfies the Jacobi identity, so that it defines bona fide Lie brackets. This implies that the equations $\partial f/\partial t+\{\mathcal{H},f\}=0$ and $\partial g/\partial t+\{\mathcal{H},g\}=0$ are Hamiltonian for any choice of Hamiltonian function $\mathcal{H}$.

Suppose now that the Hamiltonian function $\mathcal{H}$ is chosen as corresponding to the "average" energy associated with the particle Hamiltonian H, i.e.,

$$\mathcal{H}=\int d\Gamma_x d\Gamma_q fgH$$
$$=\int d\Gamma_q f(\{q_A,p_A\},t)H_S+\int d\Gamma_x g(\{x_i,v_i\},t)H_B$$
$$+\int\int d\Gamma_q d\Gamma_x f(\{q_A,p_A\},t)g(\{x_i,v_i\},t)H_I\ . \quad (28)$$

It is then easily verified that

$$\frac{\delta\mathcal{H}}{\delta f}=H_S+\int d\Gamma_x gH_I$$
$$=H_s(\{q_A,p_A\},t)+\langle H_I(\{q_A,p_A\},t)\rangle \quad (29)$$

and

$$\frac{\delta\mathcal{H}}{\delta g}=H_B+\int d\Gamma_q f H_I$$
$$=H_s(\{x_i,v_i\},t)+\langle H_I(\{x_i,v_i\},t)\rangle\ , \tag{30}$$

so that one recovers the self-consistent equations in the form

$$\frac{\partial f}{\partial t}=[H_S+\langle H_I(q,p)\rangle,f] \tag{31}$$

and

$$\frac{\partial q}{\partial t}=[H_B+\langle H_I(x,v)\rangle,g]\ . \tag{32}$$

III. APPLICATIONS

A number of different applications of this general approach are easily envisioned. Supposed, as the simplest possible example, that one is concerned with a small subsystem embedded in a bath which, in an appropriate sense, is large. To the extent that the subsystem has only a minimal effect on the bath, one can assume that the bath distribution function g is independent of the subsystem and specified independently, e.g., as a thermal distribution $g_{\rm th}$. One can then calculate the evolution of f, assuming that f evolves in the average Hamiltonian $H_S+\langle H_I(q,p)\rangle$ associated with $g_{\rm th}$.

A simple generalization of that most naive approximation would entail allowing for a subsystem and bath of comparable size, but assuming that their mutual interaction is weak. In this case, one might suppose that, in the absence of any coupling, the subsystem and bath have specified distribution functions f_0 and g_0, and that the coupling induces perturbative correlations δf and δg.

Another generalization would involve the case of a single large self-gravitating object moving through an environment comprised of many smaller objects. Because the single object is large, it will presumably have a substantial effect on the surrounding environment, so that one cannot assume that the environment is described adequately by a fixed distribution function g. However, one may still expect that, at least in a first approximation, the evolution of the larger object is governed by the collective effects of the environment, and that detailed correlations between the larger object and its surroundings are relatively small.

Perhaps the quintessential example of a mean field description is provided by the Vlasov-Poisson system, as applied to a plasma or a galaxy. Suppose, for example, that one is interest in a galaxy, idealized as a collection of N point mass objects. In this case, the natural particle Hamiltonian takes the form

$$H=\sum_{i=1}^{N}\frac{1}{2m_i}\mathbf{p}_i^2-\sum_{i<j=1}^{N}G\frac{m_i m_j}{|\mathbf{q}_i-\mathbf{q}_j|}\ . \tag{33}$$

It is convenient to view this as a Hamiltonian involving N different one-particle systems and $N(N-1)/2$ pairings between these systems. In this context, the self-consistent field approximation means that μ factorizes into a product of N one-particle distributions $f(\mathbf{q}_i,\mathbf{p}_i,t)\equiv f(i)$. The natural brackets then reduce to

$$\{\mathcal{A},\mathcal{B}\}=\sum_{i=1}^{N}\int d^3q_i d^3p_i f(i)\left[\frac{\delta\mathcal{A}}{\delta f(i)},\frac{\delta\mathcal{B}}{\delta f(i)}\right] \tag{34}$$

and the Hamiltonian function for the distribution functions $\{f(i)\}$ becomes

$$\mathcal{H}=\sum_{i=1}^{N}\int d^3q_i d^3p_i\frac{1}{2m_i}\mathbf{p}_i^2 f(i)$$
$$-G\sum_{i<j=1}^{N}\int d^3q_i d^3p_i d^3q_j d^3p_j\frac{m_i f(i) m_j f(j)}{|\mathbf{q}_i-\mathbf{q}_i|}\ . \tag{35}$$

For the special case in which all the particles are treated as identical, the $f(i)$'s all have the same functional form, so that one recovers the standard [13] mean field Hamiltonian

$$\mathcal{H}=N\int d^3q d^3p\frac{1}{2m}\mathbf{p}^2 f$$
$$-\frac{GN(N-1)}{2}\int d^3q\, d^3p\, d^3q' d^3p'$$
$$\times\frac{m f(\mathbf{q},\mathbf{p}) m f(\mathbf{q}',\mathbf{p}')}{|\mathbf{q}-\mathbf{q}'|} \tag{36}$$

and the standard brackets [12,14]

$$\{\mathcal{A},\mathcal{B}\}=\int d^3q d^3p\, f\left[\frac{\delta\mathcal{A}}{\delta f},\frac{\delta\mathcal{B}}{\delta f}\right]\ . \tag{37}$$

Yet another example corresponds to a collection of charged particles interacting via electromagnetic field, which is viewed as a collection of "oscillators." Working in a gauge where A_t, the time component of the vector potential, vanishes identically, the Hamiltonian may of course be written in the form [15]

$$H=\sum_{i=1}^{N}[(\mathbf{p}_i-e\,\mathbf{A}_i)^2+m_i^2]^{1/2}-4\pi^2\int d^3k\,\mathbf{a}^\dagger(\mathbf{k})\cdot\mathbf{a}(\mathbf{k})\ , \tag{38}$$

where, in terms of creation and annihilation operators $\mathbf{a}(\mathbf{k})$ and $\mathbf{a}^\dagger(\mathbf{k})$,

$$\mathbf{A}(\mathbf{q}_i)=\int d^3k\, k^{-1}[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{q}_i}+\mathbf{a}^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{q}_i}]\ . \tag{39}$$

A more conventional description in terms of a collection of oscillators then follows from the introduction of the conjugate variables:

$$\mathbf{Q}(\mathbf{k})=\frac{\pi\sqrt{2}}{k}[\mathbf{a}^\dagger(\mathbf{k})+\mathbf{a}(\mathbf{k})]$$

and (40)

$$\mathbf{P}(\mathbf{k})=i\pi\sqrt{2}[\mathbf{a}^\dagger(\mathbf{k})-\mathbf{a}(\mathbf{k})]\ .$$

The fundamental object in such a description is a distribution function $\mu(\{\mathbf{q}_i,\mathbf{p}_i\},\{Q(\mathbf{k}),P(\mathbf{k})\},t)$, defined as a joint probability for finding the particles with specified phase space coordinates $\{\mathbf{q}_i,\mathbf{p}_i\}$ and the oscillators with specified $\{Q(\mathbf{k}),P(\mathbf{k})\}$. Suppose now for convenience that the system is confined spatially, so that the modes

are in fact discrete. In this case, the full distribution function can be written in the form

$$\mu=\prod_{i=1}^{N} f(\mathbf{q}_i,\mathbf{p}_i,t)\prod_{k=1}^{\infty} g(Q_k,P_k,t)+\mu_c \ , \tag{41}$$

where the $f(i)$'s refer to particles and the $g(k)$'s to the modes of the field. The assumption that $\mu_c=0$ then leads to a Hamiltonian description in which the $f(i)$'s satisfy a Vlasov equation and the equations for the $g(k)$'s imply the dynamical Maxwell equations [16].

It should be stressed that this Hamiltonian formulation of mean field electrodynamics is fundamentally different from, and more general than, the Hamiltonian formulation of the Vlasov-Maxwell system provided, e.g., by Marsden and Weinstein [17]. In that approach, the fundamental objects are the particle distribution f and the electromagnetic field, as characterized by electric and magnetic field strengths, **E** and **B**, or by the vector potential **A** and the conjugate momentum Π. Here instead the fundamental objects are the particle distributions $f(i)$ and the oscillator distributions $g(k)$, which are used to generate the "average" **E** and **B** entering into the conventional Vlasov description. A description in terms of $g(k)$ permits one to compute the probability that **E** and **B** assume any given value, whereas the Vlasov description involves only the average values of the fields.

The preceding construction can also be generalized to nonlinear field theories as well. Even a nonlinear field can be decomposed, at least formally, into a sum of coupled oscillators, and the neglect of all correlations between particles and/or oscillators will again lead to a Hamiltonian mean field theory. The only potential problem in this case is that, for a strongly nonlinear theory, there may exist no preferred mode decomposition, and that the physical content of neglecting mode-mode correlations may depend on the choice of modes.

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[1] Cf. A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983); Ann. Phys. (N.Y.) **149**, 374 (1983); B. L. Hu and H. E. Kandrup, Phys. Rev. D **35**, 1776 (1987); H. E. Kandrup, *ibid* **39**, 2253 (1989).

[2] Cf. A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics* (Springer, Berlin, 1992) for a discussion of the special features associated with Hamiltonian systems. The significance of a non-Hamiltonian evolution has been discussed in the context of quantum field theory by, e.g., B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992); **47**, 1576 (1993); S. Habib and R. Laflamme, *ibid.* **42**, 4056 (1990); J. P. Paz and S. Sinha, *ibid.* **44**, 1038 (1991), and numerous references contained therein.

[3] Cf. L. D. Landau and E. M. Lifshitz, *Statistical Physics, Part 1* (Pergamon, Oxford, 1981); R. K. Pathria, *Statistical Mechanics* (Pergamon, Oxford, 1972).

[4] Cf. R. K. Pathria, *Statistical Mechanics* (Pergamon, Oxford, 1972).

[5] Cf. the recent review by M. Hillary, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984). A systematic approach to the definition of Wigner functions in a curved spacetime has been presented by E. A. Calzetta, S. Habib, and B. L. Hu, Phys. Rev. D **37**, 2901 (1988).

[6] This system was originally formulated in galactic dynamics by J. H. Jeans, Mon. Not. R. Astron. Soc. **76**, 71 (1915). It was introduced in plasma physics by A. Vlasov, Zh. Eksp. Teor. Fiz. **8**, 291 (1938).

[7] Cf. the independent oscillator model of G. W. Ford, M. Kac, and P. Mazur, J. Math. Phys. **6**, 504 (1965), as generalized by S. Habib and H. E. Kandrup, Phys. Rev. D **46**, 5303 (1992), as well as Calderia and Leggett [1].

[8] The particular implementation of projection operators used here is based on an approach due to C. R. Willis and R. H. Picard, Phys. Rev. A **9**, 1343 (1974), as adapted to the case of a general interacting subsystem and environment by H. E. Kandrup, Phys. Rev. D **39**, 2253 (1989).

[9] L. Boltzmann, Wien. Ber. **66**, 275 (1872).

[10] Cf. K. Ogorodnikov, *Dynamics of Stellar Systems* (Macmillan, New York, 1965); J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University, Princeton, 1987).

[11] See, e.g., V. Guillemin, *Symplectic Techniques in Physics* (Cambridge University, Cambridge, England 1984) or J. E. Marsden, *Lectures on Mechanics* (Cambridge University, Cambridge, England, 1992) for a detailed mathematical discussion of this general approach to Hamiltonian dynamics. A more terse introduction is provided in V. I. Arnol'd, *Mathematical Methods of Classical Mechanics,* 2nd ed. (Springer, Berlin, 1989). Another, more elementary, introduction is provided by A. Dragt, in *Physics of High Energy Particle Accelerators,* edited by R. A. Carrigan, F. R. Huson, and M. Month, AIP Conf. Proc. No. 87 (AIP, New York, 1982).

[12] P. J. Morrison, Phys. Lett. **80A**, 383 (1980). These brackets also arise in other contexts as well, including the case of two-dimensional perfect fluids. Cf. P. J. Morrison, in *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems,* edited by M. Tagor and Y. M. Treve, AIP Conf. Proc. No. 99, (AIP, New York, 1982).

[13] Cf. D. Lynden-Bell and N. Sanitt, Mon. Not. R. Astron. Soc. **143**, 167 (1969).

[14] H. E, Kandrup, Astrophys. J. **351**, 104 (1990).

[15] W. Heitler, *The Quantum Theory of Radiation* (Clarendon, Oxford, 1954).

[16] N. Rostoker and M. N. Rosenbluth, Phys. Fluids **3**, 245 (1960).

[17] J. E. Marsden and A. Weinstein, Physica D **4**, 394 (1982).