Hamiltonian of SU(2) lattice gauge theory in approximate tridiagonal form

Lloyd C. L. Hollenberg
School of Physics, Research Centre for High Energy Physics, University of Melbourne, Parkville, Victoria 3052, Australia
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Employing Hamiltonian moments of SU(2) lattice gauge theory, with respect to the strong coupling vacuum, the matrix elements of the Lanczos tridiagonal form are written down from the plaquette expansion to order $1/N_p^2$ in the number of plaquettes, N_p. The consequences of this approximate tridiagonal form are studied by computing the vacuum energy density and the specific heat in the infinite lattice limit, for strong to weak coupling. The results at this order appear to reach beyond the strong to weak transition point at $g_c^2 \approx 2.0$, as indicated by the peaking behavior of the specific heat, down to $g^2 \approx \sqrt{2}$.

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I. INTRODUCTION

With the nonperturbative study of gauge field theories in the path integral formalism well and truly in the realm of the supercomputer there has recently been a revival of the Hamiltonian formalism [1–4]. Early semianalytic calculations in Hamiltonian lattice gauge theory relied on strong coupling perturbation theory where nonperturbative information in the continuum limit was obtained by Padé extrapolation of the strong coupling series to weak coupling. Although from the outset reasonable results for such quantities as the hadron spectrum were obtained [5], the need for extrapolation undermines these calculations. More recently, the t expansion [6] was introduced as a useful nonperturbative method of calculation in the Hamiltonian lattice formalism. Since Hamiltonian moments are central to this method the bulk of the calculation is analytic. Results of good quality have been obtained for various gauge theories [7–9]; however, the method relies on Padé extrapolation to the $t \to \infty$ limit, in order to project out the ground state, which unfortunately introduces some measure of ambiguity into the results.

A new method which relies on connected Hamiltonian moments without the need for extrapolation is the plaquette expansion [10]. Here, the Hamiltonian of lattice models, for which the energy is extensive in the number of plaquettes N_p, is cast into tridiagonal form as a cluster expansion of the Lanczos basis with respect to N_p. This plaquette expansion property is a general result which was initially conjectured on the basis of direct calculation and expansion of the first few elements of the Lanczos matrix for general lattice models and has recently been proved [11] for the first two terms in $1/N_p$. The plaquette expansion relies on the calculation of relatively low orders of moments of the Hamiltonian with respect to a trial state with nonzero overlap in the sector of interest. Once the Hamiltonian of a system is written down in tridiagonal form the energy spectrum is calculated by a straightforward diagonalization for increasing N_p. This has been successfully carried out for the ground-state energy density in the infinite lattice limit for the case of the one- and two-dimensional Heisenberg models [10,12,13] to good accuracy.

In this work the Hamiltonian of SU(2) lattice gauge theory is written down to order $1/N_p^2$ using Hamiltonian moments calculated with respect to the vacuum strong coupling state by Mathews *et al.* [14] (in the context of the t expansion). From the plaquette expansion the vacuum energy density in the infinite lattice limit is computed for a range of couplings. Of particular interest is the question of the ability of the plaquette expansion to match onto the strong coupling results [15,16] and pass beyond the strong to weak coupling transition [15,17].

The paper is organized as follows. In Sec. II the plaquette expansion of the Lanczos recursion is outlined. The Hamiltonian of SU(2) is written down in the Lanczos tridiagonal form in Sec. III and diagonalized. Conclusions are drawn in Sec. IV.

II. PLAQUETTE EXPANSION

By applying the Lanczos recursion with respect to some initial trial state $|v_1\rangle$,

$$|v_n\rangle = \frac{1}{\beta_{n-1}}[(H-\alpha_{n-1})|v_{n-1}\rangle - \beta_{n-2}|v_{n-2}\rangle] , \tag{1}$$

a basis $\{|v_n\rangle\}$ is constructed in which the Hamiltonian H is tridiagonal, in the sector corresponding to the symmetries of $|v_1\rangle$, with matrix elements α_n and β_n:

$$\alpha_n = \langle v_n|H|v_n\rangle, \quad \beta_n = \langle v_{n+1}|H|v_n\rangle . \tag{2}$$

Calculation of the Lanczos matrix elements α_n and β_n is typically a formidable task: a direct application of the recursion becomes prohibitively complex well before the first ten or so iterations. One of the features of the Lanczos construction is that for the lowest states of the system only a small fraction of the complete set of states $\{|v_n\rangle\}$ is required for convergence. In this sense the construction is optimal. However, in practice this sufficient number of states can only be constructed numerically in matrix representation for small lattices (small basis sizes). Certainly if one is interested in the bulk limit, $N_p \to \infty$, the limit $n \to \infty$ must also be taken, which, either due to computational limitations or algebraic complexity is not

possible using the exact Lanczos recursion.

However, it is possible to make a cluster expansion of α_n and β_n in the number of plaquettes on the lattice which has the remarkable property that the various terms in the expansion parameter are related by a polynomial n dependence. The plaquette expansions take on a particularly simple form:

$$\alpha_n = N_p \left[a_0 + (n-1)\frac{a_1}{N_p} + (n-1)(n-2)\frac{a_2}{N_p^2} + \cdots \right] , \tag{3}$$

$$\beta_n^2 = N_p \left[nb_0 + n(n-1)\frac{b_1}{N_p} + n(n-1)(n-2)\frac{b_2}{N_p^2} + \cdots \right] ,$$

where the coefficient functions are given in terms of the connected Hamiltonian moments $\langle H^n \rangle_c \equiv c_n N_p$, with respect to the trial state. The first few are

$$\begin{aligned} a_0 &= c_1 , \\ a_1 &= \frac{c_3}{c_2} , \\ a_2 &= \frac{3c_3^3 - 4c_2c_3c_4 + c_2^2c_5}{4c_2^4} , \\ b_0 &= c_2 , \\ b_1 &= \frac{c_2c_4 - c_3^2}{2c_2^2} , \\ b_2 &= \frac{-12c_3^4 + 21c_2c_3^2c_4 - 4c_2^2c_4^2 - 6c_2^2c_3c_5 + c_2^3c_6}{12c_2^5} . \end{aligned} \tag{4}$$

The plaquette expansion was originally inferred [10] by direct application of the recursion up to the sixth iteration. The proof to $O(1/N_p)$ was carried out [11] using the Hankel determinant representation of the α_n and β_n. Although possible in principle, the proof has not, for reasons for complexity, been carried out to higher order; however, the higher terms were shown to persist to the ninth iteration in the case of the one-dimensional Heisenberg model [12].

III. SU(2) LATTICE HAMILTONIAN

The Kogut-Susskind SU(2) lattice Hamiltonian in 3+1 dimensions is given by [18]

$$H = \frac{g^2}{2}\sum_l \mathbf{E}_l^2 + \frac{2}{g^2}\sum_p (2 - \mathrm{tr}U_p) , \tag{5}$$

where $\mathbf{E}_l$ is the color electric flux operator corresponding to the link l, $\mathrm{tr}U_p$ is the magnetic flux operator corresponding to the plaquette p, and g is the coupling constant. In the context of t expansion, Mathews *et al.* [14] have calculated connected moments in the vacuum sector up to $\langle H^{12} \rangle_c$ with respect to the strong coupling state as a function of the coupling constant. Using these expressions for c_n, up to $n=8$, we find the following for the Lanczos tridiagonal form of the SU(2) lattice Hamiltonian in the vacuum sector up to order $1/N_p^2$:

$$\alpha_n(y) = 2yN_p + \frac{3}{y}(n-1) + \frac{1}{N_p}\frac{1}{2y}(n-1)(n-2) - \frac{1}{N_p^2}(n-1)(n-2)\left[\frac{y^2(80n-249)+76n-228}{36y^5} \right] + \cdots , \tag{6}$$

$$\beta_n^2(y) = ny^2N_p - n(n-1)\left[\frac{y^2}{2} \right] + \frac{1}{N_p}n(n-1)(n-2)\left[\frac{8y^4+11}{6y^2} \right] - \frac{1}{N_p^2}n(n-1)(n-2)\left[\frac{4y^8(223n-717) - 4y^4(329n-933) - 611n + 1833}{288y^6} \right] + \cdots , \tag{7}$$

where $y = 2/g^2$.

The Hamiltonian matrix defined by $\alpha_n(y)$ and $\beta_n(y)$ is diagonalized numerically and the lowest eigenvalue, corresponding to the ground state (vacuum), is examined. In order to be able to systematically study the effect of increasing the number of terms in the expansion and the limit $N_p \to \infty$ the following procedure is adopted. For the plaquette expansion to order $1/N_p^r$ the lowest eigenvalue of the Hamiltonian matrix of the lth Lanczos iteration (i.e., the $l \times l$ matrix) is denoted by $\lambda_0^{(r)}(l, N_p, y)$. The energy of the ground state $E_0^{(r)}(N_p, y)$ corresponding to the converged value of $\lambda_0^{(r)}(l, N_p, y)$ (always to be taken at the minimum slope with respect to l) is computed for increasing N_p. As expected one finds that the ground-state energy density $E_0^{(r)}(N_p, y)/N_p$ is finite in the infinite lattice limit. We define this quality as

$$\mathcal{E}_0^{(r)}(y) \equiv \lim_{N_p \to \infty} \frac{E_0^{(r)}(N_p, y)}{N_p} . \tag{8}$$

Typical behavior of the ground-state eigenvalue with the Lanczos iteration l for the orders $r=0$, 1, and 2 is shown in Fig. 1 for strong and weak coupling at large N_p. For the $r=0$ expansion the eigenvalue converges for all

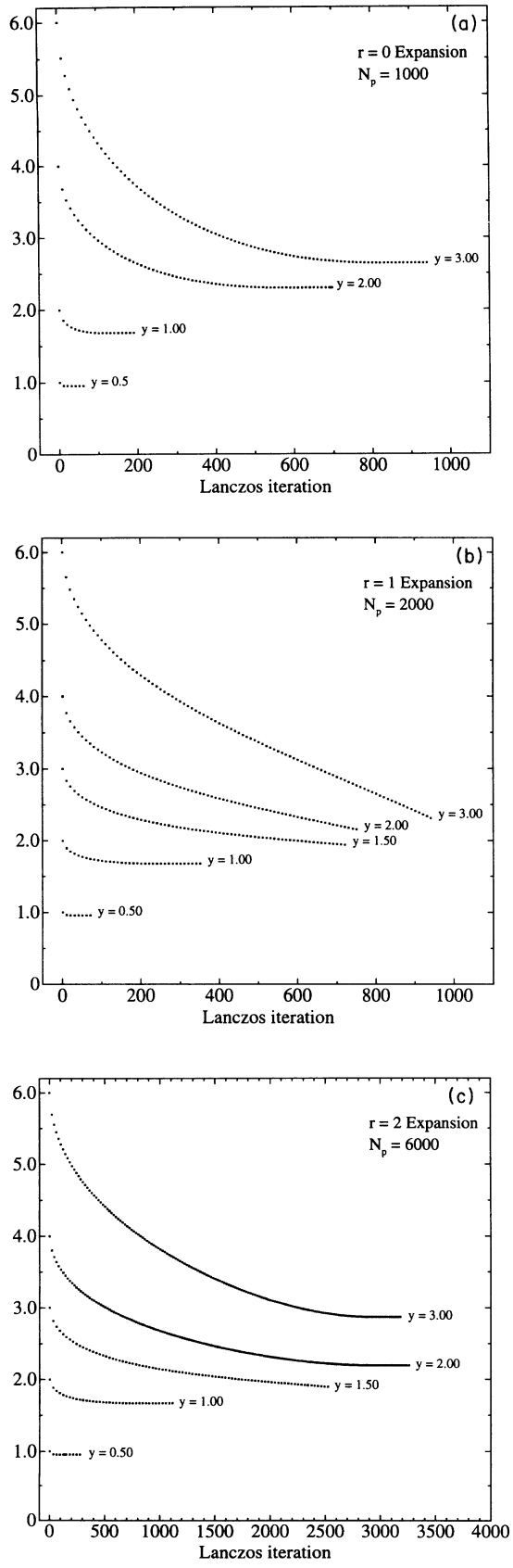

FIG. 1. Ground-state eigenvalue density $\lambda_0^{(r)}(l,N_p,y)/N_p$ as a function of Lanczos iteration, l, for various couplings (a) $r=0$ expansion ($N_p=1000$), (b) $r=1$ expansion ($N_p=2000$), and (c) $r=2$ expansion ($N_p=6000$).

coupling. This is not the case, however, for the higher-order expansions where after the transition from strong to weak coupling (near $y=1$) the character of the Lanczos convergence changes and a point of inflection develops indicating a breakdown of the expansion for fixed N_p. As $N_p\rightarrow\infty$ the convergence region flattens out. The energy $E_0^{(r)}(N_p,y)$ is taken to correspond to the value of the eigenvalue at the minimum slope. Any ambiguity associated with this choice will become irrelevant in the bulk limit. The convergence of the vacuum energy density $E_0^{(r)}(N_p,y)$ in the bulk limit $N_p\rightarrow\infty$ is shown in Fig. 2.

The plaquette expansion to order $r=0$ has been solved exactly [19] for the energy density in the infinite lattice limit using theorems on the bounds of the lowest zeros of orthogonal polynomials. One finds [19]

$$\mathcal{E}_0^{(0)}=c_1+\frac{c_2^2}{c_2c_4-c_3^2}(\sqrt{3c_3^2-2c_2c_4}-c_3)\ , \tag{9}$$

which from the connected moments for SU(2) gives

$$\mathcal{E}_0^{(0)}(y)=\frac{1}{y}(2y^2+3-\sqrt{9+2y^4})\ . \tag{10}$$

The vacuum energy density in the bulk limit for each of the expansion orders is shown in Fig. 3 together with a Monte Carlo calculation by Chin *et al.* [15], which has the correct strong and weak coupling limits, for comparison. The weak coupling limit derived from the variational calculation by Hofsäss and Horsley [17] is also shown for comparison. The "specific heat" $C_v\equiv-\partial^2\mathcal{E}_0/\partial y^2$, derived from the analytic result for $r=0$ [19], peaks at the correct transition point, $g_c^2\approx2.0$ [7]. The results for the energy density at order $r=0$ begin to diverge just past the transition point, whereas the higher orders $r=1,2$ tend to match the weak coupling behavior further. In general the plaquette expansion results will diverge from the exact result at a coupling where the overlap of the trial state with the true ground state becomes small. The nature of the departure from

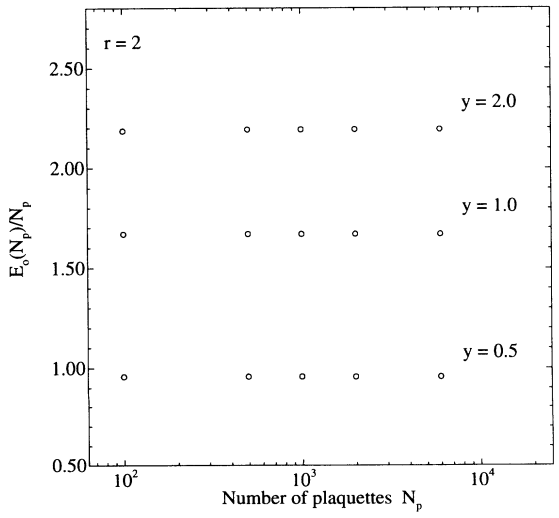

FIG. 2. Behavior of the $r=2$ expansion ground-state energy density, $E_0^{(2)}(N_p,y)/N_p$ in the large lattice limit, for various couplings.

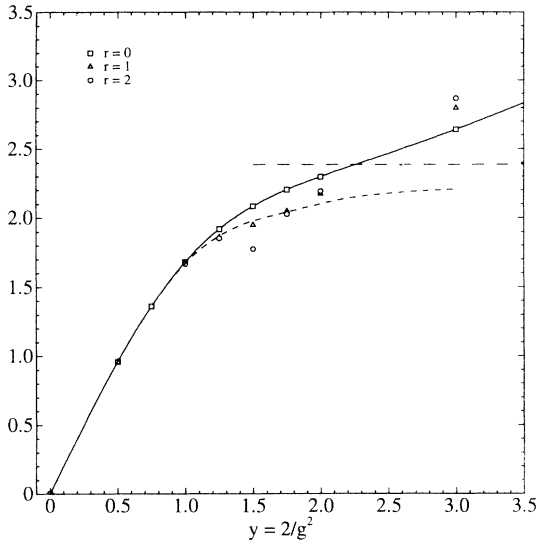

FIG. 3. The large lattice limit of the vacuum energy density, $\mathcal{E}_0^{(r)}(y)$ for $r=0$, 1, and 2 expansion orders. The short-dashed line corresponds to the calculations by Chin *et al.* [15] and the horizontal long-dashed line corresponds to the weak coupling limit calculated by Hofsäss and Horsley [17].

the exact result will be different for each expansion order. One would expect that as the expansion order is increased the deviations from the exact result would occur further into the weak coupling region and be less dramatic than that of the lower orders calculated here. In this regard the behavior of the $r=2$ expansion near $y \approx 1.5$ is somewhat surprising; however, this should not be viewed as a problem with the method so much as a lack of understanding of the method; particularly as this behavior might be rectified at higher orders.

IV. CONCLUSION

In this work the Hamiltonian of SU(2) has been cast into approximate tridiagonal form in the vacuum sector and diagonalized for the vacuum energy density in the bulk limit. On the whole, the results demonstrate that the plaquette expansion method to the order considered here is able to describe the physics of the vacuum at the strong to weak transition and just beyond. In principle, one expects the results to be improved by increasing the plaquette expansion order and/or choosing a better trial state (as was done in the calculations of Chin *et al.* [15]). In the calculations carried out here the $r=1$ results are clearly an improvement over the analytic $r=0$ result. However, the next order is, if anything, worse than the $r=1$ result. This may be a potential problem for the method as it might signify that no systematic improvement of the results is possible. On the other hand, the convergence of the plaquette expansion with expansion order may not necessarily be monotonic. In any case, higher orders of Hamiltonian moments are required to approximate the weak coupling behavior of the theory and so convergence of the method may still occur at higher orders in the expansion. While the accuracy of the results presented here is not quite of the caliber of the t expansion or Monte Carlo calculations, the method used here is simpler and, since it is semianalytic and not dependent on extrapolation, is to a large extent cleaner. Furthermore, this framework allows for the calculation of excited states and so physically interesting quantities such as glueball masses and the string tension could be calculated by diagonalizing the plaquette expansion in the relevant sector. As such, further work on this method, particularly with regard to gauge field theories, is merited.

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