Relativistic plasma in a homogeneous cosmological background

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(Received 11 January 1994)

The linearized theory of a relativistic plasma in a radiation background, proposed recently by Holcomb and Tajima, is extended for a more general background with the metric components being an arbitrary power function of time ($\sim t^n$). It is noted that the electric field falls off faster and the redshift of the electromagnetic radiation frequency is also greater in a more rapidly expanding background universe. The dispersion relation for the transverse vibration of the electromagnetic field, however, appears to be independent of the parameter n. The wave equation for the electric field is solved for a de Sitter background also.

PACS number(s): 98.80.Hw, 04.40.Nr, 52.60.+h

The $3+1$ formalism adopted in expressing the general relativistic Maxwell equations enables one to write them in terms of electric and magnetic fields completely analogous to their flat space presentation. Recent works on black hole electrodynamics (Thorne and Macdonald [1]) and relativistic magnetohydrodynamics (Evans and Hawley [2], Sloan and Smarr [3], Zhang [4]) are based on the $3+1$ formalism. In this formalism, the spacetime is decomposed into time slices, each of which is labeled by a coordinate time t. The space coordinates x^i are defined on the slice.

The three-metric of the spacelike slice is

$$\gamma_{\mu\nu}=g_{\mu\nu}+n_\mu n_\nu \tag{1}$$

and the spatial coordinates x^i are propagated along

$$t^\mu=\alpha n^\mu+\beta^\mu . \tag{2}$$

Here n^μ is the unit timelike vector normal to the slice and α and β^μ are the lapse function and shift vector respectively. The usual four-metric can be written as

$$ds^2=-c^2\alpha^2dt^2+\gamma_{ij}(dx^i+\beta^i dt)(dx^j+\beta^j dt) . \tag{3}$$

In this discussion, greek indices run from 0 to 3 and lattice indices from 1 to 3.

Recently Holcomb and Tajima [5] have developed the linear theory of cosmological plasma including electromagnetic waves in the background of radiation-dominated Friedmann-Robertson-Walker (FRW) spacetime where self-gravity of matter is ignored. Some of the interesting conclusions arrived at are related to the time-dependent redshift of photons in the background of the expanding Universe, the dispersion relation in the plasma and also the decay of the frequency of the plasma oscillations. It is natural to ask what happens if the background is no longer a purely radiation universe but is characterized by a different scale factor. In particular, interesting physical situations where the background is a mixture of radiation and matter with some equation of state may be very important in the discussion of relativistic plasma in the early Universe. It may also be worthwhile to investigate the situation when the background happens to be a Zeldovich fluid (stiff fluid, $p=\rho$). With the aim of studying, in general, the electromagnetic waves propagating in a plasma with different background metrics, this Brief Report generalizes the work of Holcomb and Tajima on free photons as well as photons in plasma. Here the FRW scale factor R is assumed to be an arbitrary power function time in the form $R\sim t^n$. The expressions obtained by Holcomb and Tajima will follow as special cases from our general equations where the parameter n appears explicitly. The results obtained in this Brief Report clearly indicate that the electric field decays faster in a more rapidly expanding universe and the redshift of the electromagnetic wave is also higher. It is interesting to note that the dielectric constant, however, remains independent of time in all cases and does not seem to vary with the choice of the scale factor. The same calculations are also carried out for a de Sitter–type background where the scale factor is an exponential function of time.

We take the line element in the form

$$ds^2=-c^2dt^2+A^2(t)(dx^2+dy^2+dz^2) , \tag{4}$$

for which $\alpha=1$, $\beta^i=0$, and the three-metric

$$\gamma_{ij}=\mathrm{diag}(A^2,A^2,A^2) . \tag{5}$$

The trace of the extrinsic curvature tensor $K_{\mu\nu}$ is given by

$$K=-\theta=-3\dot{A}/A \tag{6}$$

where θ is the expansion scalar. With the metric (4) Maxwell equations look like

$$\nabla\cdot\mathbf{E}=4\pi\rho_e , \tag{7}$$

$$\nabla\cdot\mathbf{B}=0 , \tag{8}$$

$$\frac{\partial\mathbf{E}}{\partial t}=K\mathbf{E}+cA^{-1}\nabla\times\mathbf{B}-4\pi\mathbf{J} , \tag{9}$$

$$\frac{\partial\mathbf{B}}{\partial t}=K\mathbf{B}-cA^{-1}\nabla\times\mathbf{E} , \tag{10}$$

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where $\mathbf{E},\mathbf{B},\mathbf{J}$ are three-vectors representing the electric field, magnetic field, and current density, respectively, ρ_e is the electric charge density, and ∇ is the usual flat space three-dimensional gradient operator. The equation for charge conservation and the particle equation of motion will become

$$\frac{\partial \rho_e}{\partial t}=K\rho_e-\nabla\cdot\mathbf{J}\ , \tag{11}$$

$$\frac{d\mathbf{p}}{dt}=\frac{2}{3}K\mathbf{p}+q\left[\mathbf{E}+A^{-1}\left[\frac{\mathbf{v}}{c}\right]\times\mathbf{B}\right] \tag{12}$$

respectively (see Holcomb and Tajima [5] and references therein). The momentum three-vector, the velocity three-vector, and the charge on the particle are represented by $\mathbf{p}$, $\mathbf{v}$, and $\mathbf{q}$, respectively. The wave equation for the electric field vector can be deduced from Eqs. (9) and (10) in the form

$$\frac{1}{c^2}[A^2\partial_t^2\mathbf{E}+A(\dot{A}-2AK)\partial_t\mathbf{E}+(K^2A^2-A\dot{A}K-A^2\dot{K})\mathbf{E}]$$
$$=-\nabla\times\nabla\times\mathbf{E}-\frac{4\pi A}{c^2}(\partial_t-K)A\mathbf{J}\ . \tag{13}$$

Now, we choose the metric components $\gamma_{11}=\gamma_{22}=\gamma_{33}=A^2=t^{2n}$. If $n=\frac{1}{2}$, we get the FRW solution for a radiation ($p=\frac{1}{3}\rho$) universe and $n=\frac{2}{3}$ gives the solution for a dust ($p=0$) universe. With

$$A=t^n \text{ and } K=-\frac{3\dot{A}}{A}=-\frac{3n}{t}\ , \tag{14}$$

Eq. (13) for free photons (i.e., $\mathbf{J}=0$ and $\nabla\cdot\mathbf{E}=0$) will become

$$\frac{1}{c^2}[t^{2n}\partial_t^2\mathbf{E}+7nt^{2n-1}\partial_t\mathbf{E}+3n(4n-1)t^{2n-2}\mathbf{E}]=\nabla^2\mathbf{E}\ . \tag{15}$$

This equation can be solved by the usual separation of variables technique. We write $\mathbf{E}(\mathbf{r},t)$ as a product,

$$\mathbf{E}(\mathbf{r},t)=\phi(r)\psi(t)\ , \tag{16}$$

which leads to the equation

$$\frac{1}{c^2\psi}[t^{2n}\partial_t^2\psi+7nt^{2n-1}\partial_t\psi+3n(4n-1)t^{2n-2}\psi]$$
$$=\frac{\nabla^2\phi}{\phi}=-k_i^2 \tag{17}$$

where k_i^2 is the separation constant. It is easily seen that the space dependence remains exactly the same as that in the Newtonian case. As the background metric in this case is spatially homogeneous and isotropic, this result is quite expected. For the time part, we substitute

$$\psi=t^{(1-7n)/2}u(z)\ , \tag{18}$$

where

$$z=\left[\frac{k_ic}{1-n}\right]t^{1-n}\ , \tag{19}$$

This substitution gives the equation

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}+(z^2-\tfrac{1}{4})u=0 \tag{20}$$

for $u(z)$, which is easily recognized as the Bessel equation of order $\frac{1}{2}$. The solution can be written in form of Hankel functions, $u(z)=H_{1/2}^{(2)}(z)$, so that the complete solution for the electric field is

$$\mathbf{E}=\mathbf{E}_0\mathbf{e}t^{(1-7n)/2}H_{1/2}^{(2)}\left[\left[\frac{k_ic}{1-n}\right]t^{1-n}\right]e^{i\mathbf{k}_i\cdot\mathbf{r}}\ , \tag{21}$$

where $\mathbf{e}$ is the unit polarization vector, $\mathbf{E}_0$ is a constant giving the amplitude, and Hankel function of order $\frac{1}{2}$ is given by

$$H_{1/2}^{(2)}(z)=i\left[\frac{2}{\pi z}\right]^{1/2}e^{-iz}\ . \tag{22}$$

It may be mentioned that the magnetic field in this case (i.e., $\rho_e=0$ and $\mathbf{J}=0$) will have an exactly similar solution. If one combines all the factors of time in this solution, then

$$|\mathbf{E}|\sim t^{-3n}\ , \tag{23}$$

where $|\mathbf{E}|$ represents the amplitude of the electric field, which has a periodic time-dependent part $\exp[-i[k_ic/(1-n)]t^{1-n}]$.

The expansion of the background universe affects the magnitude of the electric field which decreases with time at a faster rate if n is larger. If we set $\omega_i=k_ic$ in the usual way, where the subscript refers to the initial time, the redshift of a photon in the FRW background is

$$\omega=\omega_it^{-n}\ . \tag{24}$$

The photon frequency drops down with the expansion of the Universe and this occurs obviously at a different rate for different values of n. In the radiation background, $n=\frac{1}{2}$ and all our equations, such as (21), (23), and (24), coincide with results obtained by Holcomb and Tajima [5] for $n=\frac{1}{2}$.

The wave equation (13) can also be solved in other types of background. For example, for a de Sitter background, the metric components will be $\gamma_{11}=\gamma_{22}=\gamma_{33}=e^{2\alpha t}$ and $K=-3\alpha$. The wave equation for a free electromagnetic wave in this case will be [from Eq. (13)]

$$\frac{1}{c^2}e^{2\alpha t}[\partial_t^2\mathbf{E}+7\alpha\partial_t\mathbf{E}+12\alpha^2\mathbf{E}]=\nabla^2\mathbf{E}\ .$$

As usual, after separation of variables, the spatial part remains the same as the Newtonian solution and some suitable transformation of variables again leads the time part to form Bessel equation of order $\frac{1}{2}$. But the argument of the Hankel function solution now contains an exponential of t instead of a simple power function of t. The complete solution for $\mathbf{E}$ looks like

$$\mathbf{E}=\mathbf{E}_0\mathbf{e}e^{i\mathbf{k}\cdot\mathbf{r}}e^{-(7/2)\alpha t}H_{1/2}^{(2)}\left[\frac{ck}{\alpha}e^{-\alpha t}\right]\ .$$

The redshift will be given by

$$\omega = \frac{\omega_i}{\alpha t e^{\alpha t}} .$$

Now we shall consider the propagation of small amplitude electromagnetic waves in a plasma with the background metric = diag$(-1, t^{2n}, t^{2n}, t^{2n})$. If we linearize Eq. (12) and ignore $\mathbf{v}\times\mathbf{B}$ terms, the equation of motion for a particle becomes

$$\frac{d}{dt}(\mathbf{p}t^{2n}) = qt^{2n}\mathbf{E} . \quad (25)$$

With the help of Eq. (21), Eq. (25) can be integrated as:

$$\mathbf{p}t^{2n} = q\int t^{2n}\mathbf{E}\,dt$$

$$= qE_0\mathbf{e}e^{i\mathbf{k}_i\cdot\mathbf{r}}\int t^{(1-3n)/2}H_{1/2}^{(2)}\left[\left[\frac{k_i c}{1-n}\right]t^{1-n}\right]dt$$

$$= \frac{iq}{k_i c}E_0\mathbf{e}e^{i\mathbf{k}_i\cdot\mathbf{r}}t^{(1-5n)/2}H_{1/2}^{(2)}\left[\left[\frac{k_i c}{1-n}\right]t^{1-n}\right] .$$

Thus,

$$\mathbf{p} = \frac{iq}{k_i c}t^n\mathbf{E} = \frac{iq}{\omega_i}t^n\mathbf{E} \quad (26)$$

and from the relation

$$\mathbf{p} = \mu\Gamma\mathbf{v} ,$$

the three-velocity vector

$$\mathbf{v} = \frac{iq}{\mu\Gamma\omega_i}t^n\mathbf{E} \quad (27)$$

where μ and Γ are the rest mass of the particle and Lorentz boost factor respectively. The current density vector is

$$\mathbf{J} = \sum\rho_e\mathbf{v} = \sum\frac{\rho_e}{\mu\Gamma}\mathbf{p} = \sum\frac{qn_0}{\mu\Gamma}\mathbf{p} . \quad (28)$$

Inserting this in Maxwell equation (9) and assuming e^+e^- plasma, we obtain

$$\frac{t^n}{c}\left[(\partial_t\mathbf{E}_T - K\mathbf{E}_T) + \frac{8\pi n_0 e}{\mu\Gamma}\mathbf{p}_T\right] = (\nabla\times\mathbf{B})_T , \quad (29)$$

where the subscript T stands for transverse waves. Now using equation (21) and (26) in (29), one can deduce, after some straightforward calculation, the relation

$$-\frac{i\omega_i}{c}\epsilon(\omega)\mathbf{E}_T = (\nabla\times\mathbf{B})_T , \quad (30)$$

where the dielectric constant $\epsilon(\omega)$ is given by

$$\epsilon(\omega) = 1 - \frac{\omega_{pT}^2}{\omega^2} , \quad (31)$$

with

$$\omega_{pT}^2 = \frac{8\pi n_0 e^2}{\mu T} , \quad (32)$$

which is nothing but the natural frequency of transverse vibration of the plasma distribution. We note from Eq. (11) that the background number density of particles diminishes like t^{-3n} and also from Eqs. (23) and (26) we find that the amplitude of the momentum three-vector $|\mathbf{p}| \sim t^{-2n}$. So the demand that Eq. (9) be true for all time leads us to the conclusion that $\mathbf{v} \sim t^{-n}$ and $\Gamma \sim t^{-n}$. Thus we see that $\omega_{pT} \sim t^{-n}$ and hence $\epsilon(\omega)$ does not depend on time. The same result was previously obtained by Holcomb and Tajima [5] in a more restricted case. An important observation at this point is that in the general case, although the temporal behavior of the electromagnetic fields, Lorentz boost, momentum etc. are modified with the parameter n, the dielectric constant of the plasma distribution remains independent of time. The background spacetime does not in anyway affect the constancy of ϵ.

In order to find the dispersion relation, we take curl on both sides of Eq. (10) and replace $\nabla\times\mathbf{B}$ by Eq. (30) to get the relation

$$\left[\frac{\partial}{\partial t} - K\right]\left[-\frac{i\omega_i}{c}\epsilon(\omega)\mathbf{E}\right] = -cA^{-1}\nabla^2\mathbf{E} \qquad [\text{with}\,\nabla\cdot\mathbf{E}=0] ,$$

which, in view of Eq. (16), yields

$$\left[k_i^2 c - \epsilon(\omega)\frac{\omega_i^2}{c}\right]\mathbf{E} = 0 ,$$

or

$$\frac{k_i^2c^2}{\omega_i^2} = 1 - \frac{\omega_p^2}{\omega^2} ,$$

or

$$k^2c^2 = \omega^2 - \omega_p^2 ,$$

i.e.,

$$\omega^2 = \omega_p^2 + k^2c^2 .$$

This is the dispersion relation for the transverse electromagnetic waves in a plasma and expresses the relation between ω and the wave number k. It should be mentioned that the dielectric constant, calculated in the de Sitter background, is also independent of time.

To sum up, we have extended the work of Holcomb and Tajima for a more general background to see if the nature of the background is responsible for any qualitative change in plasma dynamics. We got the interesting result that the electric field decays more rapidly if the rate of expansion of the Universe is faster. This is quite an expected result as the frictionlike term [i.e., $7nt^{2n-1}\partial_t\psi$ in Eq. (17)] is greater for higher values of n. The fact that the dielectric constant ϵ is independent of time even in a curved background *whatever the choice of the metric* is the most important result of this work. However, its full implications have to be worked out in more detail before any definite conclusion can be made in this regard.

N.B. and A.S. would like to thank the University Grants Commission, New Delhi, for financial support.

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