Exact superpotentials in four dimensions

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Supersymmetric gauge theories in four dimensions can display interesting nonperturbative phenomena. Although the superpotential dynamically generated by these phenomena can be highly nontrivial, it can often be exactly determined. We discuss some general techniques for analyzing the Wilsonian superpotential and demonstrate them with simple but nontrivial examples.

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I. INTRODUCTION

There are three motivations to study supersymmetric field theories. First, theories with dynamical supersymmetry breaking can be used to solve the hierarchy problem. Second, they are relevant to topological field theories. Finally, they are tractable and can thus be used as testing grounds for various ideas about the dynamics of four-dimensional quantum field theories.

In four-dimensional quantum field theory, exact results, aside from those which follow directly from symmetries, are very hard to come by. Supersymmetric theories, however, are different. The combination of the holomorphy of the Wilsonian superpotential $W_{\rm eff}$ with the symmetries and selection rules provides powerful constraints. These constraints should be viewed as "kinematics." When combined with approximate dynamical information about the asymptotic behavior of the superpotential we can sometimes determine it exactly [1].

In this paper we continue this line of reasoning and apply it to more complicated systems. Unlike the models analyzed in Ref. [1], where the $W_{\rm eff}$ turned out to be rather simple functions, here we find highly nontrivial effective superpotentials. These reflect interesting new nonperturbative effects.

We will always be interested in the Wilsonian effective action. If supersymmetry is broken we limit ourselves to scales above the breaking scale, where supersymmetry is linearly realized. We will integrate out the massive modes and focus on the dynamics of the light fields. In this respect we follow the point of view of Refs. [2,3]. An alternate approach [4,5] uses an effective Lagrangian which also depends on some of the massive fields. We discuss generally how to integrate these massive fields into the low energy theory.

In Sec. II we summarize our techniques. The low energy superpotential is constrained by the symmetries and holomorphy as in Ref. [1]. The dynamical analysis can proceed in two different ways: we can analyze the asymptotic behavior of the superpotential in various limits, control it singularities, and thus completely determine it. Alternatively, we can derive differential equations that the superpotential satisfies as a function of various coupling constants and thus solve for it.

In Sec. III we give a brief review of the dynamics of supersymmetric QCD. Sections IV and V are devoted to examples demonstrating our techniques.

In Sec. IV we study an $SU(2)_1\times SU(2)_2$ gauge theory with matter fields in the representations $Q=(\mathbf{2},\mathbf{2})$ and $L_i=(\mathbf{1},\mathbf{2})$ for $i=1,\ldots,2n$. In terms of the gauge singlet composites $X=Q^2$ and $V_{ij}=L_iL_j$, we find the superpotentials

$$W_{n=0}=\frac{\left[\Lambda_1^{5/2}\pm\Lambda_2^{5/2}\right]^2}{X}\,,$$
$$W_{n=1}=\frac{\Lambda_1^5V_{12}}{XV_{12}-\Lambda_2^4}\,, \qquad (1.1)$$
$$W_{n=2}=-\frac{X\,\mathrm{Pf}V}{\Lambda_2^3}\pm2\left[\frac{\Lambda_1^5\,\mathrm{Pf}V}{\Lambda_2^3}\right]^{1/2}\,,$$

where Λ_1 and Λ_2 are the scales of the $SU(2)_1$ and $SU(2)_2$ gauge theories, respectively. Note that for $n=2$ the fields V are classically constrained by Pf $V=0$. However, quantum mechanically, Pf V is a massive field whose expectation value satisfies $\langle \mathrm{Pf}\ V\rangle=\Lambda_1^5\Lambda_2^3/X^2$. The low energy effective Lagrangian, after Pf V is integrated out, is

$$\tilde{W}_{n=2}=\frac{\Lambda_1^5}{X}\,. \qquad (1.2)$$

The $\pm$ signs in Eqs. (1.1) label distinct low energy ground states, differing in the expectation value of a massive field which is not included in the low energy effective action. The $\pm$ sign in the superpotential $W_{n=2}$ in (1.1) corresponds to two different branches of the square root; they are related by a discrete symmetry of the theory and therefore describe equivalent physics. On the other hand, the $\pm$ sign in the superpotential $W_{n=0}$ in (1.1) labels two inequivalent (unrelated by a symmetry) low energy ground states. The low energy theory includes then both continuous fields and discrete labels. A similar phenomenon was observed in Ref. [6].

In Sec. V we consider an $SO(5)\times SU(2)$ gauge theory with a matter field in the representation $F=(\mathbf{4},\mathbf{2})$ with or without two fields $L_{1,2}=(\mathbf{1},\mathbf{2})$. In terms of the gauge singlet fields $X=F^2$ and $Y=L_1L_2$, we find the superpotentials to be

$$W_{n=0}=\frac{2\Lambda_5^4}{\sqrt{X\pm 2\Lambda_2^2}}\ , \tag{1.3}$$

$$W_{n=1}=\frac{\Lambda_5^4}{\sqrt{X}}g\left[v=\frac{\Lambda_5^4\Lambda_2^3}{X^{5/2}Y}\right]\ ,$$

$$\text{with } g=\tfrac{1}{2}h(5-h^2) \text{ and } v=\tfrac{1}{2}(h^{-3}-h^{-5})\ .$$

For $W_{n=1}$ we were unable to find a closed form expression from the parametric solution in (1.3). The sign choice in $W_{n=0}$ in (1.3) is, again, a discrete label in the low energy theory. The ground states differing by this sign choice are here related by a symmetry and are thus physically equivalent.

As is clear from Eqs. (1.1) and (1.3), the superpotentials are quite complicated. They are generated by a variety of dynamical mechanisms. For example, the large field behavior of $W_{n=1}$ in (1.1) arises from an infinite sum over instantons whereas the large field behavior of $W_{n=0}$ in (1.3) arises from an interplay between gaugino condensation in the two groups and an infinite number of instantons. The dynamics leading to the superpotential generally depends upon the region of field space considered; the holomorphic superpotential smoothly interpolates between them.

We conclude in Sec. VI with an outlook and various speculations.

II. TECHNIQUES

Our general framework is a supersymmetric field theory based on a gauge group $\mathcal{G}$ and matter superfields ϕ^i transforming in representations R_i of $\mathcal{G}$. The tree level superpotential is

$$W_{\text{tree}}=\sum_r g_r X^r(\phi^i)\ , \tag{2.1}$$

where X^r are gauge invariant polynomials in the fundamental fields. Apart from the tree level couplings g_r, we also have gauge couplings: every simple factor $\mathcal{G}_s$ in $\mathcal{G}=\prod_s \mathcal{G}_s$ is characterized by a scale Λ_s.

Our analysis proceeds in several steps.

(I) We first set the tree level superpotential to zero, i.e., $g_r=0$. At the classical level there are then "flat directions" in field space where all the gauge D terms vanish. The expectation values of the scalar components of ϕ^i in these classical ground states spontaneously break the gauge symmetry. We refer to the space of classical ground states as "the classical moduli space." Instead of using the fundamental fields ϕ^i as coordinates on this space we can use gauge invariant combinations X^r. The X^r are the light superfields in the leading approximation; the classical low energy superpotential for them vanishes. It is sometimes the case that the fields X^r are constrained classically [7]. In this situation, we can represent the constraint with a Lagrange multiplier in the effective superpotential.

(II) Next we turn on the coupling constants g_r and Λ_s; i.e., consider the full quantum theory. If it is clear that some of the fields X^r are massive we can either keep them in our description or look for an effective Lagrangian after they have been integrated out. The full quantum superpotential W_{eff} is constrained by two kinematic constraints [1].

(1) Holomorphy: W_{eff} is a holomorphic function of the fields X^r and the coupling constants g_r and Λ_s. Holomorphy in the coupling constants follows from thinking about them as background fields. A related discussion of holomorphy in coupling constants of various expectation values may be found in Refs. [5,8].

(2) Symmetries: W_{eff} is invariant under all the symmetries in the problem. If a symmetry is explicitly broken by the coupling constants we can assign transformation laws to these constants such that W_{eff} is invariant under the combined transformation on the fields X^r and the coupling constants. Anomalous symmetries should be viewed as explicitly broken. However, by assigning appropriate transformation laws to the scales Λ_s of the gauge groups, they also lead to selection rules.

(III) The dynamics enters through the analysis of W_{eff} at various asymptotic values of its arguments. In Ref. [1] the weak coupling limit of small g_r, small Λ_s, and large fields X^r was powerful enough to completely determine W_{eff}. In our new examples these constraints do not uniquely determine W_{eff} and therefore we also need to study other limits. Among these limits will be strong coupling and small fields X^r. The key fact is that W_{eff}, by holomorphy, is completely determined by its behavior at various asymptotics and by its singularities.

A special limit that is often useful is when one of the matter fields is very heavy. Its mass m is one of the coupling constants g_r. When it is large the massive field can be integrated out. We can do this either in the microscopic gauge theory or in the effective low energy theory. The first of these yields a new microscopic gauge theory with fewer matter fields and whose coupling constants, g_r and Λ_s, depend on m. The low energy effective superpotential of this theory should be the same as the one obtained by integrating out the appropriate fields in the effective Lagrangian of the original theory.

It is often the case that the two kinematic conditions and the dynamics at small g_r constrain the effective superpotential to be of the form

$$W_{\text{eff}}=W_{\text{eff}}(g_r=0)+\sum_r g_r X^r$$
$$=W_{\text{dyn}}(\Lambda_s,X^r)+W_{\text{tree}}\ ; \tag{2.2}$$

i.e., it is linear in the g_r's (W_{dyn} includes the Lagrange multiplier terms for the various constraints that the composite fields X^r should satisfy). This is the case in all of our examples. When this is not the case we conjecture that it is always possible and natural to redefine the fields X_r as a function of the g_r to bring the superpotential to the form (2.2) (for a related discussion see Ref. [9]). Now let us integrate out some field, say X^0. The resulting superpotential $\widetilde{W}_{\text{eff}}$ is obtained by solving

$$\frac{\partial W_{\text{eff}}}{\partial X^0}(\langle X^0\rangle)=0 \tag{2.3}$$

for $\langle X^0 \rangle$ as a function of all the other fields X^r ($r \neq 0$) and all the coupling constants g_r, and substituting back into $W_{\rm eff}$. Clearly $\widetilde{W}_{\rm eff}$ is not linear in g_0. To see that it is linear in all the other g_r's, note that

$$\frac{\partial \widetilde{W}_{\rm eff}}{\partial g_r} = \frac{\partial W_{\rm eff}}{\partial g_r}(\langle X^0 \rangle) + \frac{\partial \langle X^0 \rangle}{\partial g_r}\frac{\partial W_{\rm eff}}{\partial X^0}(\langle X^0 \rangle)$$
$$= X^r \quad \text{for } r \neq 0 \ . \tag{2.4}$$

This suggests the definition of $\widetilde{W}_{\rm dyn}$,

$$\widetilde{W}_{\rm eff} = \widetilde{W}_{\rm dyn} + \sum_{r \neq 0} g_r X^r \ , \tag{2.5}$$

which depends on the light fields X^r ($r \neq 0$), the scales Λ_s and g_0. An equation similar to (2.4) for $r=0$ is

$$\frac{\partial \widetilde{W}_{\rm eff}}{\partial g_0} = \frac{\partial \widetilde{W}_{\rm dyn}}{\partial g_0} = \langle X^0 \rangle \ . \tag{2.6}$$

A slight generalization of the previous discussion involves the gauge coupling constants. Unlike the g_r, our effective Lagrangians do not involve any field which couples linearly to the gauge coupling constants. The reason for this is that the corresponding fields $S_s = -(W_\alpha^2)_s$ [with this sign the lowest component of S_s is $+(\lambda\lambda)_s$] are always massive and thus do not have to be included in a low energy Wilsonian effective action. However, by repeating the previous discussion with g_0 replaced by $\ln \Lambda_s^{n_s}$, where n_s is determined by the one loop β function [e.g., for SU(N_c) gauge theory with N_f quark flavors in the fundamental and antifundamental representations, $n = 3N_c - N_f$], we learn that

$$\frac{\partial W_{\rm eff}}{\partial \ln \Lambda_s^{n_s}} = \frac{\partial W_{\rm dyn}}{\partial \ln \Lambda_s^{n_s}} = \langle S_s \rangle \ . \tag{2.7}$$

In deriving (2.7) we are assuming that the effective superpotential with the S_s included is linear in $\ln \Lambda_s^{n_s}$, as with the other couplings g_r in (2.5). This is the case in all our examples and, as with the other g_r, we conjecture that it is always true.

To summarize, we conjecture that at every scale the superpotential has the form

$$W_{\rm eff} = W_{\rm dyn} + \sum_r g_r X^r \ , \tag{2.8}$$

where $W_{\rm dyn}$ depends on the fields X^r and on the coupling constants of the fields X^0 and S_s which have been integrated out such that

$$\frac{\partial W_{\rm dyn}}{\partial g_r} = 0 \quad \text{for } r \neq 0 \ ,$$
$$\frac{\partial W_{\rm dyn}}{\partial g_0} = \langle X^0 \rangle \ , \tag{2.9}$$
$$\frac{\partial W_{\rm dyn}}{\partial \ln \Lambda_s^{n_s}} = \langle S_s \rangle \ .$$

These equations can be used in two different ways.

(1) If we know the expectation values $\langle X^0 \rangle$ or $\langle S_s \rangle$ as a function of the other fields and coupling constants we can use Eq. (2.9) to solve for $W_{\rm dyn}$. This leads to differential equations for the superpotential.

(2) If we know the g_0 dependence of $W_{\rm dyn}$ at some scale we can find the expectation value of the massive field $\langle X^0 \rangle$ and using this information we can find the superpotential before it has been integrated out (we will refer to this procedure as "integrating in"). As explained in point 2 above, we can use the "integrating in" procedure to construct an effective Lagrangian similar to that of Ref. [4] involving the massive fields S_s. However, since the fields S_s are always massive and our effective actions are Wilsonian, the meaning of such an effective action for the S_s is not clear to us. Its only virtue is that it allows one to determine the $\langle S_s \rangle$ by their equations of motion.

The third equation in (2.9) allows us to derive the Konishi anomaly [5]

$$\left\langle \phi^i \frac{\partial W_{\rm tree}}{\partial \phi^i} \right\rangle - \sum_s \mu_i^s \langle S_s \rangle = 0 \quad \text{for every } i \ , \tag{2.10}$$

where μ_i^s is the index of the representation of the field ϕ^i under the $\mathcal{G}_s$ gauge group. To do this, consider the U(1)$_i$ transformation $\phi^i \to e^{i\alpha}\phi^i$ under which the composite field X^r has charge q_r^i. This symmetry is broken both by the coupling constants g_r and by the anomaly. However, if we also assign charge $-q_r^i$ to g_r and charge μ_i^s to $\Lambda_s^{n_s}$, the invariance of the superpotential states that

$$\sum_r q_r^i X^r \frac{\partial W_{\rm eff}}{\partial X^r} - \sum_r q_r^i g_r \frac{\partial W_{\rm eff}}{\partial g_r} + \sum_s \mu_i^s \Lambda_s^{n_s} \frac{\partial W_{\rm eff}}{\partial \Lambda_s^{n_s}} = 0 \ . \tag{2.11}$$

Using Eqs. (2.8) and (2.9),

$$\sum_r q_r^i X^r \frac{\partial W_{\rm dyn}}{\partial X^r} + \sum_s \mu_i^s \langle S_s \rangle = 0 \ . \tag{2.12}$$

Imposing the equations of motion $\partial W_{\rm eff}/\partial X^r(\langle X^r \rangle)=0$, this leads to

$$\sum_r q_r^i g_r \langle X^r \rangle = \sum_s \mu_i^s \langle S_s \rangle \tag{2.13}$$

which is equivalent to (2.10). Note that (2.12) applies more generally to off-shell X_r.

III. REVIEW OF A SIMPLE EXAMPLE: SUPERSYMMETRIC QCD

In this section we illustrate some of our basic ideas and conventions in the context of a well studied example: supersymmetric SU(N_c) gauge theory with N_f flavors of matter superfields Q_{cf} and $\widetilde{Q}^{cf}$ in the representations $\mathbf{N}_c$ and $\overline{\mathbf{N}}_c$, respectively, of SU(N_c).

A. Kinematics: symmetries and holomorphy

The exact Wilsonian effective superpotential for supersymmetric QCD is completely determined by the symmetries along with holomorphy. The superpotential can only depend on the combination of fields $\Delta \equiv \det_{ff'}(Q_{cf}\tilde{Q}^{cf'})$, the unique gauge singlet which is also a single under the $SU(N_f)_L \times SU(N_f)_R$ global flavor symmetry. For each flavor f there are symmetries $U(1)_{Q_f}$ and $U(1)_{\tilde{Q}_f}$ which count the superfield Q_{cf} or $\tilde{Q}_{cf}$, respectively, with charge one and all other fields with charge zero. In addition there is an R symmetry $U(1)_R$ under which squarks have charge zero, the quark components of the chiral superfields have charge -1, and the gauginos have charge $+1$. The charge conjugate fields, which make up the antichiral superfields, of course have the opposite charges under all these symmetries.

Quantum mechanically, one linear combination of the above U(1) currents is anomalous. Rather than finding linear combinations for which the anomaly cancels, it is possible to use the anomaly to find selection rules. Following the spirit of [7], we think of $Y = 8\pi^2/g^2 + i\theta$, which is the coupling for S, as a background chiral field. It is seen that the anomaly in each of the U(1) transformations can be canceled by combining them with a transformation of $\Lambda_{N_c,N_f}^{3N_c-N_f} = \mu^{3N_c-N_f} e^{-Y(\mu)}$ (the exponent is given exactly in our Wilsonian treatment by the one loop β function [8]). The charge to be assigned to the scale in order to cancel the anomaly is related to the charge assignments of the quarks ψ_{cf} and $\tilde{\psi}^{cf}$ and the gauginos λ by

$$q(\Lambda_{N_c,N_f}^{3N_c-N_f}) = \sum_f [q(\psi_{cf}) + q(\tilde{\psi}_{cf})] + 2N_c q(\lambda) . \quad (3.1)$$

The exact superpotential must have charge zero under the $2N_f$ symmetries $U(1)_{Q_f}$ and $U(1)_{\tilde{Q}_f}$ and have charge two (for the lowest component) under the R-symmetry $U(1)_R$. Δ has charge one under each of the $2N_f$ U(1) symmetries and it has zero R charge. Using (3.1). $\Lambda_{N_c,N_f}^{3N_c-N_f}$ also has charge one under each of the $2N_f$ U(1) symmetries and it has charge $2(N_c-N_f)$ under the R symmetry. Therefore, the exact superpotentials is

$$W_{\text{exact}} = a \left[\frac{\Lambda_{N_c,N_f}^{3N_c-N_f}}{\det_{ff'}(Q_{cf}\tilde{Q}^{cf'})} \right]^{1/(N_c-N_f)} , \quad (3.2)$$

where a is a constant. For a single gauge group, our use of the additional symmetry which is broken by the anomaly (through an expectation value of Λ) only gave information which could have been obtained anyway by using dimensional analysis, as was done in Ref. [2]. In the examples considered in this paper, however, it will be crucial for disentangling effects associated with several gauge groups.

The superpotential (3.2) only makes sense for $N_f < N_c$ [3]: for $N_f = N_c$ the exponent is infinite and for $N_f > N_c$ the determinant is (classically) zero since the rank of $Q_{cf}\tilde{Q}^{cf'}$ is then only N_c. Therefore for $N_f \geq N_c$ the classical vacuum degeneracy is not removed quantum mechanically.

For $N_f \geq N_c - 1$ the gauge group can be completely broken by the expectation value of the squarks. For $N_f < N_c - 1$, there is always an unbroken $SU(N_c - N_f)$ subgroup. The superpotential (3.2) picks up a $\mathbf{Z}_{(N_c-N_f)}$ phase under shifting the theta angle by 2π. This phase labels different, though physically equivalent, vacua of the theory coming from the spontaneous breaking of a discrete symmetry (by gaugino condensation) in the low energy $SU(N_c - N_f)$ theory.

B. Dynamics: instantons or gaugino condensation

We now review the dynamics [3] leading to the superpotential (3.2). In the case where $N_f = N_c - 1$, the gauge group is completely broken by the Higgs mechanism and so instanton methods are reliable. The Λ dependence of (3.2) indicates that the superpotential for this case is associated with a single instanton in the completely Higgs-like $SU(N_c)$. An explicit instanton calculation leads to (3.2) with a nonzero coefficient a [3]. It turns out to be natural to define the scale Λ_{N_c,N_c-1} so that $a=1$ in this case. To relate this Λ to, say, $\Lambda_{\overline{\text{MS}}}$ requires a detailed instanton calculation. Fortunately, such information is unnecessary for our purposes.

Having defined our normalization convention for the case of $N_c - 1$ flavors, the constant a in (3.2) can be determined for all $N_f < N_c$ by adding mass terms for $N_c - N_f - 1$ of the flavors and integrating them out. The symmetries and holomorphy imply that the exact superpotential for the theory with the mass terms is

$$W_{\text{exact}} = \frac{\Lambda_{N_c,N_c-1}^{2N_c+1}}{\det_{ff'}(Q_{cf}\tilde{Q}^{cf'})} + \sum_{f=N_f+1}^{N_c-1} m_f Q_{cf}\tilde{Q}^{cf} . \quad (3.3)$$

For energy scales below the m_f we integrate out the massive flavors by solving for them using their equations of motion obtained from (3.3) and find

$$W_{\text{exact}} = \epsilon_{(N_c-N_f)}(N_c - N_f) \left[\frac{\Lambda_{N_c,N_f}^{3N_c-N_f}}{\det(Q\tilde{Q})} \right]^{1/(N_c-N_f)} , \quad (3.4)$$

where $\epsilon_{(N_c-N_f)}$ is a $\mathbf{Z}_{(N_c-N_f)}$ phase and where now $\det(Q_{cf}\tilde{Q}^{cf'})$ is taken only over the N_f flavors in the low energy theory. The scale Λ_{N_c,N_f} in Eq. (3.4) of the low energy theory is related to the scale Λ_{N_c,N_c-1} of the high energy theory by

$$\Lambda_{N_c,N_f}^{3N_c-N_f} = \Lambda_{N_c,N_c-1}^{2N_c+1} \prod_{f=N_f+1}^{N_c-1} m_f \quad (3.5)$$

(here we absorb a possible threshold factor into our definition of Λ_{N_c,N_f}). Note that, as in (2.6), we can take $\partial/\partial m_f$ of (3.4), using (3.5), to recover the expectation values of the fields which have been integrated out: $\partial W_{\text{exact}}/\partial m_f = \langle Q_f \tilde{Q}^f \rangle$. At this point we can forget about the massive flavors which have been integrated out; the superpotential (3.4) is the exact effective low energy superpotential for $SU(N_c)$ gauge theory with N_f light flavors.

For $N_f < N_c - 1$ the gauge group is not completely broken along the flat directions and the dynamics leading to (3.4) is associated with gaugino condensation in the unbroken $SU(N_c - N_f)$ gauge group rather than with instantons [3]. The low-energy $SU(N_c - N_f)$ pure Yang-Mills theory has a scale $\Lambda_{(N_c - N_f),0}$ which is related to the scale of the high-energy theory by matching the running coupling constant at the scale, set by the order parameter Δ, where the theory becomes broken by the Higgs mechanism:

$$\left[\frac{\Lambda_{N_c,N_f}}{E}\right]^{3N_c - N_f} = \left[\frac{\Lambda_{(N_c - N_f),0}}{E}\right]^{3(N_c - N_f)} \quad \text{at } E = (\Delta)^{1/2N_f} . \tag{3.6}$$

As before, we absorb the order one threshold coefficient into the definition of $\Lambda_{(N_c - N_f),0}$. The superpotential (3.4) is thus given by

$$W = \epsilon_{(N_c - N_f)}(N_c - N_f)\Lambda^3_{(N_c - N_f),0} , \tag{3.7}$$

where $\Lambda_{(N_c - N_f),0}$ is to be thought of as a function of Δ and the high-energy scale Λ_{N_c,N_f}. Using (2.7) in the low energy $SU(N_c - N_f)$ theory [so $n = 3(N_c - N_f)$], (3.7) gives the gaugino condensate

$$\langle S_{SU(N_c - N_f)} \rangle = \epsilon_{(N_c - N_f)} \Lambda^3_{(N_c - N_f),0} . \tag{3.8}$$

Indeed, superpotential (3.4) with Λ_{N_c,N_f} held fixed is exactly equivalent to the low-energy superpotential, obtained by inserting (3.8) into the Wess-Zumino (WZ) term

$$W_{WZ} = \int d^2\theta [3(N_c - N_f) - (3N_c - N_f)] \ln \left[\frac{\Delta^{1/2N_f}}{M}\right] S_{SU(N_c - N_f)} , \tag{3.9}$$

needed in the low energy theory to correct the β function as in (3.6), with $\Lambda_{(N_c - N_f),0}$ held fixed. Note that by starting with the instanton-induced superpotential for $N_f = N_c - 1$, which is calculated to be nonvanishing, and integrating out some of the matter fields, we have derived gaugino condensation [10].

The $\mathbb{Z}_{N_c - N_f}$ phase in (3.8) and (3.4) labels the physically equivalent vacua of $SU(N_c - N_f)$ Yang-Mills theory associated with the spontaneous breaking of the $\mathbb{Z}_{2(N_C - N_f)}$ chiral symmetry left unbroken by instantons down to $\mathbb{Z}_2$ by the gaugino condensate. The vacua are physically equivalent because they are related by a discrete, nonanomalous, R symmetry. In particular, the discrete $\mathbb{Z}_{2(N_c - N_f)}$ R symmetry under which the squarks are neutral is anomaly free. The terms in (3.4) are invariant under this symmetry but, because the superpotential has R-charge 2, the superpotential picks up a $\mathbb{Z}_{N_c - N_f}$ phase under the symmetry. Therefore, vacua differing by the phase in (3.4) are physically equivalent.

Finally note that if we add mass terms for all of the flavors and integrate them out, (2.7), along with the equations of motion and the matching condition on the scales, gives

$$\langle S_{SU(N_c)} \rangle = \epsilon_{N_c} \Lambda^3_{N_c,0} , \tag{3.10}$$

with a normalization consistent with (3.8). Using the equations of motion from (3.4) plus the added tree-level mass terms, we also find

$$m_f \langle Q_{cf} \tilde{Q}^{cf} \rangle = \epsilon_{N_c} \Lambda^3_{N_c,0} . \tag{3.11}$$

The equality $m_f \langle Q_{cf} \tilde{Q}^{cf} \rangle = \langle S \rangle$, seen from (3.10) and (3.11), is also a consequence of the Konishi anomaly; this provides a nontrivial check on our normalization conventions.

C. Continuous moduli spaces of inequivalent vacua for $N_f \geq N_c$

We can describe the theories with $N_f \geq N_c$ by starting with the theory with $N_f = N_c - 1$, "integrating in" very massive and thus decoupled matter, and then reducing the mass terms until the extra matter appears in the low energy theory. The central feature of the theories with $N_f \geq N_c$ is that, even at the nonperturbative level, they have a moduli space of vacua.

For example, when $N_f = N_c$ we see from Eq. (3.4) that no invariant superpotential exists. Thus there is a continuum of inequivalent vacua corresponding to different squark expectation values subject to the D-flatness conditions. As discussed in Ref. [7], this moduli space of vacua differs from the classical space of D-flat vacua. Classically the singlets $\Delta = \det_{ff'}(Q_{cf} \tilde{Q}^{cf'})$, $B = \det Q_{cf}$, and $\tilde{B} = \det \tilde{Q}^{cf}$ satisfy the constraint $\Delta = B\tilde{B}$. However, at the quantum level this is modified (by instantons) to

$$\Delta - B\tilde{B} = \Lambda^{2N_c} , \quad \text{Pf } V = \Lambda^4 \text{ for } N_c = 2 , \tag{3.12}$$

where for $N_c = 2$ the constraint is in terms of the SU(2) singlet fields $V_{fg} = Q_{cf} Q_{c'g} \epsilon^{cc'}$, which transforms as a 6 under the $SU(4)_F$ flavor symmetry.

For $N_f = N_c + 1$, the quantum moduli space of vacua coincides with the classical space [7]. The singularity at the origin in this case is resolved by having extra light fields come down.

IV. ILLUSTRATIVE EXAMPLES BASED ON $SU(2)_1 \times SU(2)_2$ GAUGE THEORY

In this section we illustrate some of our basic points and techniques in the context of a class of very simple examples based on $SU(2)_1 \times SU(2)_2$ gauge theory.

A. Matter content $Q=(2,2)$ and $L_{\pm}=(1,2)$

There are two independent classical D-flat directions, which can be labeled by $X=Q^2\equiv\frac{1}{2}Q_{\alpha\beta}Q_{\gamma\delta}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}$ and $Y=L_{\alpha+}L_{\beta-}\epsilon^{\alpha\beta}$. At generic values of X and Y the gauge group is completely broken. At the classical level, for $X=0$ $SU(2)_1$ is unbroken and for $Y=0$ there is an unbroken diagonal $SU(2)_D$.

The symmetries $U(1)_Q$, $U(1)_{L_\pm}$, and $U(1)_R$, with charges assigned as in (3.1) to the scales Λ_1 and Λ_2 of $SU(2)_1$ and $SU(2)_2$, determine the superpotential to be of the form

$$W=\frac{\Lambda_1^5}{X}f\left[\frac{\Lambda_2^4}{XY}\right]. \tag{4.1}$$

Note that for $\Lambda_1\to 0$ the superpotential goes to zero, which is the proper behavior for the $SU(2)_2$ gauge theory with four doublets, as discussed in the previous section.

In order to determine the function $f(u=\Lambda_2^4/XY)$ we first study the limit $u\to 0$. A term in f proportional to u^n has a Λ_1 and Λ_2 dependence characteristic of an $SU(2)_1\times SU(2)_2$ effect with instanton charges $(1,n)$. Because the gauge group is completely broken we only expect contributions associated with instantons, i.e., only terms proportional to u^n with n integer. For small u, f thus has the expansion

$$f=\sum_{n=0}^{\infty}a_n u^n . \tag{4.2}$$

If we set $\Lambda_2=0$, the theory is SU(2) with one flavor (two doublets) and (3.4) gives $a_0=1$. The term a_1u in (4.2) has the quantum numbers of a (1,1) instanton; it can be understood as follows. For $X\gg Y$ the gauge group is broken to the diagonal subgroup $SU(2)_D$. An instanton in $SU(2)_D$ then gives, according to (3.4), a superpotential Λ_D^5/Y. Matching the running coupling constant of the low energy theory, $g_D^{-2}=g_1^{-2}+g_2^{-2}$, to the high energy ones at $E=X^{1/2}$, the scales of the low and high energy theories are related by $\Lambda_D^5=\Lambda_1^5\Lambda_2^4/X^2$ (there is no finite threshold correction here in our conventions for the scales) and thus $a_1=1$ in (4.2).

To further determine the function f we temporarily set $\Lambda_1=0$. Then $SU(2)_2$ couples to four doublets and the model has an SU(4) global symmetry. The massless modes can be expressed in terms of $X=Q^2$, $Y=L_+L_-$, and two doublets $A_\pm=QL_\pm$. Classically, these six fields are constrained by $XY=A_+A_-$. Quantum mechanically, this constraint is modified as in (3.12) to

$$XY-A_+A_-=\Lambda_2^4 . \tag{4.3}$$

Now we weakly gauge $SU(2)_1$. In this limit of $\Lambda_2\gg\Lambda_1$, the theory is simply $SU(2)_1$ gauge theory with the two doublets $A_\pm$ and the two singlets X and Y, satisfying the constraint (4.3). For nonzero A_+A_- the $SU(2)_1$ gauge symmetry is thus completely broken and the light fields are only X and Y. There is an unbroken gauge symmetry at $A_+A_-=0$ which, because of $SU(2)_2$ instanton effects in (4.3), is at $XY=\Lambda_2^4$ rather than the classical value of zero. Therefore, the superpotential can only be singular at $u=1$. In particular, since the gauge symmetry is broken at $XY=0$ the superpotential cannot be singular there; the function $f(u)$ in (4.1) must thus satisfy $\lim_{u\to\infty}f(u)\le O(1/u)$.

The singularity of the superpotential at $u=1$ is given by (3.4) for $SU(2)_1$ with the two doublets $A_\pm$. We thus have in the limit

$$\Lambda_2^2\gg A_+,A_-\gg\Lambda_1^2 , \tag{4.4}$$

$$W=\frac{\Lambda_1^5\mu}{A_+A_-}=\frac{\Lambda_1^5\mu}{XY-\Lambda_2^4} , \tag{4.5}$$

where μ is a dimensionful normalization factor, needed because $A_\pm$ are not canonically normalized doublets but, rather, composites. Comparing with (4.1), it is seen that $\mu=Yg(u)$ for some function g and thus

$$W=\frac{\Lambda_1^5Yg(u)}{XY-\Lambda_2^4} . \tag{4.6}$$

By holomorphy, the superpotential must be of the form (4.6) for any values of the fields X and Y and scales Λ_1 and Λ_2. Finally, we note that the holomorphic function $g(u)$ cannot have any singularities in the entire complex u plane (including infinity); therefore, $g(u)$ must be a constant. Comparing with the known first term in (4.2) at $u=0$, we find $g(u)=1$. Therefore, the exact superpotential for this theory is

$$W=\frac{\Lambda_1^5Y}{XY-\Lambda_2^4} . \tag{4.7}$$

The superpotential (4.7) exactly sums the multi-instanton expansion (4.2).

We can rederive the superpotential (4.7) as the solution of a differential equation by adding mass terms for the matter fields and integrating them out. Adding mass terms to the superpotential (4.1), holomorphy and the symmetries determine the exact superpotential to be

$$W=\frac{\Lambda_1^5}{X}f\left[\frac{\Lambda_2^4}{XY}\right]+m_XX+m_YY \tag{4.8}$$

[note that as in Eq. (2.5), this is linear in the couplings m_X and m_Y]. Below the scales set by the masses we can integrate out the matter fields to obtain pure glue $SU(2)_1\times SU(2)_2$ Yang-Mills theory. The gaugino condensates in this low-energy theory can be expressed in terms of the high-energy couplings by taking account of the charges of these couplings under the $U(1)_Q$, $U(1)_{L_\pm}$, and $U(1)_R$ symmetries and the fact that the condensates must have charge zero under the U(1) symmetries and charge two under $U(1)_R$. This gives

$$\begin{aligned}\langle S_1\rangle&=\epsilon_1(m_X\Lambda_1^5)^{1/2}f_1\left[\frac{m_Y\Lambda_2^4}{\Lambda_1^5}\right],\\ \langle S_2\rangle&=\epsilon_2(m_Xm_Y\Lambda_2^4)^{1/2}f_2\left[\frac{\Lambda_1^5}{m_Y\Lambda_2^4}\right],\end{aligned} \tag{4.9}$$

where $\epsilon_{1,2}=\pm 1$ and f_1 and f_2 are functions. In the limits of large m_Y or small Λ_2, we can reliably determine $\langle S_1\rangle$ by using (3.10) in the low-energy $SU(2)_1$ Yang-Mills theory and matching the low-energy scale to our high-energy scales; this gives a condensate as in (4.9) with $f_1=1$. Since the argument of f_1 is independent of m_X, the function $f_1=1$ identically. Similarly, we can reliably determine that the condensate $\langle S_2\rangle$ must be independent of Λ_1 in the limit of large m_X and hence f_2 must be a constant. The limit where m_Y is also large determines $f_2=1$. Thus

$$\langle S_1\rangle=\epsilon_1(m_X\Lambda_1^5)^{1/2}$$

and (4.10)

$$\langle S_2\rangle=\epsilon_2(m_X m_Y\Lambda_2^4)^{1/2} .$$

We can use these equations together with the (assumed) relations of Eq. (2.9):

$$\Lambda_1^5\frac{\partial\langle W\rangle}{\partial\Lambda_1^5}=\langle S_1\rangle , \quad \Lambda_2^4\frac{\partial\langle W\rangle}{\partial\Lambda_2^4}=\langle S_2\rangle , \tag{4.11}$$

where, as in Sec. II, $\langle W\rangle$ means the superpotential (4.8) with X and Y integrated out, i.e., replaced with the solutions $\langle X\rangle$ and $\langle Y\rangle$ to their equations of motion, obtained from (4.8) as functions of the couplings. By varying m_X and m_Y we can change the expectation values and thereby determine the function f for all values of its argument. In particular, writing the X and Y equations of motion obtained from (4.8) as

$$m_X=\frac{\Lambda_1^5}{X^2}[f(u)+uf'(u)] , \quad m_Y=\frac{\Lambda_1^5}{\Lambda_2^4}u^2f'(u) ,$$

a comparison of (4.10) and (4.11) with the superpotential (4.8) gives differential equations for the function $f(u)$:

$$f^2=(f+uf') \text{ and } f'=(f+uf') ,$$

which uniquely determine $f=1/(1-u)$ and thus, in agreement with (4.7),

$$W=\frac{\Lambda_1^5 Y}{XY-\Lambda_2^4} .$$

This agreement can be used as further evidence for the assumption (2.7).

We also note that we can take our result (4.7) and "integrate in" the massive fields S_1 and S_2. The superpotential which satisfies (2.9) and which gives (4.7) upon integrating out S_1 and S_2 is

$$W=S_1\left[\ln\left[\frac{\Lambda_1^5}{S_1X}\right]+1\right]+S_2\ln\left[\frac{\Lambda_2^4}{XY}\right] +S_1\ln\left[\frac{S_1+S_2}{S_1}\right]+S_2\ln\left[\frac{S_1+S_2}{S_2}\right] . \tag{4.12}$$

The first two terms would be expected following the analysis of [4] for the $SU(2)_1$ and $SU(2)_2$ theories. The second two terms indicate the "interaction" between the two gauge groups. A suggestive way to write (4.12) is as

$$W=S_1\left[\ln\left[\frac{\Lambda_1^5}{S_1^2}\right]+1\right]+S_2\ln\left[\frac{\Lambda_2^4}{S_2^2}\right] +(S_1+S_2)\ln\left[\frac{S_1+S_2}{X}\right]+S_2\ln\left[\frac{S_2}{Y}\right] . \tag{4.13}$$

The first two terms in (4.13) can be associated purely with $SU(2)_1$ and $SU(2)_2$, respectively. The third term is associated with the matter field $Q=(2,2)$ and the fourth is associated with the $L_\pm$. The expression (4.13) naturally generalizes, as we will discuss.

B. Matter content $Q=(2,2)$

If we add a mass term to (4.7),

$$W=\frac{\Lambda_1^5 Y}{XY-\Lambda_2^4}+m_Y Y , \tag{4.14}$$

we can integrate out $L_\pm$ to obtain the superpotential for an $SU(2)_1\times SU(2)_2$ theory with matter content $Q=(2,2)$. Y is easily integrated out; there are two solutions to its equation of motion leading to

$$W_{\text{eff}}=\frac{1}{X}[\Lambda_1^5\pm 2(\Lambda_1\tilde{\Lambda}_2)^{5/2}+\tilde{\Lambda}_2^5] =\frac{(\Lambda_1^{5/2}\pm\tilde{\Lambda}_2^{5/2})^2}{X} , \tag{4.15}$$

where the low-energy scale is matched to the high-energy one by $\tilde{\Lambda}_2=(m_Y\Lambda_2^4)^{1/5}$. So the superpotential for the $SU(2)_1\times SU(2)_2$ theory with matter content $Q=(2,2)$ is (4.15); having integrated out $L_\pm$, we can forget about the original high-energy theory and thus drop the tilde on Λ_2.

The terms in (4.15) have a clear interpretation. Along the flat direction labeled by X, the $SU(2)_1\times SU(2)_2$ gauge symmetry is broken by the Higgs mechanism down to a diagonally embedded $SU(2)_D$. An instanton in the broken $SU(2)_1$ gives, according to (3.4), Λ_1^5/X. Likewise, an instanton in the broken $SU(2)_2$ gives Λ_2^5/X. Finally, gaugino condensation in the unbroken $SU(2)_D$ gives the superpotential (3.7) [with the factor of N_c-N_f in (3.7) replaced with 2 because the unbroken gauge group is SU(2)] which is $\pm 2\Lambda_D^3=\pm 2\Lambda_1^{5/2}\Lambda_2^{5/2}/X$. These are precisely the terms found in our exact answer (4.15).

The $\pm$ sign in (4.15) is a discrete label which labels two physically inequivalent ground states of the theory. This sign comes from the fact that the low energy theory has a $\mathbb{Z}_4$ symmetry which is spontaneously broken down to $\mathbb{Z}_2$ by gaugino condensation in the low energy $SU(2)_D$: $\langle\lambda\lambda\rangle_{SU(2)_D}=\pm\Lambda_D^3=\pm\Lambda_1^{5/2}\Lambda_2^{5/2}/X$. Because of the contributions of $SU(2)_1$ and $SU(2)_2$ instantons to the superpotential, the sign choice involved in gaugino condensation in $SU(2)_D$ label physically inequivalent vacua. For example, the potential energy as a function of X differs for the two sign choices in (4.15). Just as the $SU(N_c)$ theories with $N_f\geq N_c$ have a continuum of physically inequivalent vacua, this theory has a discrete choice of physically inequivalent vacua.

To further illuminate these two inequivalent vacua we add a mass term for the field X and consider integrating it out. Using the symmetries, the gaugino condensates in the low-energy $SU(2)_1 \times SU(2)_2$ Yang-Mills theory are of the form

$$\langle S_1 \rangle = \epsilon_1 m_X^{1/2} \Lambda_1^{5/2} f_1 \left[\frac{\Lambda_2}{\Lambda_1} \right]$$

and (4.16)

$$\langle S_2 \rangle = \epsilon_2 m_X^{1/2} \Lambda_2^{5/2} f_2 \left[\frac{\Lambda_1}{\Lambda_2} \right] ,$$

where ϵ_1 and ϵ_2 are ± 1 and f_1 and f_2 are functions. In the limit of large m_X we can reliably compute the condensates in (4.16) by using (3.10) in the low energy Yang-Mills theory and matching the low-energy scale to the scales of the high-energy theory which includes the massive field Q; this gives $f_1 = 1$ and $f_2 = 1$. Thus, there are four ground states given by the condensates

$$\langle S_1 \rangle = \epsilon_1 m_X^{1/2} \Lambda_1^{5/2}$$

and (4.17)

$$\langle S_2 \rangle = \epsilon_2 m_X^{1/2} \Lambda_2^{5/2} .$$

In the pure glue $SU(2)_1 \times SU(2)_2$ theory all four states would be related by a symmetry. Here, the two states with $\epsilon_1 = \epsilon_2$ are indeed related by the spontaneously broken $\mathbb{Z}_4$ symmetry of $SU(2)_D$. Likewise, the two states with $\epsilon_1 = -\epsilon_2$ are related by this symmetry. On the other hand, the pair of states with $\epsilon_1 = \epsilon_2$ are not related by a symmetry to the pair of states with $\epsilon_1 = -\epsilon_2$; they are physically inequivalent. They differ because of the interactions with the high energy massive sector. In particular, the massive field Q has the expectation value $m_X \langle X \rangle = \epsilon_1 m_X^{1/2} \Lambda_1^{5/2} + \epsilon_2 m_X^{1/2} \Lambda_2^{5/2}$.

Another way to understand this is the following. In the low energy theory we can perform independent rotations of the two θ parameters. The four ground states are related by $\theta_i \to \theta_i + 2\pi$ (i.e., $\tilde{\Lambda}_i^6 \to e^{2\pi i} \tilde{\Lambda}_i^6$). In the full theory which includes the field Q, the combination $\theta_1 + \theta_2$ can be rotated away but $\theta_1 - \theta_2$ is physical. Therefore, the two pairs of states related by simultaneous shifts of the two theta parameters $\theta_i \to \theta_i + 2\pi$ (i.e., $\Lambda_i^5 \to e^{2\pi i} \Lambda_i^5$) are related by a symmetry but if only one of the θ parameters is shifted by 2π inequivalent ground states are interchanged.

The low energy space includes both the continuous field Q^2 and a discrete label $\epsilon_1 \epsilon_2 = \pm 1$ which determines the sign in the superpotential.

If we integrate the massive fields S_1 and S_2 into our superpotential (4.15) we obtain

$$W = S_1 \left[\ln \left[\frac{\Lambda_1^5}{S_1 X} \right] + 1 \right] + S_2 \left[\ln \left[\frac{\Lambda_2^5}{S_2 X} \right] + 1 \right] + S_1 \ln \left[\frac{S_1 + S_2}{S_1} \right] + S_2 \ln \left[\frac{S_1 + S_2}{S_2} \right] . \quad (4.18)$$

This, again, can be written in the suggestive form

$$W = S_1 \left[\ln \left[\frac{\Lambda_1^5}{S_1^2} \right] + 1 \right] + S_2 \left[\ln \left[\frac{\Lambda_2^5}{S_2^2} \right] + 1 \right] + (S_1 + S_2) \ln \left[\frac{S_1 + S_2}{X} \right] , \quad (4.19)$$

corresponding to terms associated with $SU(2)_1$, $SU(2)_2$, and the matter field Q.

C. Matter content $Q = (2,2)$ and $L_i = (1,2)$ for $i = 1, \ldots, 4$

The basic gauge singlets are $X = Q^2$ and $V_{ij} = L_i L_j$. Under the $SU(4)_F$ flavor symmetry which rotates the L_i, V_{ij} transforms as a **6**. Since our superpotential must be built from $SU(4)_F$ singlets, it can only involve X and Pf V. Using the $U(1)_Q$, $U(1)_{L_i}$, and $U(1)_R$ symmetries, with the scales Λ_1 and Λ_2 assigned charges in accordance with (3.1), the exact superpotential is determined to be of the form

$$W = \frac{\Lambda_1^5}{X} f \left[u = \frac{\Lambda_1^5 \Lambda_2^3}{X^2 \text{Pf } V} \right] . \quad (4.20)$$

In the limit $\Lambda_2 \to 0$ we expect to find a superpotential corresponding to $f = 1$, coming from an instanton in $SU(2)_1$. On the other hand, for $\Lambda_1 \to 0$ the theory is $SU(2)_2$ with six doublets so there is a moduli space of vacua with a singularity at the origin, corresponding to the fact that there are extra light fields there [7].

In order to determine the superpotential we begin with $\Lambda_1 = 0$. The theory is then $SU(2)_2$ with the six doublets (three flavors) Q_α and L_i, where the flavor indices $\alpha = 1,2$ and $i = 1, \ldots, 4$. There is a global flavor $SU(6)_F$; the basic $SU(2)_2$ gauge singlet U transforms as the **15** of $SU(6)_F$. In terms of our original fields, U has the components $U_{\alpha\beta} = X \epsilon_{\alpha\beta}$, $U_{ij} = V_{ij}$, and $U_{\alpha i}$. As in Ref. [7], all 15 fields in U are physical fields in the spectrum. A superpotential is dynamically generated which gives six of these fields masses along a flat direction:

$$W_{SU(2)_2,\text{dyn}} = -\frac{\text{Pf}_6 U}{\Lambda_2^3} = -\frac{X \,\text{Pf}\, V + \Gamma \cdot V}{\Lambda_2^3} , \quad (4.21)$$

where Pf_6 is a Pfaffian over the SU(6) indices, Pf is taken over the SU(4) indices, $\Gamma_{ij} \equiv U_{\alpha i} U_{\beta j} \epsilon^{\alpha\beta}$, and $\Gamma \cdot V \equiv \frac{1}{4} \epsilon^{ijkl} \Gamma_{ij} V_{kl}$.

We now gauge $SU(2)_1 \subset SU(6)_F$, labeled by the index α, keeping $\Lambda_2 \gg \Lambda_1$. Below the scale Λ_2, our spectrum consists of the 15 fields U with the superpotential (4.21). The seven composite fields $U_{\alpha\beta}$ and V_{ij} are $SU(2)_1$ singlets and the fields $U_{\alpha i}$ are four $SU(2)_1$ doublets. Thus this is the situation (3.12) where there is a moduli space for the scalar components of the $U_{\alpha i}$ with a constraint which is modified by a single $SU(2)_1$ instanton to be

$$\text{Pf } \Gamma = \Lambda_1^5 \Lambda_2^3 , \quad (4.22)$$

where we again define $\Gamma_{ij} = \epsilon^{\alpha\beta} U_{\alpha i} U_{\beta j}$. The right-hand side of (4.22) follows from the symmetries up to a function of u. Inspection of various limits along with holomorphy implies *a posteriori* that this function must be unity; for simplicity then we will not retain it in the following. The constraint (4.22) can be implemented by a superpotential with a Lagrange multiplier field A:

$$W_{SU(2)_1,\mathrm{dyn}}=A(\mathrm{Pf}\,\Gamma-\Lambda_1^5\Lambda_2^3)\ . \qquad (4.23)$$

Putting together the $SU(2)_1$ and $SU(2)_2$ contributions (4.23) and (4.21) to the superpotential we obtain

$$W=-\frac{X\ \mathrm{Pf}\,V+\Gamma\cdot V}{\Lambda_2^3}+A(\mathrm{Pf}\,\Gamma-\Lambda_1^5\Lambda_2^3)\ . \qquad (4.24)$$

Along the flat direction labeled by an expectation value for V, the superpotential (4.24) gives masses to the fields $U_{\alpha i}$ which were not in our original list of fields. Thus, away from $V=0$, we can integrate the field Γ out of (4.24). Upon integrating out A to implement the constraint on Pf Γ, the Γ equation of motion gives $\langle\Gamma\cdot V\rangle=\pm2\sqrt{\Lambda_1^5\mathrm{Pf}\,V/\Lambda_2^3}$ and (4.24) becomes

$$W=\frac{-X\ \mathrm{Pf}\,V}{\Lambda_2^3}\pm2\left[\frac{\Lambda_1^5\mathrm{Pf}\,V}{\Lambda_2^3}\right]^{1/2}\ . \qquad (4.25)$$

The Pf V equation of motion obtained from (4.25) gives

$$W=\frac{\Lambda_1^5}{X}\ \text{ and }\ \langle\mathrm{Pf}\,V\rangle=\frac{\Lambda_1^5\Lambda_2^3}{X^2}\ . \qquad (4.26)$$

Result (4.26) gives the correct superpotential (3.4) for $SU(2)_1$ with its one flavor and the correct constraint (3.12) for PF V in the limit of large X, where the theory is broken to $SU(2)_D$ with $\Lambda_D^4=\Lambda_1^5\Lambda_2^3/X^2$. Using holomorphy, Eq. (4.24) is thus the exact superpotential for this theory. The complicated looking dynamics in (4.25) arises simply from having integrated out the extra fields in Γ.

The massive fields S_1 and S_2 can be integrated in, as in the previous examples. The result is

$$W=S_1\left[\ln\left[\frac{\Lambda_1^5(X\ \mathrm{Pf}\,V+\Gamma\cdot V)}{\mathrm{Pf}\,\Gamma(S_2-S_1)}\right]+1\right]+S_2\left[\ln\left[\frac{\Lambda_2^3(S_2-S_1)}{X\ \mathrm{Pf}\,V+\Gamma\cdot V}\right]-1\right]\ . \qquad (4.27)$$

If we integrate Γ out of (4.27) using the equations of motion

$$\langle X\ \mathrm{Pf}\,V+\Gamma\cdot V\rangle=\left[\frac{S_2-S_1}{S_2+S_1}\right]X\ \mathrm{Pf}\,V\ ,$$

$$\langle\mathrm{Pf}\,\Gamma\rangle=\frac{S_1^2}{(S_1+S_2)^2}X^2\mathrm{Pf}\,V\ ,$$

Eq. (4.27) becomes

$$\begin{aligned}W&=S_1\left[\ln\left[\frac{\Lambda_1^5}{S_1X}\right]+1\right]+S_2\left[\ln\left[\frac{\Lambda_2^3S_2}{X\ \mathrm{Pf}\,V}\right]-1\right]+S_1\ln\left[\frac{S_1+S_2}{S_1}\right]+S_2\ln\left[\frac{S_1+S_2}{S_2}\right]\\&=S_1\left[\ln\left[\frac{\Lambda_1^5}{S_1^2}\right]+1\right]+S_2\left[\ln\left[\frac{\Lambda_2^3}{S_2^2}\right]-1\right]+(S_1+S_2)\ln\left[\frac{S_1+S_2}{X}\right]+S_2\ln\left[\frac{S_2^2}{\mathrm{Pf}\,V}\right]\ .\end{aligned} \qquad (4.28)$$

We can rederive the exact superpotential (4.25) by adding mass terms for the L_i and requiring the result to agree with those of the previous sections upon integrating out some of the L_i. We thus consider

$$W=\frac{\Lambda_1^5}{X}f\left[\frac{\Lambda_1^5\Lambda_2^3}{X^2\mathrm{Pf}\,V}\right]+m_XX+m_VV\ . \qquad (4.29)$$

Suppose we take V and the mass terms m_V to be of the form

$$V=\begin{bmatrix}Yi\sigma_2&0\\0&Zi\sigma_2\end{bmatrix}\ ,\quad m_V=\begin{bmatrix}m_Yi\sigma_2&0\\0&m_Zi\sigma_2\end{bmatrix}\ , \qquad (4.30)$$

with $m_Z>m_Y$. At the scale m_Z we integrate out Z. We should then obtain the superpotential (4.14) in the low energy theory with only two L_i. Rewriting the scales there in terms of our high-energy scales here using the matching condition at m_Z, (4.14) becomes

$$W=\frac{\Lambda_1^5}{X}\frac{1}{1-\frac{m_Z\Lambda_2^3}{XY}}+m_XX+m_YY\ . \qquad (4.31)$$

Below the scale m_Y we can also integrate out Y; the equation of motion for Y obtained from (4.31) is

$$X\langle Y\rangle=m_Z\Lambda_2^3\pm\left[\frac{m_Z\Lambda_1^5\Lambda_2^3}{m_Y}\right]^{1/2}\ . \qquad (4.32)$$

Having integrated out V, this same result must come from the V equations of motion obtained from (4.29). The flavor $SU(4)_F$ covariant way to write (4.32) and the analogous equation for $\langle Z\rangle$ is clearly

$$X\langle V\rangle=[\Lambda_2^3\mathrm{Pf}\,m_V\pm(\Lambda_1^5\Lambda_2^3\mathrm{Pf}\,m_V)^{1/2}]\frac{1}{m_V}\ . \qquad (4.33)$$

On the other hand, the V equation of motion obtained from (4.29) is

$$X\langle V\rangle=\Lambda_1^5uf'(u)\frac{1}{m_V}\ . \qquad (4.34)$$

We know that (4.33) and (4.34) must agree. Taking the Pfaffian of (4.33) and (4.34) gives

$$u^{3/2}f'=\left[\frac{\Lambda_2^3\mathrm{Pf}\,m_V}{\Lambda_1^5}\right]^{1/2}$$

and

$$u^{-1/2}=\left[\frac{\Lambda_2^3 \text{Pf}\, m_V}{\Lambda_1^5}\right]^{1/2}\pm 1 \ .$$

Comparing we get $f'=u^{-2}\pm u^{-3/2}$, which gives $f=-u^{-1}\pm 2u^{-1/2}$, in agreement with our previous result (4.25).

Note that the V equation of motion in the theory with $m_V\neq 0$ moves V off of the original constraint manifold, Eq. (4.26), to

$$\langle \text{Pf}\ V\rangle=\frac{\Lambda_1^5\Lambda_2^3}{X^2}\left[1\pm\left[\frac{\Lambda_2^3 \text{Pf}\, m_V}{\Lambda_1^5}\right]^{1/2}\right]^2 . \tag{4.35}$$

Also, note that if we integrate out X we are left with a low energy $SU(2)_2$ theory with the four doublets L_i. The equation of motion obtained from (4.25) upon integrating out X gives Pf $V=m_X\Lambda_2^3=\tilde{\Lambda}^4$, where $\tilde{\Lambda}$ is the scale of the low energy $SU(2)_2$ theory, in agreement with (3.12).

V. EXAMPLES WITH SO(5)×SU(2) SYMMETRY AND GAUGINO CONDENSATION

A. Matter content $F=(4,2)$

The gauge singlet combination is $X=\frac{1}{2}F_{ir}J^{ij}F_{js}\epsilon^{rs}$, with J the SO(5) invariant tensor $i\sigma_2\times 1$. Along the classical flat direction labeled by X, the gauge group is broken down to $SU(2)'\times SU(2)_D$ where $SU(2)'\subset SO(5)$ and $SU(2)_D$ is diagonally embedded. Since the gauge group is not completely broken we expect to find nonperturbative effects associated with gaugino condensation in the unbroken gauge groups rather than with instantons.

The discussion of Sec. III generalizes to other gauge theories very simply: $2N_c$ is replaced in the various formulas with the index of the adjoint representation of the gauge group and $2N_f$ is replaced with the sum of the indices of the matter representations. In particular, for SO(5) with two 4's we replace N_c with 3 and N_f with 1. Using the $U(1)_F\times U(1)_R$ symmetries with (3.1) and its SO(5) analogue, the superpotential is found to have the form

$$W=\left[\frac{\Lambda_5^8}{X}\right]^{1/2} f\left[u=\frac{\Lambda_2^4}{X^2}\right] . \tag{5.1}$$

Along the flat direction with large X, the low energy theory is just the unbroken $SU(2)'\times SU(2)_D$ Yang-Mills theory with the field X. As in (3.7) gaugino condensation in these two Yang-Mills theories gives a superpotential $W\simeq\pm 2\Lambda^3_{SU(2)'}\pm 2\Lambda^3_{SU(2)_D}$ with the signs of the two condensates independent and with the scales of the low energy theories related to the high energy ones by the matching condition, as in (3.6), at the scale X where the theory becomes broken by the Higgs mechanism: $\Lambda^3_{SU(2)'}\simeq\Lambda_5^4/X^{1/2}$ and $\Lambda^3_{SU(2)_D}\simeq\Lambda_5^4\Lambda_2^2/X^{3/2}$. In the small u limit, the superpotential is thus given by (5.1) with

$$f(u)=\pm 2\pm 2u^{1/2}+O(u) \ . \tag{5.2}$$

The overall sign of the superpotential corresponds to two physically equivalent branches of the square root $(\Lambda_5^8/X)^{1/2}$ in (5.1). We can thus take the first sign choice in (5.2) to be positive. The relative sign choice between the first two terms is a discrete label, associated with massive fields, which is needed in the low energy theory to specify the ground state. As we will see, it is related to a spontaneously broken discrete symmetry.

To further examine the function f, let us turn off Λ_5 for the moment and go to the region of strong SU(2) coupling. The basic SU(2) singlet combinations, $V_{ij}=F_{ir}F_{js}\epsilon^{rs}$, form a **6** of $SU(4)_F$. When SO(5) is gauged, we decompose this **6** as $V_{ij}=E_{ij}+\frac{1}{2}XJ_{ij}$, where X is as defined above, J is the SO(5) singlet mentioned above, and E, satisfying Tr$JE=0$, transforms as an SO(5) vector. The constraint (3.12) yields

$$\text{Pf}\ V=\text{Pf}\ E+\tfrac{1}{4}X^2=\Lambda_2^4 \ . \tag{5.3}$$

The vector E breaks SO(5) to an $SO(4)\equiv SU(2)_L\times SU(2)_R$ subgroup. This is to be compared with the $SU(2)'\times SU(2)_D$, mentioned above, which is unbroken in the weak coupling regions of field space. We see from (5.3) that, because of the modified moduli space associated with SU(2) instantons, SO(5) is unbroken at the two points $X=\pm 2\Lambda_2^2$ rather than at the classical value of zero. The superpotential (5.1) can thus only be singular at $u=\frac{1}{4}$. In particular, the function $f(u)$ must satisfy $\lim_{u\to\infty}f(u)\le O(u^{-1/4})$ to cancel the singularity in (5.1) at $X=0$.

In an SO(5) theory with a single canonically normalized vector **v**, gaugino condensation in the unbroken $SU(2)_L\times SU(2)_R$ leads to a dynamically generated superpotential

$$W=2\langle S_L\rangle+2\langle S_R\rangle = \begin{cases} \dfrac{2\Lambda_5^4}{\sqrt{\mathbf{v}^2}} & \text{for } \langle S_L\rangle=\langle S_R\rangle \\ 0 & \text{for } \langle S_L\rangle=-\langle S_R\rangle \ ; \end{cases} \tag{5.4}$$

the fact that $\langle S_L\rangle$ and $\langle S_R\rangle$ are $\pm\frac{1}{2}\Lambda_5^4/\sqrt{\mathbf{v}^2}$ is required by the normalization conventions of Sec. III (the factor $\frac{1}{2}$ arises in the matching conditions at $\mathbf{v}^2$). Our vector E differs from the canonically normalized vector **v** by some dimensionful, field-dependent normalization μ_v; in particular,

$$\mathbf{v}^2=-\mu_v^{-2}\text{Pf}\ E=\mu_v^{-2}(\tfrac{1}{4}X^2-\Lambda_2^4) \ . \tag{5.5}$$

Suppose that near one of the two points of unbroken SO(5), $X=2\eta\Lambda_2^2$ where η can be either ± 1, $\langle S_L\rangle=\langle S_R\rangle$. Using (5.4) with (5.5), the superpotential behaves in this vicinity as

$$W(X\sim 2\eta\Lambda_2^2)\sim\frac{\Lambda_5^4}{\sqrt{X-2\eta\Lambda_2^2}} \ , \tag{5.6}$$

where we have used the fact that $\mu_v\sim\Lambda_2$ in this regime. There is a unique holomorphic superpotential with the small u and large u asymptotics mentioned above and the singularity structure of Eq. (5.6):

$$W(X,\eta)=\frac{2\Lambda_5^4}{\sqrt{X-2\eta\Lambda_2^2}} . \tag{5.7}$$

The superpotential (5.7) is thus the exact effective superpotential for the theory. The phase η appearing in (5.7) is a discrete label which, comparing with (5.2), is the relative sign of the SU(2)′ and $SU(2)_D$ gaugino condensates.

The two choices of ground states labeled by η are physically equivalent: there is a discrete $\mathbb{Z}_8$ R symmetry under which $X(\theta)\to -X(e^{i\pi/4}\theta)$, which takes $W(X,\eta)\to iW(-X,\eta)=W(X,-\eta)$. For a given value of η, the superpotential (5.7) is singular at the point $X=2\eta\Lambda_2^2$ of unbroken SO(5) but it is regular at the other point $X=-2\eta\Lambda_2^2$ of unbroken SO(5). This behavior is possible because of the two branches in (5.4); if $\langle S_L\rangle=\langle S_R\rangle$ near $X=2\eta\Lambda_2^2$, we must have $\langle S_L\rangle=-\langle S_R\rangle$ near $X=-2\eta\Lambda_2^2$. The point $X=-2\eta\Lambda_2^2$ is nevertheless singular, as the normalization μ_v of vector E vanishes at this point.

The exact result presented above can be redrived from (2.9) by adding a mass term for F. With the mass term, the exact superpotential is determined by the symmetries to be

$$W=\left[\frac{\Lambda_5^8}{X}\right]^{1/2} f\left[u=\frac{\Lambda_2^4}{X^2}\right]+m_X X . \tag{5.8}$$

Equations (2.9) give

$$\Lambda_5^8\frac{\partial W}{\partial\Lambda_5^8}=\langle S_5\rangle , \quad \Lambda_2^4\frac{\partial W}{\partial\Lambda_2^4}=\langle S_2\rangle . \tag{5.9}$$

On the other hand, the gaugino condensates in the low-energy SO(5)×SU(2) pure Yang-Mills theories are given by (3.10) which, expressed in terms of the original scales using the matching conditions, are[1]

$$\langle S_5\rangle=\omega(m_X\Lambda_5^8)^{1/3} \text{ and } \langle S_2\rangle=\eta(m_X^2\Lambda_2^4)^{1/2} , \tag{5.10}$$

with $\omega^3=1$ and $\eta^2=1$. Equations (5.9) and (5.10) must agree for every m_X. Using the equations of motion from (5.8) to solve for m_X, this gives the equations

$$\left[\frac{f}{2}\right]^3=\frac{1}{2}f+2uf' , \quad uf'=\eta[u(\tfrac{1}{2}f+2uf')^2]^{1/2} , \tag{5.11}$$

which uniquely determine

$$f=\frac{2}{(1-2\eta u^{1/2})^{1/2}} .$$

So indeed

$$W=\frac{2\Lambda_5^4}{(X-2\eta\Lambda_2^2)^{1/2}} ,$$

as given in Eq. (5.7).

If we integrate the massive fields S_5 and S_2 into expression (5.7) we obtain the superpotential

$$\begin{aligned} W&=S_5\left[\ln\left[\frac{\Lambda_5^8}{S_5^2X}\right]+2\right]+S_2\ln\left[\frac{\Lambda_2^4}{X^2}\right]+S_5\ln\left[\frac{S_5+2S_2}{S_5}\right]+S_2\ln\left[\frac{(S_5+2S_2)^2}{S_2^2}\right] \\ &=S_5\left[\ln\left[\frac{\Lambda_5^8}{S_5^3}\right]+2\right]+S_2\ln\left[\frac{\Lambda_2^4}{S_2^2}\right]+(S_5+2S_2)\ln\left[\frac{S_5+2S_2}{X}\right] , \end{aligned} \tag{5.12}$$

where in the last expression the first two terms look like they arise from the SO(5) and SU(2) gauge groups and the last term from the matter field.

B. Matter content $F=(4,2)$ and $L_\pm=(1,2)$

Since in the limit $\Lambda_5\to 0$ this becomes SU(2) with six doublets, this example is similar to that of Sec. IV C. In particular, we find interesting behavior near the origin, corresponding to the extra light fields there.

In terms of the gauge singlet combinations of the superfields X (as above) and $Y=L_+^rL_-^s\epsilon_{rs}$, the symmetries determine the exact superpotential to be of the form

$$W=\frac{\Lambda_5^4}{\sqrt{X}}g\left[v=\frac{\Lambda_5^4\Lambda_2^3}{X^{5/2}Y}\right] . \tag{5.13}$$

Consider adding a mass term m_YY to the superpotential. In the limit of large mass we can integrate out Y to obtain the model of the previous subsection, a model for which we know the superpotential. Adding the mass term we have

$$W=\frac{\Lambda_5^4}{\sqrt{X}}g\left[\frac{\Lambda_5^4\Lambda_2^3}{X^{5/2}Y}\right]+m_YY .$$

[1] Actually, the second of these equations has been determined *a posteriori*. The symmetries allow the equations in (5.9) to be multiplied by holomorphic functions f_5 and f_2 of $\Lambda_5^{8/3}/m_X^{2/3}\Lambda_2^2$. The function f_5 is determined to be one for large m_X or for small Λ_2 and is therefore identically one. The function f_2 is known to be one for large m_X or small Λ_5, i.e., only when its argument is small. However, the information contained in the first equation is sufficient for what follows and, indeed, determines that $f_2=1$ identically.

Integrating out Y gives

$$W=\frac{\Lambda_5^4}{\sqrt{X}}(g+vg')\ ,$$

where v satisfies $v^2g'(v)=m_Y\Lambda_2^3/X^2$. As a function of $\tilde{\Lambda}_2^4/X^2$, with $\tilde{\Lambda}_2^4=m_Y\Lambda_2^3$ the scale of the SU(2) theory below the scale where Y has been integrated out, this superpotential must equal that of the SO(5)×SU(2) theory with matter field $F=(\mathbf{4},\mathbf{2})$, obtained in Sec. V A. Comparing with the result (5.7), g must therefore satisfy

$$\frac{2}{\sqrt{1-2\eta v\sqrt{g'}}}=g+vg'\ , \qquad (5.14)$$

with $\eta=\pm1$. This equation, along with some regularity conditions, can be used to determine $g(v)$. We are only able to provide a parametric solution to (5.14):

$$g=\tfrac{1}{2}h(5-h^2)\ ,\quad v=\tfrac{1}{2}\alpha(h^{-3}-h^{-5})\ , \qquad (5.15)$$

with $\alpha^{1/2}=\eta$. The solution $g(v)$ is, then, independent of the sign choice η.

Consider explaining (5.14) or (5.15) in the region of small v: $g(v)=\sum_n a_n v^n$. The nth term has the quantum numbers to be associated with SO(5)×SU(2) "instantons" with charges $[(n+1)/2,n]$, where terms with fractional instanton charges are presumably associated with gaugino condensation. Using (5.14) or (5.15) we find

$$g=2+v+3v^2+14v^3+O(v^4)\ . \qquad (5.16)$$

The first term in (5.16) is exactly what we expect from gaugino condensation in the SU(2)′⊂O(5) which remains unbroken by the Higgs mechanism along the flat directions. The second term can be understood along the flat direction with $X\gg Y$ where the theory is broken down to an $SU(2)_D$ diagonally embedded in SO(5)×SU(2). The $SU(2)_D$ theory has one light flavor and so an $SU(2)_D$ instanton, which is a (1,1) instanton in the high-energy theory, gives a superpotential as in (3.4) of Λ_D^5/Y. Matching Λ_D to the high energy scales, this gives exactly the term v in (5.16). The higher order terms in (5.16) are associated with more involved dynamics.

We can also expand (5.15) for large v (small h):

$$g(v\to\infty)=\tfrac{5}{2}(-2v)^{-1/5}-(-2v)^{-3/5}+\cdots\ . \qquad (5.17)$$

Equations for the superpotential equivalent to (5.14) and (5.15) can be rederived by using (2.9) in the theory with mass terms added for both X and Y:

$$W=\frac{\Lambda_5^4}{\sqrt{X}}g\left[v=\frac{\Lambda_5^4\Lambda_2^3}{X^{5/2}Y}\right]+m_XX+m_YY\ . \qquad (5.18)$$

We require

$$\Lambda_5^8\frac{\partial W}{\partial\Lambda_5^8}=\langle S_5\rangle\ ,\quad \Lambda_2^3\frac{\partial W}{\partial\Lambda_2^3}=\langle S_2\rangle\ , \qquad (5.19)$$

where the gaugino condensates, expressed in terms of the scales of the high-energy theory are given by[2]

$$\langle S_5\rangle=\omega(m_X\Lambda_5^8)^{1/3}\ ,\quad \langle S_2\rangle=\epsilon(m_X^2m_Y\Lambda_2^3)^{1/2}\ , \qquad (5.20)$$

with $\omega^3=1$ and $\epsilon^2=1$. Using the equation of motion obtained from (5.18),

$$m_X=\frac{\Lambda_5^4}{2X^{3/2}}(g+5vg')\ ,\quad m_Y=\frac{X^2}{\Lambda_2^3}v^2g'\ ,$$

and requiring (5.19) and (5.20) to agree we obtain equations which may be written as

$$h^3-2vg'=h\ ,\quad h^6=g'(v)\ , \qquad (5.21)$$

where we define $2h^3=g+5vg'$. These equations imply the parametric solution (5.15).

The massive fields S_5 and S_2 can be integrated into this theory yielding

$$\begin{aligned}W&=S_5\left[\ln\left[\frac{\Lambda_5^8}{S_5^2X}\right]+2\right]+S_2\left[\ln\left[\frac{\Lambda_2^3S_2}{X^2Y}\right]-1\right]+S_5\ln\left[\frac{S_5+2S_2}{S_5}\right]+S_2\ln\left[\frac{(S_5+2S_2)^2}{S_2^2}\right]\\&=S_5\left[\ln\left[\frac{\Lambda_5^8}{S_5^3}\right]+2\right]+S_2\left[\ln\left[\frac{\Lambda_2^3}{S_2^2}\right]-1\right]+(S_5+2S_2)\ln\left[\frac{S_5+2S_2}{X}\right]+S_2\ln\left[\frac{S_2}{Y}\right]\ .\end{aligned} \qquad (5.22)$$

Integrating S_5 and S_2 out of (5.22) gives

$$W=2\langle S_5\rangle-\langle S_2\rangle\ \ \text{with}\ \langle S_{5,2}\rangle=\frac{\Lambda_5^4}{\sqrt{X}}h_{5,2}(v)\ ,\quad h_2=vh_5^6\ ,\quad h_5^2=1+2vh_5^5\ . \qquad (5.23)$$

The superpotential (5.23) is seen to be equivalent to (5.13), with the parametric equation (5.15) for g with $h_5=h$.

VI. CONCLUSIONS

To conclude, some of the nontrivial, nonperturbative dynamics involved in supersymmetric gauge theories can be explored by a study of their superpotentials. Symmetries, holomorphy, and decoupling of heavy fields provide powerful tools which can often be used to obtain highly nontrivial superpotentials exactly.

We have demonstrated the power of these techniques in a variety of models. Some of our techniques, for exam-

[2]Again, the second of these equations is determined *a posteriori.*

ple, adding mass terms to decouple fields, are particular to theories with matter fields in real representations of the gauge group. Others are more general.

We have discussed the unusual procedure of "integrating in"—adding massive fields to the low energy theory. Usually, such a procedure is ambiguous because there are many theories with a massive field leading to the same low energy theory. With the assumption that the theory with the massive field is linear in its source the ambiguity is resolved. In all of our examples this assumption was true.

Using this assumption we could also integrate in the fields S_s. We noticed that in all our examples the resulting superpotential is of the form

$$W=\sum_s S_s\left[\ln\left(\frac{\Lambda_s^{(3G_s-\mu_s)/2}}{S_s^{G_s/2}}\right)+\frac{1}{2}(G_s-\mu_s)\right] +\sum_t F_t\ln\left(\frac{F_t^{d_t}}{Y_t(X^r)}\right), \tag{6.1}$$

where G_s is the index of the adjoint of gauge group $\mathcal{G}_s$ and $\mu_s=\sum_i\mu_i^s$, with μ_i^s the index of the representation R_i^s of the matter field ϕ_i in the gauge group $\mathcal{G}_s$. $Y_t(X^r)$ are polynomials in X^r which are invariant under all the non-Abelian global symmetries (e.g., Pf V in the example of Sec. IV C). F_t are linear combinations of the S_s satisfying $\sum_t q_i(Y_t)F_t=\sum_s\mu_i^s S_s$, where $q_i(Y_t)$ is the $U(1)_i$ charge of Y_t and $d_t=\frac{1}{2}\sum_i q_i(Y_t)$. The first term in (6.1) can be interpreted as arising from the gauge sector and the second term is from the matter fields. It is easy to check that (6.1) is invariant under all the global symmetries, including the anomalous ones, and leads to the Konishi anomaly, Eq. (2.10). Clearly, we do not have a proof of Eq. (6.1). However, given that it was observed to be satisfied in a variety of examples, we conjecture that it is true under some wide range of circumstances, thus generalizing the effective Lagrangians of Ref. [4].

It should be noted that Eq. (6.1) is sometimes of limited use. In some models it is valid but only if more fields X^r than those which are obvious from the classical flat directions are included. Also, symmetry considerations might not be powerful enough to determine the polynomials Y_t and F_t. In these cases, Eq. (6.1) is still correct but additional dynamical information, along the lines presented in this paper, is necessary to obtain the correct superpotential.

Several of the phenomena which we have observed and the tools which we have used are similar to those which have been encountered in two-dimensional $N=2$ supersymmetric field theories. For example, our superpotentials are sometimes given by an infinite sum over instantons similar to the Yukawa couplings in Calabi-Yau compactifications. One of the techniques which allowed us to perform the sum is the use of differential equations. These are somewhat reminiscent of the differential equation of Ref. [11] and the tt^* equations of Ref. [12]. Also, the fact that we can "integrate in" fields is similar to the situation in 2D gravity coupling to minimal model matter where the Korteweg–de Vries (KdV) equation allows one to "flow up" the renormalization group trajectory. Since all these two-dimensional phenomena are related to an underlying topological field theory, it is natural to conjecture that our exact results also have topological interpretations.

Although our techniques rely crucially on supersymmetry we hope that the exact results we obtain will be useful in gaining general insight concerning the dynamics of four-dimensional gauge theories. Finally, it is worth mentioning that exact results about the superpotentials of supersymmetric gauge theories are also essential for finding a viable model of dynamical supersymmetry breaking.

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