

Nonperturbative solutions and scaling properties of vector and axial-vector electrodynamics in 1+1 dimensions

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We study by nonperturbative techniques a vector and axial-vector theory characterized by a parameter which interpolates between pure vector and chiral Schwinger models. The main results are two ranges in the space of parameters which exhibit acceptable solutions. In the first range we find a free massive and a free massless bosonic excitation and interacting left-right fermions endowed with asymptotic states, which feel however a long range interaction. In the second range the massless bosonic excitation is a negative norm state which can be consistently expunged from the "physical" Hilbert space; fermions are confined. An intriguing feature of our model occurs in the first range where we find that fermionic correlators scale at both short and long distances, but with different critical exponents. The infrared limit in the fermionic sector is nothing but a dynamically generated massless Thirring model.

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I. INTRODUCTION

Quantum field theories in 1 space and 1 time dimensions have been intensively studied in recent years owing to their peculiarity of being exactly solvable both by functional and by operatorial techniques. From a practical point of view they find interesting applications in string models, while behaving as useful theoretical laboratories in which many features, present also in higher dimensional theories, can be directly tested. In addition two-dimensional models possess a quite peculiar infrared structure on their own.

Historically the first two-dimensional model was proposed by Thirring [1], describing a pure fermionic current-current interaction. The interest suddenly increased 4 years later, when Schwinger [2] was able to obtain an exact solution for two-dimensional electrodynamics with massless spinors. This model is so rich in interesting and surprising features, such as, e.g., dynamical generation of a mass for the vector field, fermion confinement, etc., that, after 30 years, it is still the subject of several investigations.

Chiral generalizations of this model were studied by Hagen [3] and, more recently by Jackiw and Rajaraman [4]. The latter authors draw very important conclusions concerning theories with "anomalies," i.e., the occurrence of symmetry breakings by quantum effects. They were able to show that, taking advantage of the arbitrariness in the (nonperturbative) regularization of the fermionic determinant, it was possible to recover a unitary theory even in the presence of a gauge anomaly.

The literature on the subject is so huge that it is impossible to refer it adequately; we just quote the book by Abdalla, Abdalla, and Rothe [5] where many references can be found.

In this paper we study in two-dimensional space with a trivial topology a family of theories which interpolate between vector and chiral Schwinger models according to a parameter r , which tunes the ratio of the axial vector to vector coupling. Our treatment will therefore depend on two parameters: r and a , a being the constant involved in the regularization of the fermionic determinant.

In Sec. II we obtain, by means of a functional approach, the correlation functions for bosons, fermions, and fermionic composite operators. We find two allowed ranges for the parameters r and a . The first range was also partially studied in a similar context in [6,7]. In this range the bosonic sector consists of two "physical" quanta, a free massive and a free massless excitation. The fermionic sector is much more interesting: both left and right spinors exhibit a propagator decreasing at very large distances, indicating the presence of asymptotic states which however feel the long-range interaction mediated by the massless boson.

The solution interpolates between two conformal invariant theories at small and large distances, respectively, with different critical exponents. This very interesting feature of our model is under investigation and the results will be reported in a forthcoming paper.

For $r=0$ one recovers the vector Schwinger model; for $r=\pm 1$ one gets the chiral model, where, in particular, one of the fermions is free.

The second range is characterized in the bosonic sector by a “physical” massive excitation and by a massless negative norm state (“ghost”). Both quanta are free; one can define a stable Hilbert space of states in which the “ghost” does not appear. However no asymptotic states for fermions are available in this case; their correlation function increases with distance, giving rise to a confinement phenomenon.

All those features are confirmed and further elucidated in subsequent sections: in Sec. III the bosonic sector is investigated by means of operators which are canonically quantized according to a Dirac brackets formalism [8]; the structure of Hilbert space of states is discussed. Section IV deals with the fermionic sector: fermionic operators are explicitly constructed, quantized, and correlation functions are examined, also in connection with the relevant equations of motion. We also discuss their behavior under symmetries and related charges.

In Sec. V we show that the fermionic correlation functions of our model at long distances exactly become those of a massless Thirring model, which is the conformal invariant infrared limit of our theory. This deep relation is present in the expression of operator fields and charges.

Section VI contains final conclusions, while some technical details are deferred to the Appendixes. Obviously our method works as we eventually deal with infraparticles; for more general theories with a nontrivial S matrix, one should possibly resort to algebraic techniques.

II. THE PATH-INTEGRAL FORMULATION

The model, characterized by the classical Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\partial\psi + e\bar{\psi}\gamma^\mu\psi A_\mu + re\bar{\psi}\gamma^\mu\gamma^5\psi A_\mu \quad (2.1)$$

where

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{ae^2}{2\pi}A_\mu A^\mu + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \\ & + \frac{e}{\sqrt{\pi}}A^\mu(\tilde{\partial}_\mu - r\partial_\mu)\phi, \end{aligned} \quad (2.7)$$

ϕ being a scalar field we have introduced in order to have a local \mathcal{L}_{eff} and a the subtraction parameter reflecting the well-known regularization ambiguity of the fermionic determinant [4].

The quantity $S(x, y; A_\mu)$ in (2.6) is the fermionic propagator in the presence of the potential A_μ , which will be computed later on by using standard decoupling techniques.

For the moment we let the sources η and $\bar{\eta}$ vanish and consider the bosonic sector of the model for different

will be quantized in this section following the path-integral method. In (2.1) $F_{\mu\nu}$ is the usual field tensor, A_μ the vector potential, and ψ a massless spinor. The quantity r is a real parameter interpolating between the vector ($r=0$) and the chiral ($r=\pm 1$) Schwinger models. Our notation is

$$\begin{aligned} g_{00} = -g_{11} = 1, \quad \epsilon^{01} = -\epsilon_{01} = 1, \\ \gamma^0 = \sigma_1, \quad \gamma^1 = -i\sigma_2, \quad \gamma^5 = \sigma_3, \quad \tilde{\partial}_\mu = \epsilon_{\mu\nu}\partial^\nu, \end{aligned} \quad (2.2)$$

σ_i being the usual Pauli matrices.

The classical Lagrangian (2.1) is invariant under the local transformations

$$\begin{aligned} \psi'(x) &= \exp[ie(1+r\gamma^5)\Lambda(x)]\psi(x), \\ A'_\mu(x) &= A_\mu(x) + \partial_\mu\Lambda. \end{aligned} \quad (2.3)$$

However, as is well known, it is impossible to make the fermionic functional measure simultaneously invariant under vector and axial vector gauge transformations; as a consequence, for $r \neq 0$ the quantum theory will exhibit anomalies.

The Green function generating functional is

$$W[J_\mu, \bar{\eta}, \eta] = \mathcal{N} \int \mathcal{D}(A_\mu, \bar{\psi}, \psi) \exp \left[i \int (\mathcal{L} + \mathcal{L}_s) d^2x \right], \quad (2.4)$$

where \mathcal{N} is a normalization constant and

$$\mathcal{L}_s = J_\mu A^\mu + \bar{\eta}\psi + \bar{\psi}\eta, \quad (2.5)$$

J_μ , η , and $\bar{\eta}$ being vector and spinor sources, respectively.

The integration over the fermionic degrees of freedom can be performed, leading to the expression

$$W[J_\mu, \eta, \bar{\eta}] = \mathcal{N} \int \mathcal{D}(A_\mu, \phi) \exp \left[i \int d^2x \mathcal{L}_{\text{eff}}(A_\mu, \phi) \right] \exp \left[i \int d^2x J_\mu A^\mu \right] \exp \left[-i \int d^2x d^2y \bar{\eta}(x) S(x, y; A_\mu) \eta(y) \right], \quad (2.6)$$

values of the parameters r and a . In this sector the effective Lagrangian is quadratic in the fields; this means an essentially free (although nonlocal) theory.

First functionally integrating over ϕ and then over A_μ , we easily obtain

$$W[J_\mu, 0, 0] = \exp \left[-\frac{1}{2} \int d^2x J^\mu (K^{-1})_{\mu\nu} J^\nu \right], \quad (2.8)$$

where

$$\begin{aligned} K_{\mu\nu} = & g_{\mu\nu} \left[\square + \frac{e^2}{\pi}(1+a) \right] - \left[1 + \frac{e^2}{\pi} \frac{1+r^2}{\square} \right] \partial_\mu\partial_\nu \\ & + \frac{e^2}{\pi} \frac{r}{\square} (\tilde{\partial}_\mu\partial_\nu + \tilde{\partial}_\nu\partial_\mu) \end{aligned} \quad (2.9)$$

and, consequently,

$$\begin{aligned}
(K^{-1})_{\mu\nu} &\equiv D_{\mu\nu} \\
&= \frac{1}{\square + m^2} \left[g_{\mu\nu} + \frac{\square + \frac{e^2}{\pi}(1+r^2)}{\frac{e^2}{\pi}(a-r^2)} \frac{\partial_\mu \partial_\nu}{\square} \right. \\
&\quad \left. + \frac{r}{r^2 - a} \frac{1}{\square} (\tilde{\partial}_\mu \partial_\nu + \tilde{\partial}_\nu \partial_\mu) \right]. \quad (2.10)
\end{aligned}$$

We have introduced the quantity

$$m^2 = \frac{e^2}{\pi} \frac{a(1+a-r^2)}{a-r^2}, \quad (2.11)$$

which is to be interpreted as a dynamically generated mass in the theory; $D_{\mu\nu}$ has a pole there $\sim(k^2 - m^2 + i\epsilon)^{-1}$, with causal prescription, as usual. We note that $D_{\mu\nu}$ exhibits also a pole at $k^2=0$.

Equations (2.10) and (2.11) generalize the well-known results of the vector and chiral Schwinger models. As a matter of fact, setting first $r=0$ and then $a=0$ we recover for m^2 the value e^2/π of the (gauge invariant version of the) vector Schwinger model. The kinetic term $K_{\mu\nu}$ becomes a projection operator

$$K_{\mu\nu}(a=0, r=0) = (\square + m^2) \left[g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\square} \right], \quad (2.12)$$

which can only be inverted after imposing a gauge fixing. In other words the limit $r=0, a=0$ in (2.10) is singular, as it should, as gauge invariance is indeed recovered.

When $r=\pm 1$ we obtain the two equivalent formulations of the chiral Schwinger model; (2.11) becomes

$$m^2 = \frac{e^2}{\pi} \frac{a^2}{a-1}. \quad (2.13)$$

To avoid tachyons we must require $a > 1$. Gauge invariance is definitely lost, and (2.10) becomes

$$\begin{aligned}
D_{\mu\nu} &= \frac{1}{\square + m^2} \left[g_{\mu\nu} + \frac{1}{a-1} \left[\frac{\pi}{e^2} + \frac{2}{\square} \right] \partial_\mu \partial_\nu \right. \\
&\quad \left. \mp \frac{1}{a-1} \frac{\tilde{\partial}_\mu \partial_\nu + \tilde{\partial}_\nu \partial_\mu}{\square} \right]. \quad (2.14)
\end{aligned}$$

The limit $a \rightarrow 1$ is singular in (2.13). Nevertheless a definite expression can be obtained for the propagator

$$\begin{aligned}
D_{\mu\nu}|_{a=1} &= \frac{\pi}{e^2} \left[\left[\frac{\pi}{e^2} + \frac{2}{\square} \right] \partial_\mu \partial_\nu \mp \frac{\tilde{\partial}_\mu \partial_\nu + \tilde{\partial}_\nu \partial_\mu}{\square} \right] \\
&= \frac{\pi}{e^2} \frac{(\partial_\mu + \tilde{\partial}_\mu)(\partial_\nu + \tilde{\partial}_\nu)}{\square}, \quad (2.15)
\end{aligned}$$

where in the last equality noncovariant ‘‘contact terms’’ have been disregarded. They are related to the usual ambiguity affecting the time-ordered product definition.

In the limit $a = \kappa r^2$, $r^2 \rightarrow \infty$, $er = \epsilon = \text{fixed}$, (2.11) becomes

$$m^2 = \frac{\kappa}{\pi} e^2 r^2 = \frac{\kappa}{\pi} \epsilon^2, \quad (2.16)$$

a value which resembles the Schwinger’s mass. However the presence of the axial anomaly prevents a trivial decoupling of the vector interaction and the parameter κ is related to the arbitrariness in the choice of coboundaries. To recover gauge invariance we should choose $\kappa=1$, which however can only be done after the limit $r^2 \rightarrow \infty$.

In order to further clarify this point, let us go back to (2.7) and take the limit $r^2 \rightarrow \infty$, $er = \epsilon = \text{fixed}$. If a is kept finite we get

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\epsilon}{\sqrt{\pi}} A^\mu \partial_\mu \phi, \quad (2.17)$$

which is neither equivalent to the Schwinger model nor is gauge invariant. We can instead consider the limit in which $ae^2 = \kappa \epsilon^2$ is fixed; in this way a mass is radiatively generated. If $\kappa \neq 1$, gauge invariance is broken and the kinetic term in (2.9) can be inverted, leading to (2.16). When $\kappa=1$ gauge invariance is recovered and the expression in (2.9) becomes a projection operator. In order to obtain the Schwinger’s propagator a gauge fixing must be introduced. This is the reason why the limit $\kappa \rightarrow 1$ in (2.16) can be performed only after the limit $r^2 \rightarrow \infty$ has been taken.

Going back to the general expression (2.11) we remark that the condition $m^2 > 0$, which is necessary to avoid the presence of tachyons in the theory, allows two ranges:

$$\begin{aligned}
(1) \quad &a > r^2, \\
(2) \quad &0 < a < r^2 - 1 \quad \text{or} \quad r^2 - 1 < a < 0,
\end{aligned} \quad (2.18)$$

for the parameters (a, r) . Only the first range has been considered so far in the literature, to our knowledge.

By taking in (2.10) the residue at the pole $k^2 = m^2$, one gets

$$\text{Res} D_{\mu\nu}|_{k^2=m^2} = \frac{1}{m^2} T_{\mu\nu}(k), \quad (2.19)$$

$T_{\mu\nu}$ being a positive semidefinite degenerate quadratic form in k_μ , involving the parameters (a, r) . One eigenvalue vanishes, corresponding to a decoupling of the would-be related excitation, the other is positive and can be interpreted in both ranges as the presence of a vector particle with a rest mass given by the positive square root of (2.11) and positive residue at the pole in agreement with the unitary condition. This state decouples in the limit $a = r^2$. There is also a massless degree of freedom with

$$\text{Res} D_{\mu\nu}|_{k^2=0} = \frac{\pi}{e^2 a (1+a-r^2)} [(1+r^2)k_\mu k_\nu - r(\tilde{k}_\mu k_\nu + \tilde{k}_\nu k_\mu)]|_{k^2=0}. \quad (2.20)$$

One can easily realize that again the quadratic form at the numerator is positive semidefinite for any value of r . The poles at $k^2=m^2$ and $k^2=0$ exhaust the singularities of $D_{\mu\nu}$.

Let us consider the situation in the two ranges of parameters. The first range does not deserve particular comments at this stage. No ghost is present at $k^2=0$, as one eigenvalue of the residue matrix vanishes and the other is positive, corresponding to a “physical” excitation. The second range does entail no news concerning the state with mass m . The situation is different however when considering the pole at $k^2=0$. We have indeed a negative residue in this case corresponding to a “ghost” excitation (particle with a negative probability). The theory can be accepted only if this excitation can be consistently excluded from a positive norm Hilbert space of states, which is stable under time evolution. This point will be reconsidered when we shall solve the model in the framework of a canonical quantization.

To draw definite conclusions from this path-integral approach, it is worth considering at this stage the fermionic sector. The bosonic world is rather dull indeed, consisting only of free excitations.

We go back to the general expression (2.6) in which fer-

mionic sources are on. We have now to consider the fermionic propagator in the field A_μ , which obeys the equation

$$[i\partial + e(1-r\gamma^5)A]S(x,y;A_\mu) = \delta^2(x-y), \quad (2.21)$$

with causal boundary conditions. Let us also introduce the free propagator S_0 ,

$$i\partial S_0(x) = \delta^2(x), \quad (2.22)$$

with the solution

$$S_0 = \int \frac{d^2k}{(2\pi)^2} \frac{\not{k}}{k^2+i\epsilon} e^{-ikx} = \frac{1}{2\pi} \frac{\gamma_\mu x^\mu}{x^2-i\epsilon}. \quad (2.23)$$

If we remember that any vector in two dimensions can be written as a sum of a gradient and a curl part,

$$A_\mu = \partial_\mu \alpha + \tilde{\partial}_\mu \beta, \quad (2.24)$$

the following change of variables in (2.4),

$$\psi = \exp[ie(\alpha + \gamma^5\beta + r\beta + r\alpha\gamma^5)]\chi, \quad (2.25)$$

realizes the decoupling of the fermions, leading to the expression for the “left” propagator (see Appendix A):

$$\begin{aligned} S^L(x-y) &\equiv \int \mathcal{D}(A_\mu, \phi) S^L(x,y;A_\mu) \exp\left[i \int d^2z \mathcal{L}_{\text{eff}}(A_\mu, \phi)\right] \\ &= S_0^L(x-y) Z_L \exp\left\{-\frac{1}{4} \frac{(1-r^2)^2}{a(a+1-r^2)} \ln\{\tilde{m}^2[-(x-y)^2+i\epsilon]\}\right. \\ &\quad \left.-i\pi \frac{a+1-r^2}{a(a-r^2)} \left[r - \frac{a}{a+1-r^2}\right]^2 D(x-y, m)\right\}, \end{aligned} \quad (2.26)$$

where $\tilde{m} = me^\gamma/2$, D is the scalar Feynman propagator: $D \equiv D_0$, with

$$\begin{aligned} D_{1-\omega}(x, m) &= -(\lambda^2)^{1-\omega} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{e^{-ikx}}{k^2 - m^2 + i\epsilon} \\ &= \frac{2i}{(4\pi)^\omega} \left[\frac{\lambda^2 \sqrt{-x^2}}{2m} \right]^{1-\omega} K_{1-\omega}(m\sqrt{-x^2+i\epsilon}), \end{aligned} \quad (2.27)$$

γ being the Euler-Mascheroni constant. For further developments it is useful to consider 2ω dimensions and to introduce a mass parameter λ to balance dimensions. Z_L is a (dimensionally regularized) ultraviolet renormalization constant for the fermion wave function

$$Z_L = \exp\left[i \frac{\pi(r-1)^2}{a-r^2} D_{1-\omega}(0, m)\right]. \quad (2.28)$$

The “right” propagator can be obtained from (2.26) simply by replacing S_0^L with S_0^R and changing the sign of the parameter r .

The Fourier transform in the momentum space of Eq. (2.26) cannot be obtained in closed form; however it exhibits the singularities related to the thresholds at $p^2=0$ and $p^2=(nm)^2, n=1, 2, 3, \dots$

First of all, we notice that for $r=1$ the “left” fermion is free. The same happens to the “right” fermion when

$r=-1$. Moreover we notice from (2.26) that the long-range interaction completely decouples for $r^2=1$. As a consequence the interacting fermion (for instance, the “right” one for $r=1$) asymptotically behaves like a free particle.

In general, at small values of x^2 , the propagator S^L has the behavior

$$S^L \underset{x^2 \rightarrow 0}{\sim} C_0 x^+ (-x^2+i\epsilon)^{-1-A} \quad (2.29)$$

with

$$A = \frac{1}{4} \frac{(1-r)^2}{a-r^2} \quad (2.30)$$

and C_0 a suitable constant. We remark that the ultraviolet behavior of the left fermion propagator can be directly obtained from the ultraviolet renormalization constant

$$\gamma_{\psi_L} = \lim_{\omega \rightarrow 1} \frac{1}{2} \left[\lambda \frac{\partial}{\partial \lambda} \ln Z_L \right] = - \frac{(1-r)^2}{4(a-r^2)} \quad (2.31)$$

and, of course, it coincides with the one of the explicit solution (2.26). It can be obtained from the renormalization group equation in the ultraviolet limit in which the mass-dependent term is disregarded.

For large values of x^2 we get instead

$$S^L \underset{x^2 \rightarrow -\infty}{\sim} C_\infty x^+ (-x^2 + i\epsilon)^{-1-B}, \quad (2.32)$$

where

$$B = \frac{1}{4} \frac{(1-r^2)^2}{a(a+1-r^2)} \quad (2.33)$$

and C_∞ another constant.

We shall see in Sec. V that (2.32) exactly coincides with the fermionic propagator of the massless Thirring model.

In the first range ($a > r^2$), both A and B are positive. The propagator decreases at infinity indicating the possible existence of asymptotic states for fermions, which however feel the long-range interaction mediated by the massless excitation which is present in the bosonic spectrum. The situation in the second range is much more intriguing. Here both A and B are negative. Moreover

$$1+B = \frac{(2a+1-r^2)^2}{4a(a+1-r^2)} < 0 \quad (2.34)$$

$$\mathcal{H}(x) = \exp \left\{ -4i\pi \left[\frac{a}{(a-r^2)(a-r^2+1)} [D(x, m) - D_{1-\omega}(0, m)] + \frac{1-r^2}{a-r^2+1} [D_{1-\omega}(0, 0) - D_{1-\omega}(x, 0)] \right] \right\} \quad (2.37)$$

and

$$Z = \exp \left\{ 4i\pi \frac{r^2}{a-r^2} D_{1-\omega}(0, m) \right\}. \quad (2.38)$$

Dimensional regularization is understood.

Let us now discuss the quantity $Z^{-1}\mathcal{H}$, which represents the deviation from the free theory result

$$Z^{-1}\mathcal{H} = \exp \left[\frac{2a}{(a-r^2)(a-r^2+1)} K_0(m\sqrt{-x^2+i\epsilon}) + \frac{1-r^2}{a-r^2+1} \ln[\tilde{m}^2(-x^2+i\epsilon)] \right]. \quad (2.39)$$

For small values of x^2 we get

$$Z^{-1}\mathcal{H} \underset{x^2 \rightarrow 0}{\sim} \tilde{C}_0 (-x^2+i\epsilon)^{-r^2/(a-r^2)}, \quad (2.40)$$

whereas, for large negative x^2 ,

$$Z^{-1}\mathcal{H} \underset{x^2 \rightarrow -\infty}{\sim} \tilde{C}_\infty (-x^2+i\epsilon)^{(1-r^2)/(a-r^2+1)}, \quad (2.41)$$

$\tilde{C}_0, \tilde{C}_\infty$ being suitable constant quantities. Again the ultraviolet behavior can be recovered from the anomalous dimension related to Z .

In the first range ($a > r^2$) we have a singular behavior at short distances [negative exponent in (2.40)] and, since $-1+(1-r^2)/(a-r^2+1) < 0$, a decreasing behavior of the correlation function at infinity. We interpret this phenomenon as the existence of a long-range interaction mediated by the massless bosonic excitation. If $r^2=1$ we

leading to a propagator which increases when $x^2 \rightarrow -\infty$. We interpret this phenomenon as a sign of confinement. We recall indeed that gauge invariance is broken and therefore the fermion propagator is endowed of a direct physical meaning. The unphysical massless bosonic excitation, which occurs in this region of parameters, produces an antiscreening effect of a long-range type. Nevertheless no asymptotic freedom is expected ($A \neq 0$).

All this analysis will be confirmed and reinterpreted in a deeper way when following a canonical procedure.

Propagators are not suitable to discuss the limiting case $a=r=0$ (vector Schwinger model) in which gauge invariance is restored. There is however another interesting quantity which can be easily discussed in a path-integral approach. Let us introduce the scalar fermion composite operator

$$S(x) = N [\bar{\psi}(x)\psi(x)], \quad (2.35)$$

where N means the finite part, after divergences have been (dimensionally) regularized and renormalized. By repeating standard techniques, it is not difficult to get the expression

$$\langle 0 | T[S(x)S(0)] | 0 \rangle = - \frac{Z^{-1}}{2\pi^2(x^2-i\epsilon)} \mathcal{H}(x), \quad (2.36)$$

where

see from (2.36) and (2.41) that the correlation function of the composite operator decreases at infinity as in the free theory. We know indeed that, in this case, one of the fermions with a definite chirality is free and the other one has only a short-range interaction, as the long range massless excitation decouples in this case.

In the second range, both exponents $-1-r^2/(a-r^2)$ and $-1+(1-r^2)/(a-r^2+1)$ are positive. The correlation decreases at short distances and increases when $x^2 \rightarrow -\infty$. This is again a sign of confinement. In the correlation function for the S operator we can take first the limit $r \rightarrow 0$ and then $a \rightarrow 0$, thereby recovering the result we expect in the gauge invariant Schwinger model. We obtain a correlation function which goes to a constant at infinity, as expected, since fermions are confined in that model. We defer the discussion concerning

currents and the related charges to the canonical treatment in the sequel.

We end this section by remarking about the nontrivial behavior of this model under a scale transformation. We notice that conformal invariance is recovered both in the ultraviolet and in the infrared limit, with different scale coefficients.

III. OPERATORIAL APPROACH: THE BOSONIC SECTOR

In this section we canonically implement the quantum dynamics of the model described by the effective Lagrangian (2.7) using a Dirac-brackets formalism [8]. Actually, this procedure only concerns the bosonic sector of the theory (2.1); nevertheless the scalar degrees of freedom will appear as the “building blocks” in the explicit construction of a fermionic operator solving the equations of motion derived from (2.1). The possibility of constructing fermionic operators in terms of bosonic ones

(bosonization) is a well-known property of the two-dimensional world [9] and it turns out to be essential in our solution and interpretation of the model.

From the Lagrangian (2.7) we obtain the momenta canonically conjugate to the coordinates A^0 , A^1 and ϕ (we call $e^2/\pi = \hat{e}^2$):

$$\Omega_1 \equiv \Pi_0 = 0, \quad (3.1)$$

$$\Pi_1 = F_{01}, \quad (3.2)$$

$$\Pi_\phi = \partial_0 \phi - \hat{e} r A_0 - \hat{e} A_1, \quad (3.3)$$

where Ω_1 is the primary constraint. The usual total Hamiltonian is

$$H = H_0 + \int dx^1 \xi_1(x^1) \Omega_1(x^1) \quad (3.4)$$

with the introduction of the Lagrange multiplier ξ_1 and the expression

$$H_0 = \int dx^1 \left[\frac{1}{2} \Pi_1^2 + (\partial_1 A_0) \Pi_1 + \frac{1}{2} \Pi_\phi^2 + \frac{1}{2} (\partial_1 \phi)^2 + \frac{\hat{e}^2}{2} A_0^2 (r^2 - a) + \frac{1}{2} \hat{e}^2 (a+1) A_1^2 - \hat{e} r (\partial_1 \phi) A_1 - \hat{e} (\partial_1 \phi) A_0 + \hat{e} r A_0 \Pi_\phi + \hat{e} A_1 \Pi_\phi + \hat{e}^2 r A_0 A_1 \right], \quad (3.5)$$

derived from (2.7) by a Legendre transformation. Requiring that the primary constraint persists in time, we find the secondary constraint

$$\Omega_2(x^1) \equiv \partial_1 \Pi_1 - \hat{e}^2 (r^2 - a) A_0 + \hat{e} \partial_1 \phi - \hat{e} r \Pi_\phi - \hat{e}^2 r A_1 = 0. \quad (3.6)$$

No new constraint arises for $a \neq r^2$: the Poisson brackets

$$\{\Omega_1(x^1), \Omega_2(y^1)\} = \hat{e}^2 (r^2 - a) \delta(x^1 - y^1) \quad (3.7)$$

do not vanish and hence the condition $\Omega_2(x^1) = 0$ only determines the Lagrange multiplier $\xi_1(x^1)$: we are in presence of a system with second class constraints. The discussion of the limiting case $a = r^2$ is deferred to Appendix B.

Following the standard procedure we introduce the Dirac brackets, derived from (3.7),

$$\{Q(x^1), P(y^1)\}_D = \{Q(x^1), P(y^1)\} - \frac{1}{\hat{e}^2 (r^2 - a)} \int dz^1 \left[-\{Q(x^1), \Omega_1(z^1)\} \{\Omega_2(z^1), P(y^1)\} + \{Q(x^1), \Omega_2(z^1)\} \{\Omega_1(z^1), P(y^1)\} \right], \quad (3.8)$$

leading to the canonical structure (we report only the nonzero brackets)

$$\begin{aligned} \{A_1(x^1), \Pi_1(y^1)\}_D &= \delta(x^1 - y^1), \quad \{A_0(x^1), A_1(y^1)\}_D = -\frac{1}{\hat{e}^2 (r^2 - a)} \partial_{x^1} \delta(x^1 - y^1), \\ \{\phi(x^1), \Pi_\phi(y^1)\}_D &= \delta(x^1 - y^1), \quad \{A_0(x^1), \Pi_\phi(y^1)\}_D = \frac{1}{\hat{e}^2 (r^2 - a)} \partial_{x^1} \delta(x^1 - y^1), \\ \{A_0(x^1), \Pi_1(y^1)\}_D &= -\frac{r}{r^2 - a} \delta(x^1 - y^1), \quad \{A_0(x^1), \phi(y^1)\}_D = \frac{r}{\hat{e} (r^2 - a)} \delta(x^1 - y^1). \end{aligned} \quad (3.9)$$

In the Dirac-Bargmann formalism, the equations of motion can be written as

$$\dot{g} = \{g, H_{\text{red}}\}_D \Big|_{\Omega_i=0}, \quad (3.10)$$

g being any function of canonical variables. H_{red} is obtained from H_0 , by expressing A_0 as the solution of the constraint $\Omega_2 = 0$:

$$\begin{aligned} H_{\text{red}} = \int dx^1 \left[\frac{1}{2} \Pi_1^2 + \frac{1}{2} \frac{a}{a-r^2} \Pi_\phi^2 + \frac{1}{2} (\partial_1 \phi)^2 \frac{a+1-r^2}{a-r^2} + \frac{1}{2} \hat{e}^2 \frac{a(a+1-r^2)}{a-r^2} A_1^2 - \hat{e} r \frac{a+1-r^2}{a-r^2} A_1 \partial_1 \phi \right. \\ \left. + \hat{e} \frac{a}{a-r^2} A_1 \Pi_\phi + \frac{1}{2} \frac{1}{\hat{e}^2} \frac{1}{a-r^2} (\partial_1 \Pi_1)^2 + \frac{1}{\hat{e} (a-r^2)} \right. \\ \left. \times \partial_1 \phi \partial_1 \Pi_1 + \frac{1}{\hat{e}} \frac{r}{a-r^2} \partial_1 \Pi_1 \Pi_\phi - \frac{r}{a-r^2} A_1 \partial_1 \Pi_1 - \frac{r}{a^2 - r^2} \partial_1 \phi \Pi_\phi \right]. \quad (3.11) \end{aligned}$$

The quantization is now performed by taking the constraints as operatorial equations, identifying Dirac brackets with equal time (ET) commutators and using a symmetrical ordering in the product of operators.

We remark that the breaking of gauge invariance appears in the canonical treatment of the effective theory (2.7) as a change of the constraint structure: they belong to a second-class system reflecting the absence of a local symmetry.

Using (3.10), the Heisenberg equations are easily obtained and they are completely equivalent to the Lagrange equations derived from (2.7), which was not to be “*a priori*” expected:

$$\partial_\mu F^{\mu\nu} = -\hat{e}^2 a A^\nu + \hat{e} r \partial^\nu \phi - \hat{e} \tilde{\partial}^\nu \phi, \quad (3.12a)$$

$$\square \phi = r \hat{e} \partial_\mu A^\mu - \hat{e} \tilde{\partial}_\mu A^\mu. \quad (3.12b)$$

The most general solution of these equations is

$$A^\mu = -\frac{r}{\hat{e} a (1+a-r^2)} \partial^\mu \sigma - \frac{(a-r^2)}{\hat{e} a (1+a-r^2)} \tilde{\partial}^\mu \sigma + \frac{1}{\hat{e} a} (r \partial^\mu - \tilde{\partial}^\mu) h, \quad (3.13a)$$

$$\phi = -\frac{1}{(1+a-r^2)} \sigma + h, \quad (3.13b)$$

with

$$(\square + m^2) \sigma = 0, \quad (3.14)$$

$$\square h = 0, \quad (3.15)$$

and m^2 given by (2.11); σ and h describe the bosonic degrees of freedom of the theory. In order to show the equivalence with the path-integral results, we are left with computing their equal-time commutation relations, which in turn will exhibit their effective independence and will provide us with the unitarity conditions.

From the identification $\sigma = \hat{e} \Pi_1$ we get

$$[\sigma(x), \sigma(y)]_{\text{ET}} = 0, \quad (3.16a)$$

$$[\sigma(x), \dot{\sigma}(y)]_{\text{ET}} = i \frac{m^2}{\hat{e}^2} \delta(x^1 - y^1), \quad (3.16b)$$

where we have used the Heisenberg equation for Π_1 :

$$\dot{\Pi}_1 = -\hat{e}^2 a A_1 + \hat{e} r \partial_1 \phi - \hat{e} \partial_0 \phi. \quad (3.17)$$

Equation (3.13b) gives the remaining commutation relations:

$$[h(x), \sigma(y)]_{\text{ET}} = 0, \quad (3.18a)$$

$$[h(x), \dot{\sigma}(y)]_{\text{ET}} = 0, \quad (3.18b)$$

$$[h(x), h(y)]_{\text{ET}} = 0, \quad (3.19a)$$

$$[h(x), \dot{h}(y)]_{\text{ET}} = i \frac{a}{1+a-r^2} \delta(x^1 - y^1). \quad (3.19b)$$

In particular Eqs. (3.18) show the independence of massive and massless degrees of freedom. The request of the absence of tachyons from the spectrum forces the parameters a and r to range in the two regions (2.18).

In the first one ($a > r^2$) the commutation relations (3.16b) and (3.19) are physical, so that σ and h generate a Fock space with a positive defined metric. We remark that, from a rigorous point of view, the positivity of the massless sector is achieved only after a Krein extension of the original Fock topology derived from (3.13b) [10]; the realization of such nontrivial extension is also essential in order to prove the existence of the operators that, in the next section, we will construct to describe the fermionic degrees of freedom of the theory.

In the other range ($0 < a < r^2 - 1$ or $r^2 - 1 < a < 0$) h is a “ghost,” having the negative sign in its commutation relations. We can define a physical Hilbert space imposing the subsidiary condition

$$h^+(x) |\Phi_{\text{phys}}\rangle = 0, \quad (3.20)$$

which however possesses a nonlocal character with respect to A_μ . This condition is stable under time evolution, because of the free character of h . Obviously (3.20) selects the physical operators of the theory: in other words it imposes a restriction on the operators representing the fermionic sector, as we will see in the next section.

Now we try to discuss some limiting situations on the parameters a and r , but the case $a = r^2$ that involves a doubling of the constraints and is deferred to Appendix B.

The commutation relations (3.19) are singular in the limit $a = r^2 - 1$; nevertheless, if we come back to equations of motion (3.12) and we put $a = r^2 - 1$, we can solve for A_μ and ϕ without the occurrence of any singularity. The solution is

$$A_\mu = \hat{e} \frac{1}{\square} (\tilde{\partial}_\mu - r \partial_\mu) \sigma - \frac{1}{\hat{e} (r^2 - 1)} \tilde{\partial}_\mu \sigma + \frac{r}{\hat{e}} \tilde{\partial}_\mu h + \frac{1}{\hat{e} (r^2 - 1)} \partial_\mu h, \quad (3.21)$$

$$\phi = \hat{e}^2 (1 - r^2) \frac{1}{\square} \sigma + h, \quad (3.22)$$

where

$$\square \sigma = \square h = 0 \quad (3.23)$$

and $1/\square$ is the inverse of the d'Alembert operator. Clearly the relation among A_μ , ϕ , and σ is not local (due to the presence of an integral operator) and the theory seems to lose its local character. The other limiting case is $a = 0$ ($r \neq 0$): this limit corresponds to a “would-be” gauge invariant regularization of the theory, and it can be performed starting from the second interval of parameters. The mass vanishes and we recognize a situation similar to the one in the case $a = r^2 - 1$:

$$A^\mu = \hat{e} \frac{1}{\square} \partial^\mu \sigma - \hat{e} \frac{1}{\square} \tilde{\partial}^\mu \sigma + h_1^\mu, \quad (3.24)$$

$$\square \sigma = 0,$$

$$\square h_1^\mu = 0, \quad (r \partial_\mu - \tilde{\partial}_\mu) h_1^\mu = 0,$$

while

$$\square \phi = 0. \quad (3.25)$$

Again the properties of the theory are not transparent, due to the nonlocal relation with the basic degrees of freedom. The situation is reminiscent of the chiral Schwinger model for $a=0$ studied in [11]. We observe that it is possible to put $r=0$: it would correspond to having a Schwinger model regularized in a gauge dependent way.

For $a > 0$ we have two positive metric field: σ with mass $\hat{e}^2(1+a)$ and h massless; for $-1 < a < 0$, h becomes a “ghost.” These theories are not equivalent to the original Schwinger model: the introduction of the gauge-breaking counterterm

$$\frac{\hat{e}^2 a}{2} A_\mu A^\mu \quad (3.26)$$

cannot be interpreted as a gauge fixing and the model rather resembles to the Stueckelberg electrodynamics in two dimensions.

In conclusion we have recovered in an operatorial formalism, the results of the path-integral approach, concerning the bosonic sector. In particular the propagator

$$\langle 0 | T(A_\mu(x) A_\nu(y)) | 0 \rangle$$

can be computed and coincides with (2.10), apart from irrelevant noncovariant contact terms. Moreover the structure of the Hilbert space of states has been clarified in the various cases.

A last remark concerns the singularity of the solutions when $\hat{e} \rightarrow 0$: our results are truly nonperturbative as we

$$\psi(x) = \exp \frac{i\sqrt{\pi}}{a} \left[- \left[r + \frac{a}{1+a-r^2} \gamma^5 \right] \sigma(x) + (r^2-1) \gamma^5 h(x) \right] \psi_0(x), \quad (4.4)$$

where $\psi_0(x)$ obeys to the free Dirac equation. To obtain an operator solution we define $\mathcal{A}\psi(x)$ by normal ordering $:\mathcal{A}\psi:(x)$ and introduce the trial solution

$$\psi_\alpha(x) = C \left[\frac{\mu}{2\pi} \right]^{1/2} : \exp - \frac{i\sqrt{\pi}}{a} \left[\left[r + \frac{a}{a+1-r^2} \gamma_{\alpha\alpha}^5 \right] \sigma + (a+1-r^2) \tilde{h} - a \gamma_{\alpha\alpha}^5 h \right] : , \quad (4.5)$$

where $\partial_\mu \tilde{h} = \tilde{\delta}_\mu h$, μ is an infrared regulator associated to h , carrying the correct balance of canonical dimension and C a normalization constant to be determined later on. We know from (3.12a) that

$$\partial_\mu F^{\mu\nu} = \hat{e} \tilde{\delta}^\nu \sigma ; \quad (4.6)$$

we define the currents J_\pm^μ by a point splitting procedure as

$$J_\pm^\mu(x) = \lim_{\epsilon \rightarrow 0} U_\pm^{-1}(\epsilon) \{ \bar{\psi}(x+\epsilon) \gamma^\mu P_\pm \exp \left[i\sqrt{\pi} \hat{e} \int_x^{x+\epsilon} dz_\nu [A^\nu + r \tilde{A}^\nu] \right] \psi(x) - \text{VEV} \} , \quad (4.7)$$

where $\epsilon^2 < 0$, VEV stands for “vacuum expectation value,” and $P_\pm = (1 \pm \gamma^5)/2$; $U_\pm(\epsilon)$ are some ultraviolet renormalization constants. A first request is the infrared finiteness of such currents, that will fix the value of the constant C . Then we introduce a quantum current $\hat{J}_\mu(x)$,

$$\hat{J}^\mu(x) = (1+r) J_+^\mu(x) + (1-r) J_-^\mu(x) + \alpha \frac{\hat{e}}{\sqrt{\pi}} A^\mu(x) + \beta \frac{\hat{e}}{\sqrt{\pi}} \tilde{A}^\mu(x), \quad (4.8)$$

which has to be eventually consistent with (4.6): the quantities α and β parametrize the ambiguity related to the regularization procedure.

We remark that we have introduced in the string the “classical” gauge invariant expression of the currents; would we have introduced a more general expression, where A^μ and \tilde{A}^μ appear multiplied by arbitrary weights K_1 and K_2 , re-

do not introduce any gauge fixing which would be necessary to build a free propagator to start with.

IV. OPERATORIAL APPROACH: THE FERMIONIC SECTOR

In order to establish a definitive link with the path-integral formalism we have to construct the fermionic operator that solves the equations derived from (2.1). Actually we will go further, finding a conserved charge that allows us to identify a fermionic sector on the Hilbert space of the model: in this way the difference between the two ranges of parameters (2.18) will be fully enlightened, confirming the analysis of the path-integral formulation.

Let us come back to the original Lagrangian (2.1) and obtain the Maxwell and Dirac equations

$$\partial_\mu F^{\mu\nu} = -e J^\nu, \quad (4.1)$$

$$i \not{\partial} \psi + e \mathcal{A}(a + r \gamma^5) \psi = 0, \quad (4.2)$$

with the “classical” current J^μ defined as

$$J^\mu = \bar{\psi} \gamma^\mu (1 + r \gamma^5) \psi. \quad (4.3)$$

In solving these equations we need a regularization procedure to give a meaning to the composite operators $\mathcal{A}\psi(x)$ and $J^\mu(x)$: we seek consistency with the results of the bosonic sector. In so doing we are able to express ψ as a well-defined functional of the bosonic degrees of freedom σ and h .

Taking the expression (3.13a) of A_μ into account, it is easy to verify that a classical solution of (4.2) is

spectively, this would simply entail a change in α and β , namely, this arbitrariness is already taken into account by the polynomial subtractions which are, by the way, related to the loss of gauge invariance.

We begin by considering J_+^0 :

$$J_+^0(x) \simeq U_+^{-1}(\epsilon) \{ \psi^\dagger(x+\epsilon) P_+ \psi(x) + i\sqrt{\pi} \hat{\epsilon} \psi^\dagger(x+\epsilon) P_+ \psi(x) \epsilon^\mu (A_\mu + r \tilde{A}_\mu) + O(\epsilon^2) - \text{VEV} \}. \quad (4.9)$$

Standard Wick's techniques lead to

$$\begin{aligned} \psi^\dagger(x+\epsilon) P_+ \psi(x) &= \frac{\mu}{2\pi} C^2: \exp - \frac{i\sqrt{\pi}}{a} \epsilon^\mu \partial_\mu \left[- \left[r + \frac{a}{a+1-r^2} \right] \sigma \right. \\ &\quad \left. + (a+1-r^2)\tilde{h} - ah \right]: \exp \frac{\pi}{a^2} \left\{ \left[r + \frac{a}{a+1-r^2} \right]^2 a \frac{a+1-r^2}{a-r^2} D^+(\epsilon, m) \right. \\ &\quad \left. + \frac{a}{a+1-r^2} [(a+1-r^2)^2 + a^2] D^+(\epsilon, \mu) - 2a^2 \tilde{D}^+(\epsilon) \right\} \end{aligned} \quad (4.10)$$

with

$$\begin{aligned} D^+(x, m) &= \frac{1}{2\pi} K_0(m\sqrt{-x^2 + ix^0\delta}), \quad D^+(\epsilon, \mu) = -\frac{1}{4\pi} \ln(-\mu^2\epsilon^2 + ix^0\delta), \\ \tilde{D}^+(\epsilon) &= \frac{1}{4\pi} [\ln(\epsilon^- - i\delta) - \ln(\epsilon^+ - i\delta)], \quad \epsilon^\pm = \epsilon^0 \pm \epsilon^1, \quad \delta > 0. \end{aligned} \quad (4.11)$$

Then we choose

$$C = \left[\frac{\mu}{\tilde{m}} \right]^{(1/4)(1-r^2)^2/a(a+1-r^2)} \quad (4.12)$$

in order to obtain a result independent of the infrared regulator:

$$[\psi^\dagger(x+\epsilon) P_+ \psi(x) - \text{VEV}] = U_+(\epsilon) \left\{ \frac{1}{2\sqrt{\pi}a} \frac{\epsilon^\mu}{\epsilon^-} \partial_\mu \left[\left[r + \frac{a}{a+1-r^2} \right] \sigma + (a+1-r^2)\tilde{h} - ah \right] + O(\epsilon) \right\}, \quad (4.13)$$

$$U_+(\epsilon) = \exp \frac{\pi}{a^2} \left\{ \left[r + \frac{a}{a+1-r^2} \right]^2 \frac{a(a+1-r^2)}{a-r^2} D^+(\epsilon, m) - \frac{1}{4\pi} \frac{a(r^2-1)^2}{a+1-r^2} \ln(-\tilde{m}^2\epsilon^2 + i\epsilon^0\delta) \right\}. \quad (4.14)$$

One should compare this expression with the fermion wave function renormalization constant in (2.28). Apart from the different regularization, they manifestly exhibit the same behavior $U_+(\epsilon) \sim Z_R^{-1}$, which is rooted in the fact that the current J_+^μ does not undergo renormalization.

In the same way we get

$$i\sqrt{\pi} \hat{\epsilon} \psi^\dagger(x+\epsilon) P_+ \psi(x) \epsilon^\mu (A_\mu + r \tilde{A}_\mu) - \text{VEV} = U_+(\epsilon) \left\{ \frac{\hat{\epsilon}}{2\sqrt{\pi}} \frac{\epsilon^\mu}{\epsilon^-} (A_\mu + r \tilde{A}_\mu) + O(\epsilon) \right\}. \quad (4.15)$$

Collecting all terms we end up with

$$J_+^0(x) = \lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{2\sqrt{\pi}} \frac{\epsilon^\mu}{\epsilon^-} (\partial_\mu - \tilde{\partial}_\mu) \left[\frac{1}{1+a-r^2} \sigma - h \right] \right\}. \quad (4.16)$$

We notice that the limit $\epsilon \rightarrow 0$ does not depend on the direction. Then, taking into account that $\gamma^0 P_+ = \gamma^1 P_+$, we get

$$J_+^\mu = \frac{1}{2\sqrt{\pi}} (\tilde{\partial}^\mu - \partial^\mu) \left[h - \frac{1}{a+1-r^2} \sigma \right], \quad (4.17a)$$

$$J_-^\mu = \frac{1}{2\sqrt{\pi}} (\tilde{\partial}^\mu + \partial^\mu) \left[h - \frac{1}{a+1-r^2} \sigma \right]. \quad (4.17b)$$

As we have predicted, these currents are not conserved: by requiring the consistency of the Maxwell equations we obtain

$$\alpha = a, \quad \beta = 0, \quad (4.18)$$

namely,

$$\partial_\mu F^{\mu\nu} = \hat{e} \hat{\delta}^\nu \sigma = -\hat{e} \sqrt{\pi} \hat{J}^\nu = -\hat{e} \sqrt{\pi} \left\{ (1+r) J_+^\nu + (1-r) J_-^\nu + \frac{a \hat{e}}{\sqrt{\pi}} A^\nu \right\} \tag{4.19}$$

and our trial solution in (4.5) turns out to be fully consistent. We notice that it is essential that the fermionic operator does not factorize as a product of a free spinor times a short-range interacting term.

We can recover the known results for the chiral Schwinger model putting $r = 1$:

$$\psi_R = \psi_R^0(x), \quad \psi_L(x) =: \exp -2i\sqrt{\pi} \sigma : \psi_L^0(x) . \tag{4.20}$$

In this case the interacting solution factorizes into an interaction piece, depending on the massive component, and a free fermion $\psi^0(x)$; its asymptotic behavior is the one of a free Dirac theory. We remark that only for $r = \pm 1$ the free part is factorized.

One can easily check that the Green functions

$$\begin{aligned} \langle 0 | T(\psi_\alpha(x) \psi_\beta^\dagger(y)) | 0 \rangle , \\ \langle 0 | T(\bar{\psi}(x) \bar{\psi}(y)) | 0 \rangle , \end{aligned}$$

computed using (4.5), coincide with the path-integral ones (2.26), (2.36), apart from the wave-function renormalization constant for the field ψ (our solution is here renormalized).

Now we want to discuss the properties of the solution (4.5). As starting point we remark that gauge invariance is completely broken; hence $\psi_\alpha(x)$ is not affected, in principle, by any gauge ambiguity.

The first investigation concerns the electric charge of this solution: integrating the zero component of the conserved current $\hat{J}_\mu(x)$ (that couples to the Gauss's law), we

$$\begin{aligned} [h^+(x), \psi_\alpha(y)] &= -\frac{i\sqrt{\pi}}{a} [h^+(x), (a+1-r^2)\bar{h}^-(y) - a\gamma_{\alpha\alpha}^5 h^-(y)] \psi_\alpha(y) \\ &= -\frac{i\sqrt{\pi}}{a} \psi_\alpha(y) \left[a \left[\bar{D}^+(x-y) - \frac{i}{4} \right] - \frac{a^2}{a+1-r^2} \gamma_{\alpha\alpha}^5 D^+(x-y, \mu) \right] \neq 0 , \end{aligned} \tag{4.24}$$

shows that $\psi_\alpha(x)$ fails to be physical.

This analysis agrees with the path-integral one and is confirmed by the inspection of the (anti) commutation relations. We study

$$\{ \psi_\alpha(x), \psi_\beta^\dagger(0) \}_{\text{ET}} . \tag{4.25}$$

For $\alpha \neq \beta$ is straightforward to show (using the standard properties of Green functions in 1+1 dimensions), that the result is zero. For $\alpha = \beta$ the computation gives

$$\begin{aligned} \{ \psi_\alpha(x), \psi_\alpha^\dagger(0) \}_{\text{ET}} =: \exp \left\{ -\frac{i\sqrt{\pi}}{a} \left[\left[r + \frac{a}{a+1-r^2} \gamma_{\alpha\alpha}^5 \right] \sigma(x) + (a+1-r^2)\bar{h}(x) - a\gamma_{\alpha\alpha}^5 h \right. \right. \\ \left. \left. - \left[r + \frac{a}{a+1-r^2} \gamma_{\alpha\alpha}^5 \right] \sigma(0) - (a+1-r^2)\bar{h}(0) + a\gamma_{\alpha\alpha}^5 h(0) \right] \right\} : A_{\alpha\alpha}(x) B_{\alpha\alpha}(x) , \end{aligned} \tag{4.26}$$

where

$$A_{\alpha\alpha}(x) = C^2 \exp \frac{\pi}{a^2} \left[\left[r + \frac{a}{a+1-r^2} \gamma_{\alpha\alpha}^5 \right]^2 \frac{(a+1-r^2)a}{a-r^2} D^+(x, m) + \frac{a(1-r^2)^2}{a+1-r^2} D^+(x, \mu) \right] , \tag{4.27a}$$

get the generator

$$\hat{Q} = \int dx^1 \hat{J}_0(x^1) . \tag{4.21}$$

A simple calculation gives the commutation rule

$$[\hat{Q}, \psi_\alpha(x)] = 0 , \tag{4.22}$$

showing that $\psi_\alpha(x)$ is electrically neutral; actually, in order to be rigorous, one should smear \hat{Q} with a test function of compact support f_R and prove that

$$\lim_{R \rightarrow \infty} [\hat{Q}_R, \psi_\alpha(x)] = 0$$

with $\hat{Q}_R = \int dx^1 \hat{J}_0(x^1) f_R(x^1)$.

The electric charge of the original fermion is totally screened: this is true for any value of r and a .

At this point we recall that, in the first range ($a > r^2$), ψ is a well-defined operator on the Hilbert space of σ and h with the prescription of taking the limit $\mu \rightarrow 0$ on its correlation functions; moreover ψ generates a positive norm Hilbert space, whose properties will be specified in the next section.

On the other hand, in the second range we have to impose on the physical operators the condition (3.14) equivalent to

$$[h^+(x), \Phi_{\text{phys}}] = 0 . \tag{4.23}$$

A short calculation

$$B_{\alpha\alpha}(x) = \frac{\mu}{2\pi} \exp 2\pi \{ [D^+(x, \mu) - \gamma_{\alpha\alpha}^5 \tilde{D}^+(x)] + [D^+(-x, \mu) - \gamma_{\alpha\alpha}^5 \tilde{D}^+(-x)] \} . \quad (4.27b)$$

For $x^0=0$ we get $B_{\alpha\alpha}(x) = \delta(x^1)$, so that

$$\begin{aligned} \{\psi_\alpha(x), \psi_\alpha^\dagger(0)\}_{\text{ET}} &= A_{\alpha\alpha}(0) \delta(x^1) , \\ A_{\alpha\alpha}(0) &= \exp \left[i \frac{(r + \gamma_{\alpha\alpha}^5)^2}{a - r^2} D_{1-\omega}(0, m) \right] . \end{aligned} \quad (4.28)$$

Recalling (2.28) we find

$$A_{11} = Z_R^{-1} , \quad (4.29a)$$

$$A_{22} = Z_L^{-1} . \quad (4.29b)$$

Equation (4.28) are anticommutation relations for interacting fermions (see, e.g., [12]).

In the same way

$$\{\psi_\alpha(x), \psi_\beta(0)\}_{\text{ET}} = 0 \quad \forall_{\alpha, \beta} .$$

In the next section we shall restrict ourselves to the case $a > r^2$, where we shall succeed in giving a deeper characterization of the solution in this case.

V. THE RELATION WITH THE MASSLESS THIRRING MODEL

We have seen that for $r^2=1$ the solution (4.20) factorizes into an interaction piece depending on the massive boson σ and a free spinor: the asymptotic behavior of the Green functions is the one of free chiral fermions. The solution is electrically neutral and carries the fermion number associated with the free conserved current [11] $J_0^\mu(x) = \bar{\psi}_0 \gamma^\mu \psi_0(x)$:

$$\hat{Q}^{(0)} = \int dx^1 J_0^{(0)}(x^1), \quad [\hat{Q}^{(0)}, \psi_\alpha(x)] = \psi_\alpha(x) . \quad (5.1)$$

The conclusion is that a free massless fermion exists as asymptotic state.

In the general situation $r^2 \neq 1$, as we have seen, we cannot draw a similar conclusion, due to the long-range character of the interaction. Nevertheless a solution of the Dirac equation exists, carrying the correct anticommutation relation: we try to find what kind of states are linked to this operator. Because of the independence of σ and h we can factorize ψ_α as

$$\psi_\alpha(x) = C \left[\frac{\mu}{2\pi} \right]^{1/2} : \exp - \frac{i\sqrt{\pi}}{a} \left[\left[r + \frac{a}{a+1-r^2} \gamma_{\alpha\alpha}^5 \right] \sigma \right] : \exp - \frac{i\sqrt{\pi}}{a} [(a+1-r^2)\tilde{h} - a\gamma_{\alpha\alpha}^5 h] : . \quad (5.2)$$

First we look at the “spin” of this operator: we study the transformation property under Lorentz boost of the correlation function

$$\begin{aligned} \langle 0 | \psi_\alpha(x) \psi_\alpha^\dagger(0) | 0 \rangle &= C^2 \exp \left\{ \pi m^2 \left[\frac{r}{a} + \frac{1}{a+1-r^2} \gamma_{\alpha\alpha}^5 \right] D^+(x, m) \right\} \\ &\times \exp \left\{ \pi \left[\frac{a+1-r^2}{a} + \frac{a}{a+1-r^2} \right] D^+(x, \mu) \right\} \exp \left[-2\pi \gamma_{\alpha\alpha}^5 \tilde{D}^+(x) \right] . \end{aligned} \quad (5.3)$$

Calling χ the parameter of the Lorentz boost $\sinh \chi = v / (\sqrt{1-v^2})$, the transformation of the massless commutators $D^+(x, \mu)$ and $\tilde{D}^+(x)$ are easily found to be

$$D^+(x, \mu) \rightarrow D^+(x, \mu) , \quad (5.4a)$$

$$\tilde{D}^+(x) \rightarrow \tilde{D}^+(x) - \frac{\chi}{2\pi} . \quad (5.4b)$$

The boost on (5.3) acts as

$$\langle \psi_\alpha(x) \psi_\alpha^\dagger(0) \rangle \rightarrow \langle \psi_\alpha(x) \psi_\alpha^\dagger(0) \rangle \exp(\gamma_{\alpha\alpha}^5 \chi) , \quad (5.5)$$

that suggests the rule

$$\psi(x) \rightarrow \exp(\frac{1}{2} \gamma^5 \chi) \psi(x) . \quad (5.6)$$

The “spin” is $s = \frac{1}{2}$ (independent of r and a); we remark we are not talking about a true spin, as no rotation group is present in two dimensions. Hence the “spin” is rather a label for the representation of the Lorentz group.

Then we turn our attention to scaling properties: the question is subtler because the existence of the field σ .

The explicit presence of a mass violates scale invariance: in the limit $x^2 \rightarrow +\infty$, when the massive components decouple from the correlation function, we can recover an exact scaling. It is not difficult to read the asymptotic scale dimension of $\psi_\alpha(x)$ from (5.3), in this limit. Under a dilatation $x_\mu \rightarrow \lambda x_\mu$,

$$D^+(x, \mu) \rightarrow D^+(x, \mu) - \frac{\lambda}{2\pi} , \quad (5.7a)$$

$$\tilde{D}^+(x) \rightarrow \tilde{D}^+(x) , \quad (5.7b)$$

giving

$$\psi_\alpha(x) \rightarrow \psi_\alpha(x) \exp \left[-\lambda \frac{1}{4} \left[(1+g) + \frac{1}{1+g} \right] \right] , \quad (5.8)$$

where

$$g = \frac{1-r^2}{a} . \quad (5.9)$$

The asymptotic scale dimension (that we identify with

the scale dimension of the asymptotic state) is

$$d = \frac{1}{4} \left[(1+g) + \frac{1}{1+g} \right]. \quad (5.10)$$

Obviously this result is fully consistent with the analysis of the anomalous dimension of the propagator for $x^2 \rightarrow -\infty$ (2.32); using the notation of Sec. II we get

$$d = \frac{1}{2} + B.$$

We notice that for $g=0$ we recover the free spinor of the chiral Schwinger model: in a precise sense, that we discuss below, g describes a kind of asymptotic interaction.

The propagator (2.32) in the large x limit is the propagator of the massless Thirring model [13,14], in the spin $\frac{1}{2}$ representation. Actually for this model the spin labels the representation of the conformal group [15]. Our asymptotic state is a massless Thirring fermion: we can write

$$\psi_\alpha(x) =: \exp \left[-i\sqrt{\pi} \left(\frac{r}{a} + \frac{1}{a+1-r^2} \gamma_{\alpha\alpha}^5 \right) \sigma \right] : \hat{\psi}_\alpha(x), \quad (5.11)$$

with

$$\begin{aligned} \hat{\psi}_\alpha(x) &= C : \exp(-i\sqrt{\pi}[(1+g)\tilde{h} - \gamma_{\alpha\alpha}^5 h]) : (x), \\ C &= \left[\frac{\mu}{2\pi} \right]^{1/2} \left[\frac{\mu}{\tilde{m}} \right]^{g^2/a(1+g)}. \end{aligned} \quad (5.12)$$

It is not difficult to show that, from the operatorial point of view, $\hat{\psi}_\alpha$ is a solution of the massless Thirring model, namely, of the equation

$$i\gamma^\mu \partial_\mu \hat{\psi} = \hat{g} : \gamma^\mu \hat{J}_\mu \hat{\psi} : , \quad (5.13)$$

where we have defined [5]

$$\begin{aligned} \hat{J}^0 &= \lim_{\epsilon \rightarrow 0} Z(\epsilon) \{ J^0(x, \epsilon) - \langle 0 | J^0(x, \epsilon) | 0 \rangle \}, \\ \hat{J}^1 &= \lim_{\epsilon \rightarrow 0} Z(\epsilon) \frac{1}{1+g} \{ J^1(x, \epsilon) - \langle 0 | J^1(x, \epsilon) | 0 \rangle \}, \\ J_\pm(x, \epsilon) &= \psi^\dagger(x, \epsilon) (1 \pm \gamma^5) \psi(x), \\ Z(\epsilon) &= (-\tilde{m}^2 \epsilon^2)^{g^2/4(1+g)}, \end{aligned} \quad (5.14)$$

and

$$\hat{g} = \pi g = \pi \frac{1-r^2}{a}. \quad (5.15)$$

The coupling constant of this “effective” Thirring model depends on r and a : for $a > r^2$ we have a dynamical generation of the Thirring theory. One can also check directly that (5.12) is a Thirring fermion (spin $\frac{1}{2}$) looking at the Klaiber manifold [14]: Eq. (5.10) is the correct dimension for the spin $\frac{1}{2}$ solution.

We can now define the charge associated to this model

$$\begin{aligned} \hat{Q}_T &= \int dx^1 \hat{J}_0(x^1), \\ \hat{J}_\mu(x) &= -\frac{1}{2\sqrt{\pi}} \partial_\mu h(x). \end{aligned} \quad (5.16)$$

\hat{Q}_T is obviously conserved and it results

$$[\hat{Q}_T, \psi_\alpha(x)] = \psi_\alpha(x). \quad (5.17)$$

In other words the solution ψ_α carries the quantum number of a Thirring fermion.

The selection rules are obtained setting $\mu \rightarrow 0$ in the correlation function. This “thermodynamic limit” is essential in order to recover the symmetries of the original theory; for example, the naive definition

$$\langle 0 | \psi_\alpha | 0 \rangle = C \left[\frac{\mu}{2\pi} \right]^{1/2} \neq 0$$

suggests the spontaneous breaking of the U(1) rigid symmetry generated by \hat{Q}_T . The vacuum is not invariant under this transformation: only when $\mu \rightarrow 0$ we recover the correct invariance. This procedure leads to selection rules equivalent to Klaiber’s ones and ensure the positivity of the Hilbert space.

At this point we remark that all our constructions are justified, from a rigorous mathematical point of view, by the fact that we can make a Krein extension of the original massless boson Hilbert space in order to obtain a representation of the fermionic algebra solving the massless Thirring model [16]. Using this technique one can define the charge operator \hat{Q}_T and prove the existence of (5.12) in a strong operatorial sense.

The invariance of the vacuum, in this formalism, is not achieved by means of the *ad hoc* infrared limit $\mu \rightarrow 0$, but by a careful construction of the fermionic vacuum in the Krein topology: uniqueness is obtained modulo zero norm vectors (that are quotiented out).

The Hilbert space of our system seems to be the tensor product of the Hilbert space of a boson of mass $m^2 = \hat{e}^2(a-r^2)/[a(a+1-r^2)]$ and of a massless Thirring model; nevertheless the situation is more intriguing due to the presence of the operator $\psi(x)$ that interpolates between two extreme situations. We recall that for $x^2 \sim 0$, its behavior is characterized by the anomalous dimension (2.31) while the infrared limit is described by the Thirring theory.

We have two critical points corresponding to conformal field theories in the short- and long-distance limits: the noncritical theory has both massive and massless degrees of freedom.

The emerging theory, in the large x limit, is not chiral: chirality is in fact screened by the interaction, as the electric charge is. The short-distance behavior, on the contrary, strongly depends on chirality (as one can see from propagators). In our case we do not know what the ultraviolet theory is, if any. One would be tempted to think that the ultraviolet theory bears some relations to the axial-vector generalization of the Thirring model [16]: an easy computation of the critical exponents shows that this is not the case. In particular one realizes that the short-distance behavior of the fermion propagator is different from the result one would expect according to a naive power counting. While the origin of this phenomenon can be traced back to our functional integration, its deep physical meaning is still not complete-

ly understood. We leave this problem open to future investigations.

VI. CONCLUDING REMARKS

In conclusion we have thoroughly studied a vector-axial vector theory in two dimensions characterized by a parameter which interpolates between pure vector and chiral Schwinger models. The theory has been completely solved by means of nonperturbative techniques, both in a functional approach and in a canonical operatorial framework.

The main results are the presence of two ranges in the space of parameters in which acceptable solutions can be obtained. The first range is characterized by a massive and a massless free bosonic excitations and by fermions which are endowed with asymptotic states, which feel however a long-range interaction, but in the chiral case. The theory in the second range of parameters has a massive free boson and a massless ghost; fermions are confined as their correlators grow with distance. Nevertheless a Hilbert space of states can be consistently singled out.

The most attractive feature is present in the first region: in this situation fermionic correlators scale at short and long distances with different critical exponents. The infrared limit fully corresponds to a massless Thirring model times a free massless bosonic sector. Field,

charges, and Hilbert space of states do indeed coincide. The ultraviolet limit leads to a conformal invariant theory with a larger number of components (in agreement with Zamolodchikov's theorem [17]), whose Lagrangian formulation, if any, is so far unknown. These aspects of our model and, more generally, its relation to conformal invariant theories will be deferred to forthcoming work.

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APPENDIX A

In this appendix we show how to derive the left propagator (2.26) in the path-integral formalism; all the other Green functions can be obtained in the same way. The first step is to integrate the fermions in (2.4) to give (2.6) (we put $J_\mu=0$). The change of variables (2.25) decouples the spinors from A_μ but has a nontrivial Jacobian $\mathcal{J}[A_\mu]$:

$$\mathcal{J}[A_\mu] = \exp \left[\int d^2x \frac{e^2}{2\pi} A_\mu \left[(1+a)g^{\mu\nu} - (1+r^2) \frac{\partial^\mu \partial^\nu}{\square} - 2r\epsilon^{\alpha\mu} \frac{\partial_\alpha \partial^\nu}{\square} \right] A_\nu \right]. \quad (\text{A1})$$

This result can be obtained, in Euclidean space, using ζ -function technique for functional determinants [18]

$$\det[D] = \exp \left[-\frac{d}{ds} \zeta_D(s) \Big|_{s=0} \right], \quad \zeta_D(s) = \int d^{2n}x \text{Tr}[K_s(D;x,x)],$$

where $K_s(D;x,x)$ is the kernel of D^{-s} , D being a pseudoelliptic operator [19]. The knowledge of the relevant Seeley-DeWitt coefficients of D and a trivial (in two dimensions) integration of the Jacobian of the infinitesimal transformation [20], lead to the above expression.

The fermionic action is now

$$\int d^2x [i\bar{\chi}\partial\chi + \bar{\eta} \exp(i\epsilon[\alpha + \gamma^5\beta + r\beta + r\alpha\gamma^5])\chi + \bar{\chi} \exp(i\epsilon[-\alpha + \gamma^5\beta - r\beta + r\alpha\gamma^5])\eta], \quad (\text{A2})$$

where χ is a free fermion and α, β are linked by (2.24) to A_μ . The diagonalization of (A2) gives the propagator $S(x,y; A_\mu)$:

$$S(x,y; A_\mu) = S_0^L(x-y) \exp \left[i \int d^2z \xi_\mu^L(z;x,y) A^\mu(z) \right] + S_0^R(x-y) \exp \left[i \int d^2z \xi_\mu^R(z;x,y) A^\mu(z) \right], \quad (\text{A3})$$

$$\xi_\mu^{L,R}(z;x,y) = e(r \pm 1) (\partial_\mu^\pm \bar{\partial}_\mu^\pm) [D(z-x) - D(z-y)],$$

where $D(x)$ is the free massless scalar propagator in $d=1+1$ and S_0^L, S_0^R the free left and right fermion propagators.

To obtain the left propagator (2.26) we derive with respect to $\bar{\eta}_L$ and η_L [the left component of the sources (2.5)] and get

$$S^L(x,y) = S_0^L(x-y) \int \mathcal{D}A_\mu \mathcal{J}[A_\mu] \exp \left[i \int d^2z \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}(z) + \xi_\mu^L(z;x,y) A^\mu(z) \right] \right]. \quad (\text{A4})$$

Using the explicit form of $\mathcal{J}[A_\mu]$ [(A1)] we can write the path integral over A_μ as

$$\int \mathcal{D}A_\mu \exp \left[i \int d^2z \left[\xi_\mu^L A^\mu + \frac{1}{2} A_\mu K^{\mu\nu} A_\nu \right] \right], \quad (\text{A5})$$

$K^{\mu\nu}$ being defined in (2.9). The Gaussian integration is trivial and gives

$$S_L(x,y) = S_0^L(x-y) \exp \left[-\frac{1}{2} \int d^2z d^2w \xi_\mu^L(z;x,y) \{K^{-1}\}^{\mu\nu}(z,w) \xi_\nu^L(w;x,y) \right]. \quad (\text{A6})$$

The explicit computation of the exponential factor gives the renormalization constant Z_L and the interaction contribution in (2.26).

APPENDIX B

We want to investigate in the space of parameters a and r , the limiting situation $a=r^2$. The Poisson brackets (3.7) vanish; hence the request $\Omega_2=0$ implies a third constraint:

$$\Omega_3 \equiv -r\Pi_1 = 0. \quad (\text{B1})$$

We note that for $r=0$ we have no other constraint in addition to $\Omega_1=0$ and $\Omega_2=0$; they are first class and therefore the theory is gauge invariant.

Obviously $a=r=0$ corresponds to the vector Schwinger model. Taking $r \neq 0$, from $\dot{\Omega}_3=0$ we get

$$\Omega_4 \equiv r[\hat{\partial}(1+r^2)A_1 - r\partial_1\phi + \Pi_\phi + \hat{\partial}rA_0] = 0. \quad (\text{B2})$$

Now, since

$$\{\Omega_4(x^1), \Omega_1(y^1)\} = \hat{\partial}r^2\delta(x^1-y^1), \quad (\text{B3})$$

we have no further constraints. We end up with a system of four second-class constraints. Introducing Dirac brackets, we get the nonvanishing relations

$$\begin{aligned} \{A_0(x^1), A_1(y^1)\}_D &= \frac{1+r^2}{r^2\hat{\partial}^2} \partial_x^1 \delta(x^1-y^1), \\ \{A_1(x^1), \phi(y^1)\}_D &= \frac{1}{\hat{\partial}} \delta(x^1-y^1), \\ \{A_0(x^1), \phi(y^1)\}_D &= -\frac{r}{\hat{\partial}} \delta(x^1-y^1), \\ \{A_1(x^1), \Pi_\phi(y^1)\}_D &= \frac{1}{r\hat{\partial}} \partial_{x^1} \delta(x^1-y^1), \\ \{A_0(x^1), \Pi_\phi(y^1)\}_D &= -\frac{1}{\hat{\partial}r^2} \partial_{x^1} \delta(x^1-y^1), \\ \{A_1(x^1), A_1(y^1)\}_D &= -\frac{2}{r\hat{\partial}^2} \partial_{x^1} \delta(x^1-y^1), \\ \{\phi(x^1), \Pi_\phi(y^1)\}_D &= \delta(x^1-y^1). \end{aligned} \quad (\text{B4})$$

The variables ϕ and Π_ϕ have a canonical structure and we can express all the other variables through the constraints to get the Hamiltonian H_{red} :

$$H_{\text{red}} = \int dx^1 \left\{ \frac{r^2}{2} \Pi_\phi^2 + \frac{1}{2r^2} (\partial_1\phi)^2 \right\}.$$

The Heisenberg equations

$$\partial_0\phi = r^2\Pi_\phi, \quad \partial_0\Pi_\phi = \frac{1}{r^2}\partial_1^2\phi, \quad (\text{B5})$$

are equivalent to

$$\square\phi = 0. \quad (\text{B6})$$

The commutation relations are

$$\begin{aligned} [\phi(x), \dot{\phi}(y)]_{\text{ET}} &= ir^2\delta(x^1-y^1), \\ [\phi(x), \phi(y)]_{\text{ET}} &= 0. \end{aligned} \quad (\text{B7})$$

The vector potential is

$$A_\mu = \frac{1}{\hat{\partial}r} \left[\partial_\mu - \frac{1}{r} \tilde{\partial}_\mu \right] \phi, \quad (\text{B8})$$

giving

$$F_{\mu\nu} = 0, \quad (\text{B9})$$

which is consistent with $\Omega_3=0$. The only degree of freedom is a massless scalar excitation. We can construct the fermionic operator solving the Dirac equation in the same way as in Sec. IV; this time the current coupled to $F^{\mu\nu}$ is zero and the fermionic sector again describes a Thirring model. The absence of a massive component in the spectrum forces scale invariance for any x^2 : our model becomes totally equivalent to a massless Thirring model.

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