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Effect of wave function renormalization in N-flavor three-dimensional QED at finite temperature

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A recent study of dynamical chiral symmetry breaking in N flavor three-dimensional QED at finite temperature is extended to include the effect of fermion wave function renormalization in the Schwinger-Dyson equations. The simple "zero-frequency" truncation previously used is found to lead to unphysical results, especially as $T \to 0$. A modified set of equations is proposed, whose solutions behave in a way which is qualitatively similar to the $T=0$ solutions of Pennington and co-workers who have made extensive studies of the effect of wave function renormalization in this context, and who concluded that there was no critical N_c (at $T=0$) above which chiral symmetry was restored. In contrast, we find that our modified equations predict a critical N_c at $T \neq 0$, and an N-T phase diagram very similar to the earlier study neglecting wave function renormalization. The reason for the difference is traced to the different infrared behavior of the vacuum polarization at $T=0$ and at $T \neq 0$.

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I. INTRODUCTION

In a recent paper with collaborators [1], we studied dynamical chiral symmetry breaking in N-flavor three-dimensional QED (QED_3) at finite temperature, in the large N approximation. Using an approximate treatment of the Schwinger-Dyson equation for the fermion self-energy, we found that chiral symmetry was restored above a certain critical temperature, which itself depended on N. The ratio r of twice the zero-momentum and zero-temperature fermion mass to the critical temperature turned out to have a value of about 10 (approximately independent of N), which is considerably larger than a typical BCS value, but consistent with previous work [2] using a momentum-independent self-energy. We found evidence for a temperature-dependent critical number of flavors, above which chiral symmetry was restored, and the N-T phase diagram for spontaneous mass generation in the theory was presented. A more extensive account of some of the relevant details, and of the possible relevance to high T_c superconductivity, is contained in [3].

The question of the existence, or not, of a critical N in analogous calculations at zero temperature is still to some extent controversial. The original calculations of Appelquist and co-workers [4] found chiral symmetry breaking only for $N < N_c$ where $N_c = 32/\pi^2$, but this work made a possibly crucial appeal to perturbation theory (in $1/N$) to justify the neglect of wave function renormalization and the use of a simple bare vertex. This step has been strongly criticized by Pennington *et al.* [5–8], who in a series of papers, using Schwinger-Dyson equations with increasingly elaborate nonperturbative vertices satisfying the Ward and Ward-Takahashi identities, have found no evidence for any N_c—rather, the fermion mass simply decreases exponentially with N. On the other hand, an alternative nonperturbative study by Atkinson *et al.* [9] suggested that chiral symmetry is restored at large enough N; but this paper is also criticized in recent work of Curtis *et al.* [8]. Finally, Kondo and Nakatani [10] have examined the effect of imposing an infrared cutoff on the Schwinger-Dyson equations, including wave function renormalization and various *Ansätze* for the vertex. For the Pennington-Webb [5] vertex, which we also shall use, Kondo and Nakatani [10] obtained a (cutoff-dependent) N_c for the case of a "large" infrared cutoff, and $N_c \to \infty$ for the case of a "small" cutoff.

Our previously mentioned calculations at finite temperature [1] made precisely the same perturbative approximation as Appelquist *et al.* [4], by neglecting wave function renormalization and using the bare vertex. The arguments and results of Pennington *et al.* [5–8] certainly provide strong motivation to go beyond that approximation, and investigate whether the conclusions of our finite-temperature study survive a better treatment of the vertex. The present paper reports the results of such an investigation, using a simplified (but not perturbative) form of the vertex, but otherwise following [1] as closely as possible. In brief, we find that the conclusions of [1] are, in fact, essentially unchanged, though the formulation of a simple extension of the formalism of [1] to include wave function renormalization turns out to be not completely straightforward. It seems that the crucial ingredient in obtaining a finite N_c is the softening of the infrared behavior at finite temperature. Our calculations therefore suggest a natural physical interpretation of the "large" infrared cutoff regime introduced phenomenologically (at $T=0$) by Kondo and Nakatani [10].

II. SIMPLE SCHWINGER-DYSON EQUATIONS AT $T \neq 0$

We shall choose a simplified *Ansatz* for the vertex, and we begin by introducing that choice within the context of the zero-temperature Schwinger-Dyson (SD) equations; then we shall pass to the finite-temperature case. In Eu-

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clidean space, the SD equation for the fermion propagator is

$$S_F^{-1}(p)=S_F^{(0)-1}(p)-e^2\int\frac{d^3k}{(2\pi)^3}\gamma^\mu S_F(k)\Delta_{\mu\nu}(q)\Gamma^\nu(k,p) \quad (1)$$

where the superscript 0 denotes the bare quantity, $S_F^{-1}(p)=[1+A(p)]\not{p}+\Sigma(p)$ is the inverse fermion propagator, and $q=k-p$. We shall not make any change here in our previous (and common) choice for $\Delta_{\mu\nu}$—namely, it is approximated by the sum of massless fermion bubbles. We shall also continue to work in Landau gauge. Turning then to Γ^ν, the Ward-Takahashi identity may be written as

$$q_\nu\Gamma^\nu(k,p)=S_F^{-1}(k)-S_F^{-1}(p)\ . \quad (2)$$

Taking the limit $q_\nu\to 0$ yields the Ward identity

$$\frac{\partial S_F^{-1}(p)}{\partial p_\nu}=\Gamma^\nu(p,p) \quad (3)$$

which ensures that the full vertex is free of kinematic singularities. Both (2) and (3) should hold to all orders of perturbation theory.

Ball and Chiu [11] have given a vertex which satisfies both these relations, but has an unconstrained transverse part (which, however, is believed to be unimportant in the infrared region [11,5–7]). In fact, a more general vertex including not only the full Ball-Chiu vertex but also a nontrivial transverse part has recently been studied by Curtis *et al.* [8], at zero temperature. This vertex satisfies (2) and (3) and also correctly reproduces the leading asymptotic behavior known from perturbation theory. For our purposes, however, the salient fact is that the results (as to dynamical mass generation, N_c, etc.) of this most recent work are in qualitative agreement with earlier studies [5,6] where a much simpler choice of vertex was made [albeit one no longer satisfying (2) and (3)]: namely,

$$\Gamma^\nu(k,p)=\gamma^\nu[1+A(k)]\ . \quad (4)$$

Similar approximations in four dimensions have been shown to give results which are in fairly good agreement with those obtained using the full vertex. It seems reasonable to hope that such agreement will persist in our finite-temperature case, and we shall therefore now adopt the Pennington-Webb [5] vertex (4).

Inserting (4) into (1) and separating the scalar and spinor parts we find [see also Eqs. (2.35) and (2.36) of [3]]

$$\mathcal{M}(p)=\frac{\alpha}{4\pi^3N}\frac{1}{1+A(p)}\int d^3k\frac{1}{q^2+\Pi(q)}\frac{\mathcal{M}(k)}{k^2+\mathcal{M}^2(k)} \quad (5)$$

and

$$A(p)=\frac{\alpha}{16\pi^3Np^2}\int d^3k\frac{(p^2-k^2)^2-(q^2)^2}{q^2[q^2+\Pi(q)]}\frac{1}{k^2+\mathcal{M}^2(k)} \quad (6)$$

where $\alpha\equiv e^2N$ is understood to be fixed as N varies, Π is the vacuum polarization, and we have introduced the mass function

$$\mathcal{M}(p)\equiv\frac{\Sigma(p)}{1+A(p)}\ . \quad (7)$$

The explicit appearance of the $1/N$ factor in (6) is, of course, the reason for the claim that to leading order in $1/N$ wave function renormalization can be ignored and A set to zero. But Pennington and co-workers have convincingly argued [5–8] that, at least at the low momenta relevant to dynamical mass generation, the integral can provide a compensating N dependence (as after all happens for Σ or $\mathcal{M}$), so that A is by no means of order $1/N$. These authors [5] have obtained numerical solutions to equations equivalent to (5) and (6), which show no sign of a critical N_c. We follow them in retaining (6), and pass now to the finite-temperature case, following the usual prescriptions. As in our previous work [1] (see also [3]) we define

$$\begin{aligned}&p=(P_0,\mathbf{p})\ ,\quad P=|\mathbf{p}|\ ,\quad p_0=(2m+1)\pi/\beta\quad(\beta=1/k_BT)\ ,\\&k=(k_0,\mathbf{k})\ ,\quad K=|\mathbf{k}|\ ,\quad k_0=(2n+1)\pi/\beta\ ,\\&q=(q_0,\mathbf{q})\ ,\quad Q=|\mathbf{q}|=|\mathbf{k}-\mathbf{p}|\ ,\quad q_0=2(n-m)\pi/\beta\ .\end{aligned} \quad (8)$$

Integrals over the temporal component of a fermion loop momentum are replaced by infinite sums over odd Matsubara frequencies, while bosonic loops are evaluated by summing over even frequencies. The vacuum polarization and the fermion functions A and Σ become functions of (the modulus of) the momentum, and of the temperature, and acquire a discrete index n corresponding to the Matsubara temporal component. Since we are here concerned with the effect of introducing the nonperturbative vertex (4), we shall follow [1] and [3] in retaining only the $\mu=\nu=0$ component of the photon propagator $\Delta_{\mu\nu}$, and ignore all but the zero-frequency ($n=0$) component, so that

$$\Delta_{\mu\nu}(q_0,Q,\beta)=\delta_{\mu0}\delta_{\nu0}/[Q^2+\Pi_0(Q,\beta)]\ , \quad (9)$$

where, to an excellent approximation [1],

$$\Pi_0(Q,\beta)=\frac{\alpha}{8\beta}\left[Q\beta+\frac{16\ln 2}{\pi}\exp\left[\frac{-\pi}{16\ln 2}Q\beta\right]\right]\ . \quad (10)$$

In the same spirit, we shall also ignore the frequency dependence of the kinematic factors in (6). In this zero-frequency limit, then, both A and Σ become independent of the Matsubara frequency index, and the sums over these indices in (5) and (6) can be performed explicitly so as to yield the following simple equations for the temperature-dependent mass function $\mathcal{M}(P,\beta)$ and wave function renormalization $A(P,\beta)$:

$$\mathcal{M}(P,\beta)=\frac{\alpha}{8N\pi^2}\frac{1}{1+A(P,\beta)}\times\int d^2\mathbf{k}\frac{\mathcal{M}(K,\beta)}{Q^2+\Pi_0(Q,\beta)}\frac{\tanh\frac{\beta}{2}\sqrt{K^2+\mathcal{M}^2(K,\beta)}}{\sqrt{K^2+\mathcal{M}^2(K,\beta)}} \quad (11)$$

and

$$A(P,\beta)=\frac{\alpha}{16N\pi^2}\int d^2\mathbf{k}\,\frac{(P^2-K^2)^2-Q^4}{P^2Q^2[Q^2+\Pi_0(Q,\beta)]}\times\frac{\tanh\frac{\beta}{2}\sqrt{K^2+\mathcal{M}^2(K,\beta)}}{\sqrt{K^2+\mathcal{M}^2(K,\beta)}}\ . \tag{12}$$

With regard to Eqs. (11) and (12), we note that the first (for $\mathcal{M}$) is the same as that for Σ in Ref. [1], except for the coefficient in front of the integral which acquires a factor $[1+A(P,\beta)]^{-1}$ in the present case. In [1], of course, A was set to zero. We also note that A does not appear under the integral in (12), but is given by a simple integral involving $\mathcal{M}$. This latter feature is a consequence of the ansatz (4); in general we would have had to deal with two coupled integral equations for $\mathcal{M}$ and A.

It would seem that all that now remains is to solve Eqs. (11) and (12) numerically. However, it turns out that there is an unsuspected problem (or so we regard it) with Eq. (12), as we now explain.

III. PROBLEM WITH THE SIGN OF A IN THE SOLUTION OF (11) AND (12)

It is a standard result in zero-temperature field theory (see for example [12]) that the complete wave function renormalization factor $Z=1+A$ satisfies the relation

$$0\le Z<1\ , \tag{13}$$

which implies

$$-1\le A<0\ . \tag{14}$$

Indeed, the angular integral in the zero-temperature expression (6) for A was evaluated analytically in [5] and found to be negative definite (which clearly implies $A<0$), and the corresponding numerically evaluated Z satisfied (13). Unfortunately, the same results do not hold for our approximation expression (12) for A at finite temperature, as we shall now see.

Consider the angular integral in (12), namely

$$I(P,K,\beta)=\frac{K}{\alpha}\int_0^{2\pi}d\phi\,\frac{(P^2-K^2)^2-Q^4}{P^2Q^2[Q^2+\Pi_0(Q,\beta)]}\ , \tag{15}$$

where $Q^2=P^2+K^2-2PK\cos\phi$, and we have included the (dimensionless) factor K/α coming from the two-dimensional phase space $d^2\mathbf{k}$. As it is not possible to evaluate I analytically, we have had to resort to numerical evaluation. Clearly it is difficult to give a complete picture of I as a function of all three variables P, K, and β, but it turns out that for wide ranges of these variables I is predominantly *positive,* although it is negative in the region around $K\approx P$. These features are illustrated in Fig. 1, which shows I versus K/α for two values of P (one "large" compared to the natural scale α, the other small), and two values of (inverse) temperature β. It is clear from (15) that I is always negative at the point $K=P$, and that the quantity Π_0 acts as a kind of regulator for the $1/Q^2$ singularity associated with the photon propagator. In particular, from Fig. 1 we see that the width of the region where I is negative decreases with increasing β (decreasing T). These latter details are understandable from

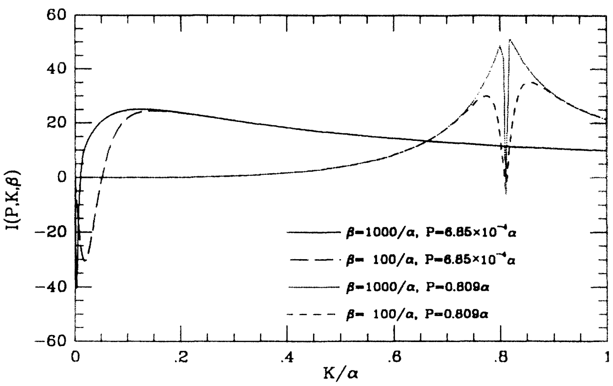

FIG. 1. Angular integral $I(P,K,\beta)$ at $P=6.85\times10^{-4}\alpha$, 0.809α and $\beta=1000/\alpha, 100/\alpha$.

the form of Π_0, Eq. (10), which has a temperature-independent linear term ($\alpha Q/8$) and another which depends on temperature as $(\alpha C/8\beta)\exp(-\beta Q/C)$ where $C=(16\ln2)/\pi$.

It is of course difficult to read off from Fig. 1 what the eventual sign of A in (12) will be, since much might depend on the relative weighting attached to the region $K\approx P$, and to the remainder, in the K integration of (12). Nevertheless, it does seem likely that, especially at the low temperatures characteristic of dynamical mass generation, the relative insignificance of the negative parts of I in Fig. 1 will imply that A in (12) turns out to be positive, contradicting (14). This is indeed the case.

We have solved (11) and (12) by an iterative procedure, as follows. In zeroth order, we took $A^{(0)}=0$ and $\mathcal{M}^{(0)}=\Sigma$, the solution of Eq. (8) of our previous paper [1] [which is the same as (11) with $A=0$]. Inserting $\mathcal{M}^{(0)}$ into the right-hand side of (12) gave the first iterate $A^{(1)}$. This was then substituted into the coefficient in front of the integral in (11), and the latter evaluated using $\mathcal{M}^{(0)}$ as the input function, so as to yield the first iterate $\mathcal{M}^{(1)}$. This was substituted back into (12) to give $A^{(2)}$, and so on. The procedure was continued until convergence to within 2% was achieved. Note that, as in [1], we work with a momentum cutoff at $\Lambda=\alpha$, and scale all momenta, temperatures and masses by α.

For the sake of illustration we consider the case $N=1$. We have obtained stable and converged solutions to Eqs. (11) and (12), with properties we now describe. First, the behavior of $\mathcal{M}(P,\beta)$ is qualitatively similar to that of $\Sigma(P,\beta)$ found in [1]—namely, $\mathcal{M}$ is constant for $P/\alpha\lesssim10^{-2}$, and falls rapidly to zero for larger P/α. Further, the zero-momentum value of $\mathcal{M}$ starts to fall rapidly when $\beta\alpha$ goes from 2000 to 1000, indicating the possibility of a finite critical temperature. Indeed a plot of $\mathcal{M}(0,T)/\alpha$ versus k_BT/α suggests a critical temperature of order $k_BT_c\sim10^{-3}\alpha$, quite similar to the T_c's found in [1]. However, all these results involve an A which is greater than zero, and hence a Z violating (13). Figure 2 shows the corresponding solution of (12) for $N=1$ as a function of momentum, for different temperatures. The behavior of the full $Z=1+A$ is quite different from that found by Pennington and Webb [5]

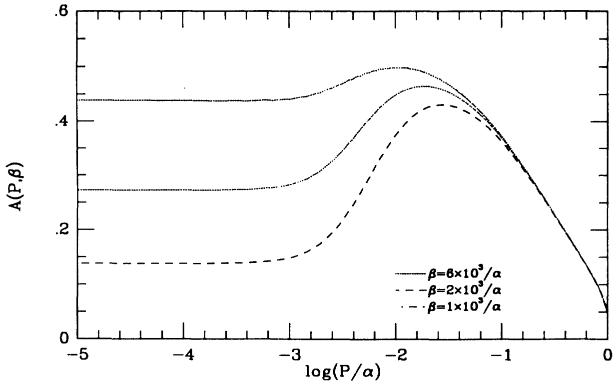

FIG. 2. The wave function renormalization as a function of scaled momenta, at various (scaled) temperatures, for $N=1$. The logarithm is to base e.

from the zero-temperature equation (6), which, as mentioned above, always satisfies (14).

We might wonder whether for very low temperatures our solution to (12) goes negative, but this does not happen. Instead, as the temperature decreases the value of $A(0,\beta)$ rises, approaching a value of approximately 0.6. This behavior bears no resemblance to the zero-temperature result of Pennington and Webb [5].

It is certainly possible that the condition (14), which relies on unitarity in Minkowski space, may not necessarily hold in the Euclidean space appropriate to $T\neq 0$. There may, for example, be "heat-bath" creation processes for temperatures above the pair-creation threshold which would cause a violation of (14). But we shall take the view here that the regime relevant to dynamical mass generation is definitely a low temperature one ($k_B T \ll \alpha$), and that consequently we should hope to find an A which is negative and qualitatively similar to that of [5]. After all, the essential aim of the present work is to examine the effect of a reasonable temperature-dependent extension of [5] on the existence or otherwise of a critical N_c. We therefore reject the solutions of (11) and (12) described above, and seek a modified Eq. (12) which will give a temperature-dependent A satisfying (14), for the (low) temperature with which we are concerned.

IV. MODIFICATION OF THE "A" EQUATION TO SECURE $-1\leq A<0$ AT $T\neq 0$

We want to understand why the zero-temperature expression (6) for A satisfies (14), while our approximate finite-temperature version of it gives $A>0$. We believe the answer lies in our dropping all but the zero frequency components in Π, Σ, and A. First, it is clear that if all components are kept, the finite-temperature equations must correctly reduce to the zero-temperature ones as $T\to 0$. More particularly, by retaining only the zero component we have effectively lost a dimension [compare (12) with (6)], and this is crucial for the following reason. Suppose we consider the zero-temperature limit of (15), in which $\Pi_0\to\alpha Q/8$. In this case the integrand of (15) is dominated by very large positive values associated with the regions close to $\phi=0$ and 2π, whereas for intermediate values of ϕ the integrand is negative but very much smaller. This is why, as noted earlier, I is predominantly positive. However, if we were simply to multiply the integrand of (15) by $\sin\phi$, so as to mimic the three-dimensional phase space at zero temperature, we would effectively eliminate the unwanted large positive contribution, and enhance the negative ones. In fact, we have checked that introducing such a factor by hand in (15), and integrating ϕ from 0 to π, renders I negative for all P and K, and hence ensures $A<0$. Hence we believe that a proper "reconstruction" of the full phase space, at least near $T=0$, would succeed in changing the sign of A as desired.

Unfortunately, it seems a formidable task to attack the fully coupled equations, including all frequency components. Instead, we shall seek here a simple modification of (15), which behaves qualitatively in as similar a way as possible to the zero-temperature kernel in (6) but is temperature dependent. In this way we hope at least to model the effect which the Pennington-Webb vertex (4) would have in a more realistic frequency-dependent calculation.

The modification we propose is motivated in part by the notion that the approximation of retaining only the zero-frequency components is best justified at high temperatures. In this (small β) limit, the quantity Π_0 in (10) reduces to the (temperature-dependent) value $(2\alpha\ln 2/\pi\beta)$, but of course we are really interested in low temperatures. However, dynamical mass generation is a low-momentum phenomenon, and furthermore the integrands in (5) or (6) are certainly enhanced for $q\approx 0$, so that we may hope that it may be reasonable to try a "small βQ" approximation, rather than simply a "small β" one. In the small βQ limit, Π_0 of (10) reduces again to $(2\alpha\ln 2/\pi\beta)$, plus $O(\beta Q)^2$ corrections. These considerations lead us to explore the result of replacing Π_0 in (12) by a constant, Δ^2 say, which we will take to be an adjustable parameter with a value of order α^2 or less. We will choose it so as to obtain a wave function renormalization as much like Ref. [5] as possible.

In replacing Π_0 in (12) by Δ^2 we are of course also altering the $Q\to 0$ behavior of the kernel, making it less singular. To "compensate" for this we might think of dropping the K in the phase-space factor, at the same time as replacing Π_0 by Δ^2. To explore these possibilities we shall study (a) the "exact" angular integral (with the K factor included) of (15), called $I(P,K,\beta)$, (b) the modified kernel with $\Pi_0\to\Delta^2$ and no K factor, namely,

$$I_\Delta(P,K)=\int_0^{2\pi}\frac{(P^2-K^2)^2-Q^4}{P^2Q^2(Q^2+\Delta^2)}d\phi\ , \tag{16}$$

and (c) the kernel with $\Pi_0\to\Delta^2$ but retaining the K factor, i.e., the quantity $K\times I_\Delta$. Of these we shall prefer the one which, on the one hand, best captures the low-momentum behavior of $I(P,K,\beta)$ (since this is the relevant region for dynamical mass generation), and on the other gives a wave function which most resembles the qualitative behavior of the $T=0$ results of Ref. [5].

It turns out that I_Δ can be integrated analytically, with the result

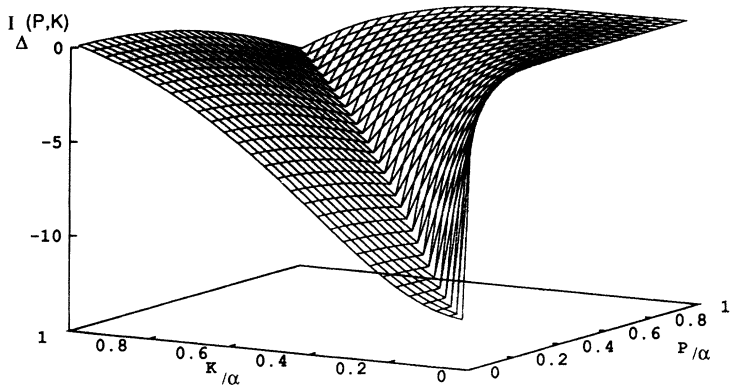

FIG. 3. Angular integral $I_\Delta(P,\lambda)$ for $\Delta^2=0.3$.

$$I_\Delta=\frac{-2\pi}{P^2}\left[1-\frac{|P^2-K^2|}{\Delta^2}+\frac{[(P^2-K^2)+\Delta^2][(P^2-K^2)-\Delta^2]}{\Delta^2\{[(P-K)^2+\Delta^2][(P+K)^2+\Delta^2]\}^{1/2}}\right]. \tag{17}$$

Figure 3 shows a plot of (17) as a function of P and K (scaled by α), for the case $\Delta^2=\alpha^2$.

We note that, most importantly, I_Δ is almost always *negative* [in contrast with I of (15)] except for a small region of large K and P values, which is not the important region for dynamical mass generation. In fact, the behavior of I_Δ is similar to that of an analogous integral which arose in an approximate treatment [6] of the three-dimensional zero-temperature equation (6): namely, a rapid variation with K for $K<P$, and a much slower variation for $K>P$, the sign being (in [6]) negative throughout. These features are true of I_Δ for a substantial range of Δ^2, and ensure $A<0$ as we shall see in the next section.

Figure 4 shows a comparison of I, I_Δ and $K\times I_\Delta$ for $P/\alpha=6.851\times10^{-4}$ and $\Delta^2=\alpha^2$. It is remarkable how well the low K behavior of I is captured by I_Δ, and how poorly $K\times I_\Delta$ performs. The large positive contribution in I for larger values of K is, as noted above, an undesirable feature and the reason for obtaining $A>0$ in Sec. III. On the other hand, if too small a value of Δ^2 is chosen (for example Δ^2 of order $0.1\alpha^2$) we find that I_Δ becomes too large and negative, with the consequence that the corresponding A begins to approach -1. From the appearance of the factor $(1+A)^{-1}$ in (11) we expect that such A's will be associated with too large a value of $\mathcal{M}$, contradicting the basic assumption $\mathcal{M}\ll\alpha$. The above considerations already strongly suggest that I_Δ will be the most satisfactory kernel at $T\neq0$, and we now proceed to discuss the results of using this kernel in place of I in (12); we shall also briefly describe the results of using $K\times I_\Delta$.

V. RESULTS USING A MODIFIED "A" EQUATION

We have solved the original Eq. (11), together with one or the other of the following modified equations for A:

$$A_\Delta(P,\beta)=\frac{\alpha}{16N\pi^2}\int_0^\alpha dK I_\Delta(P,K)\times\frac{\tanh\frac{\beta}{2}\sqrt{K^2+\mathcal{M}^2(K,\beta)}}{\sqrt{K^2+\mathcal{M}^2(K,\beta)}}, \tag{18}$$

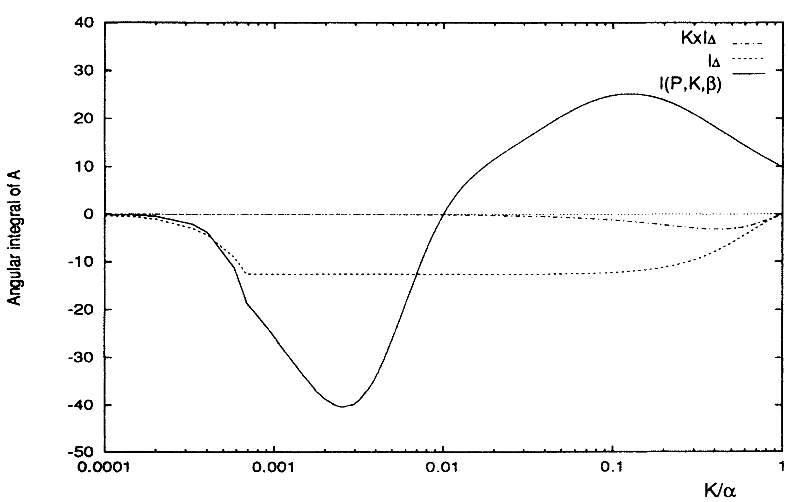

FIG. 4. Comparison of I, I_Δ and $K\times I_\Delta$ for $P/\alpha=6.851\times10^{-4}$ and $\Delta^2=\alpha^2$.

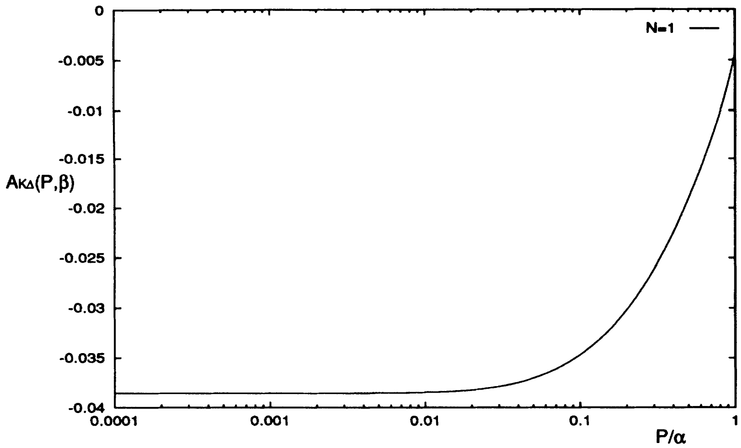

FIG. 5. $A_{K\Delta}(P,\beta)$ versus P/α for $\Delta^2=\alpha^2$ and $N=1$.

$$A_{K\Delta}(P,\beta)=\frac{\alpha}{16N\pi^2}\int_0^\alpha dKKI_\Delta(P,K)\times\frac{\tanh\frac{\beta}{2}\sqrt{K^2+\mathcal{M}^2(K,\beta)}}{\sqrt{K^2+\mathcal{M}^2(K,\beta)}}\,, \tag{19}$$

with I_Δ given by Eq. (17). We followed the same iterative procedure as described in Sec. III, obtaining stable and converged solutions for A and Σ. The results using $A_{K\Delta}$ can be quickly summarized. Figure 5 shows $A_{K\Delta}(P,\beta)$ versus P/α for $\Delta^2=\alpha^2$ and $N=1$; we note that $A_{K\Delta}$ is negative, as required, but very small in magnitude. Clearly this is a consequence of the very small (and negative) value of $K\times I_\Delta$ as seen in Fig. 4. Figure 6 shows the corresponding $\mathcal{M}(0,T)/\alpha$ versus k_BT/α, for $N=1$. This figure is very similar to the $A=0, N=1$ case shown in Fig. 3 of Ref. [1], as would be expected from the small value of $A_{K\Delta}$ shown in Fig. 5. In particular, the ratio $r=2\mathcal{M}(0,0)/k_BT_c$ remains close to 10. Qualitatively, the effect of the small negative $I_{K\Delta}$ is to produce a small upward shift of the $\mathcal{M}(0,T)/\alpha$ curve, relative to the $A=0$ case; this is easily understood as being associated with the $(1+A)^{-1}$ factor in (11). We have obtained the corresponding phase diagram using $I_{K\Delta}$, but it differs only rather slightly, and predictably, from Fig. 7 of Ref. [1].

The results using I_Δ are more interesting. For $\Delta^2\lesssim 0.1\alpha^2$, we found that A_Δ was positive for a small range of P, near $P\approx\alpha$; otherwise, $A_\Delta(P)$ was negative as required. More seriously, for this value of Δ^2 we found that $\mathcal{M}(P)$ was only about one order of magnitude smaller than α, so that the condition $\mathcal{M}\ll\alpha$ characteristic of dynamical mass generation in QED_3 (and of [5]) was no longer true (implying that the approximation of Π by the *massless* bubble would need reexamination). For $\Delta^2=\alpha^2$, however, we did obtain values for $\mathcal{M}$ and A_Δ which were in much closer qualitative agreement with the zero-temperature results, and we now concentrate on the results for this value of Δ^2.

In Fig. 7 we show $\mathcal{M}(P,\beta)$ as a function of momentum, for $N=1$ and different β's, while Fig. 8 shows the corresponding A_Δ's, all satisfying (14). These A_Δ's are all very reasonable-looking, reminiscent of those at $T=0$ [5,6]. The rapid decrease of $\mathcal{M}$ as β decreases from $\beta=500/\alpha$ to $130/\alpha$ suggests the existence of a critical temperature near $10^{-2}\alpha$. In Fig. 9 we show $\mathcal{M}(0,T)$ as a function of temperature, from which it appears that the critical temperature T_c is such that $k_BT_c\approx 7.8\times 10^{-3}\alpha$. Figures 7 and 9 can be compared with Figs. 2 and 3 of [1]; the conclusion is that the effect of including wave function renormalization in the way described is to increase both the mass $\mathcal{M}$ and the critical temperature T_c by roughly a factor of 2; again, this can be qualitatively

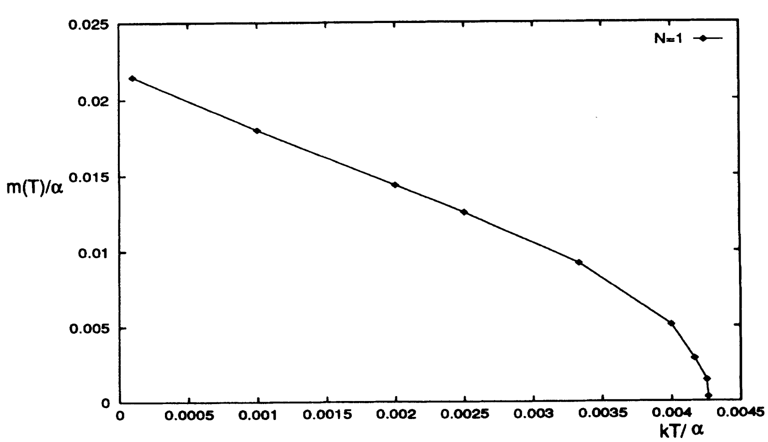

FIG. 6. The scaled self-energy $m(T)/\alpha$ [$\equiv\mathcal{M}(0,T)/\alpha$] as a function of scaled temperature for $N=1$ and $\Delta^2=\alpha^2$ using $A_{K\Delta}$.

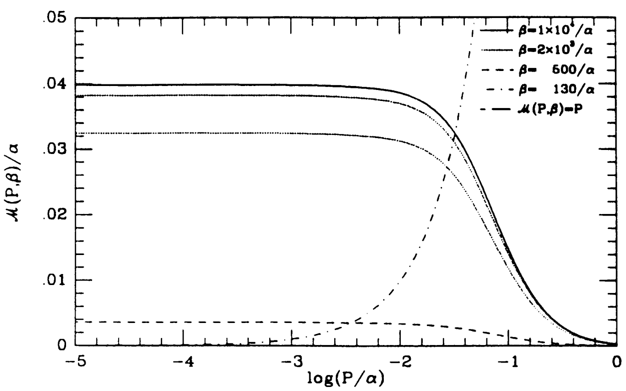

FIG. 7. The scaled self-energy as a function of scaled momentum for $N=1$ and $\beta\alpha=10^4$, 2000, 500, and 130. The parameter Δ^2 is fixed at the value $\Delta^2=\alpha^2$. The logarithm is to base e.

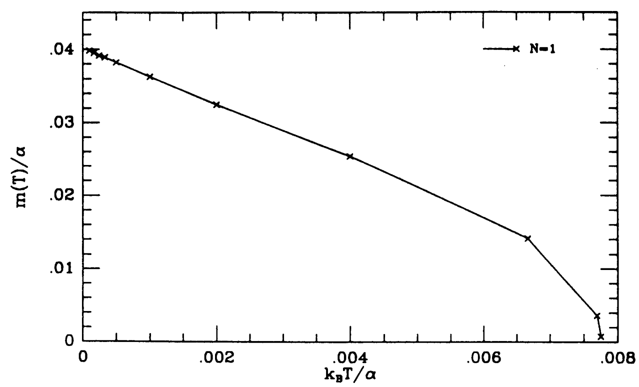

FIG. 9. The scaled self-energy $m(T)/\alpha$ [$\equiv \mathcal{M}(0,T)/\alpha$] as a function of scaled temperature for $N=1$ and $\Delta^2=\alpha^2$ using A_Δ.

understood as a consequence of the $(1+A)^{-1}$ factor in (11). Indeed, the ratio r [$=2\mathcal{M}(0,0)/k_B T_c$] comes out as 10.27, almost identical to the value for $N=1$ which we quoted in [1] without using wave function renormalization.

Figure 8 shows that the dependence of A_Δ on β is not very strong, disappearing altogether at the high-momentum end. In Fig. 10 we show A_Δ at $P=0$, as a function of T [note that (18) actually has nontrivial solutions even for $\mathcal{M}=0$, so that we can investigate the region $T>T_c$ for A_Δ]. It can be shown easily that $A_\Delta(P=0,T)\to 0$ as $T\to\infty$.

Finally, we come to the main question we originally set out to answer—the existence of a critical N_c, or otherwise. Figure 11 shows $\mathcal{M}(0,\beta)$ versus N, for various fixed temperatures. As in our previous work [1], we are not able to obtain reliable results for $\mathcal{M}/\alpha$ much below 10^{-5}. Nonetheless, as before, it seems reasonable to infer that $\mathcal{M}$ does vanish beyond some finite N_c which itself depends on T, increasing as T decreases. We can extrapolate the $\mathcal{M}/\alpha$ curves to find the critical values $N_c(T)$ for the various temperatures, and thus obtain the phase diagram shown in Fig. 12, and which is very similar to the one we obtained previously with $A=0$ [1].

Comparing Fig. 11 with Fig. 5 of [1], we notice that for a given temperature, the critical number of flavors has increased in the case where $A\neq 0$: for example, at $\beta=10^4/\alpha$, N_c was just above 1.8 in the $A=0$ case, whereas it now has a value slightly above 2.2. Although probably too much should not be made of such a relatively small difference, it is in fact simple to understand it, following an argument of Pennington and Webb [5]. Comparing our (11) above with (8) of [1], we can see that a measure of the effect of $A\neq 0$ can be obtained by replacing N in the $A=0$ case by $N(1+\langle A\rangle)$ where $\langle A\rangle$ is an average value for A. Hence a solution of the $A=0$ equations for a given $N_{A=0}$ will effectively correspond to a solution of the $A\neq 0$ equations with the identification $N_{A\neq 0}(1+\langle A\rangle)=N_{A=0}$. Since $\langle A\rangle<0$, the critical number of flavors for a given T increases in going to the $A\neq 0$ case.

Figure 13 shows A_Δ as a function of P for various values of N, at fixed T. Although we have not been able to explore as wide a range of N's as Pennington and Walsh [7], this figure is qualitatively similar to their Fig. 3, for $N\approx 2$, which encourages us to think that our modified A_Δ equation, (18), is physically reasonable.

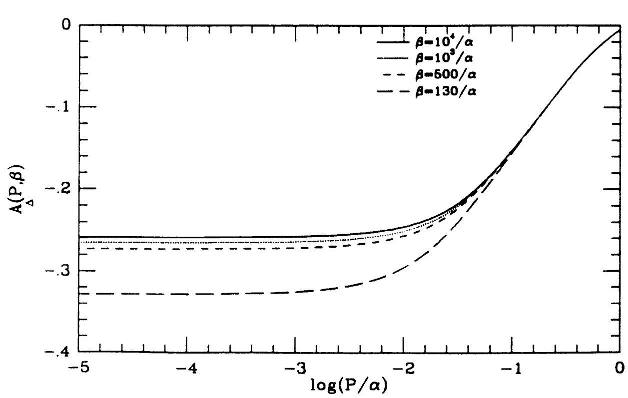

FIG. 8. $A_\Delta(P,\beta)$ as a function of scaled momentum for fixed $N=1$ and $\beta\alpha=10^4$, 2000, 500, and 130, with $\Delta^2=\alpha^2$. The logarithm is to base e.

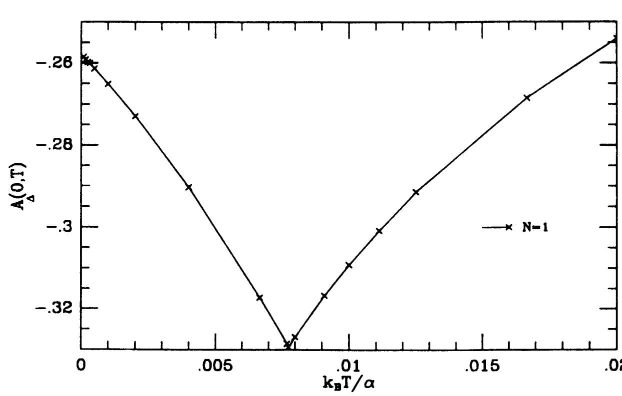

FIG. 10. $A_\Delta(0,T)$ as a function of scaled temperature for $N=1$ and $\Delta^2=\alpha^2$.

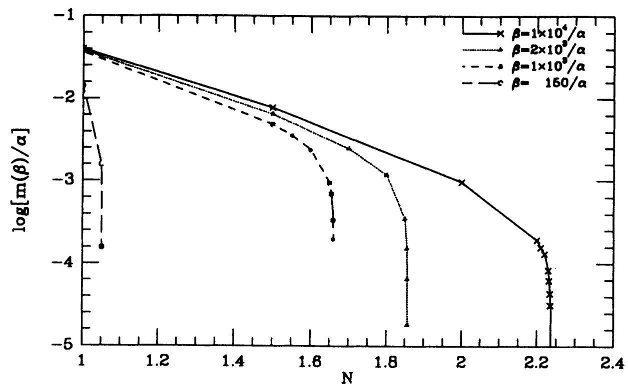

FIG. 11. The scaled self-energy $m(\beta)/\alpha$ $[\equiv \mathcal{M}(0,\beta)/\alpha]$ as a function of N at various scaled temperatures.

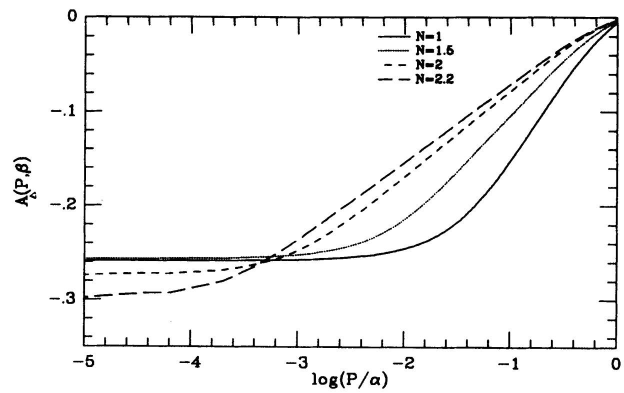

FIG. 13. The wave function renormalization $A_\Delta(P,\beta)$ as a function of scaled momenta, for $N=1$, 1.5, 2, 2.2 at $\beta\alpha=10^4$. The logarithm is to base e.

However, it is interesting to note that in the zero-temperature results of [5] and [7], $A(P=0)$ seems to approach -1 with increasing N, suggesting that the corresponding $N_{\rm eff}=(N_{A=0})/(1+\langle A\rangle)$ becomes so large that any sign of criticality in N (which might have been present in the $A=0$ case) disappears. Even for smaller N's, where $\langle A\rangle$ is considerably different from -1, the results of [5–8] do not show any indication of a sudden decrease in the dynamical mass at some critical N, such as is seen in our Fig. 11.

Nevertheless, it should be stressed that we are not able to obtain reliable numerical results for temperatures below $\sim 10^{-4}\alpha$, so that we are not able to say what the precise zero-temperature limit of our $N_c(T)$ might be, if indeed it exists at all. In addition, we must not forget the inexact nature of the "Δ^2 modification." In short, it is quite possible that $N_c(T)\to\infty$ as $T\to 0$, which would be in agreement with the conclusion of Pennington *et al.* [5–8].

If this suggestion is correct, we need to understand why our nonzero temperature results still point so clearly to a critical N, even though we have included a nonzero A. The answer seems surely to lie in the important alteration which the finite-temperature vacuum polarization Π_0 makes to the infrared regime of the fermion self-energy. As we have discussed earlier, the $Q\to 0$ limit of Π_0 in Eq. (10) is a finite temperature-dependent term $2\alpha\ln 2/\pi\beta$, a phenomenon called "thermal screening." The "Δ^2 modification" we used in Eq. (18) for A_Δ also captures this softening, for small $Q\beta$. By contrast, the zero-temperature Π of Pennington *et al.* [5–8] (inherited from Pisarski [13]) behaves as αQ for small Q and has no infrared softening. We believe that it is this infrared screening, associated with the temperature-dependent Π_0, which is crucial for the existence of an $N_c(T)$, just as it was found to be vital to the generation of a large value for r [1].

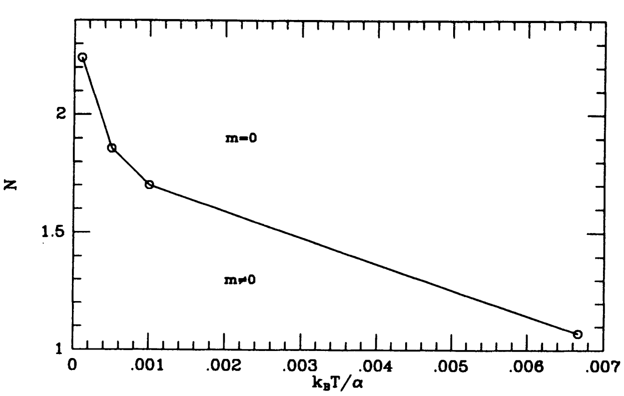

FIG. 12. The phase diagram of QED_3, in our approximation, with nontrivial wave function renormalization. The critical line separates the region where there is dynamical mass generation ($\mathcal{M}\neq 0$) from that in which there is not ($\mathcal{M}=0$).

VI. CONCLUSIONS

The work of Pennington *et al.* [5–8] strongly indicated that the existence [4] of a critical number of flavors N_c, above which chiral symmetry was unbroken in QED_3 at zero temperature, was an artifact of incorrectly ignoring wave function renormalization (via an unjustified appeal to perturbation theory in $1/N$). We have been interested in extending the study of chiral symmetry breaking in QED_3 to finite temperature. In [1] we found clear evidence for the existence of an N_c, dependent on T, but we made the approximation of ignoring wave function renormalization. The present study has been aimed at removing that approximation, and studying the effect of including wave function renormalization on chiral symmetry breaking in QED_3 at $T\neq 0$.

The simplest generalization of the model of [1] to include wave function renormalization, in which we adopted the "zero-frequency" approximation, turned out to lead to unphysical results, in that they failed to show any similarity to the zero-temperature results of [5–8] as $T\to 0$. We regarded this as reason for discarding that model, and adopting instead a modified equation for the wave function renormalization at $T\neq 0$, which gave results consistent with the $T=0$ case. We found, using the modified model (namely the kernel I_Δ of Secs. IV and V), that although wave function renormalization was now included in a way qualitatively very similar to that of [5–8], nevertheless at finite temperature we still found clear evidence for an N_c, in contrast to the results of [5–8]. The essential reason for the difference seemed to be the characteristically different infrared behavior of the vacuum polarization, at zero and at finite temperature. Indeed, the inclusion of wave function renormalization, in the manner described in Sec. IV above, gave results very little different, overall, from those of [1]. In particular, a value of about 10 for the ratio r seems to be remarkably robust, while the values of $\mathcal{M}(0,0)$ and T_c depend more on the model for A.

The conclusion that it is the infrared behavior which is crucial in obtaining a finite N_c is supported by the calculations of Kondo and Nakatani [10]. Using (among other vertices) the Pennington-Webb vertex of Eq. (4), these authors found that with an infrared cutoff ϵ of order $10^{-4}\alpha$, a fermion mass was generated for $N < N_c$, where N_c depended logarithmically on ϵ. Very roughly, we might interpret ϵ as corresponding, in our calculation, to the "thermal mass" $(\Pi_0)^{1/2}$, which is of order $T^{1/2}$. We then have a physical interpretation of the perhaps somewhat artificial cutoff introduced by Kondo and Nakatani [10].

These authors [10] also found that the behavior of the dynamically generated mass for N near N_c, using the vertex of Eq. (4), was consistent with mean field theory: that is, $\mathcal{M} \approx (N_c - N)^{1/2}$. By contrast, in our previous paper [1] we presented evidence (see Fig. 6 of [1]) which suggested a behavior of the form $\mathcal{M} \approx \exp[-C/(N_c - N)^{1/2}]$. However, we have reexamined those calculations and are now less confident that a firm conclusion can be drawn regarding the behavior near N_c, where the mass is very small and numerical accuracy increasingly difficult to control. In fact, a more conservative interpretation of both our present and previous data is that they are quite consistent with mean field theory, at least in the regime (for N not too close to N_c) where the calculations are most reliable.

The present calculation can, of course, be criticized for invoking the somewhat *ad hoc* "Δ^2 modification" for the A equation (though we stress again that this modification does the job required of it, namely to produce sensible-looking solutions at low T). It would clearly be much more satisfactory to attack the equations for the coupled $n \neq 0$ components of Σ, using the full $n \neq 0$ components of the photon propagator as given in [3]. In this way the true effects of the $n \neq 0$ components, especially at low T, could be identified. No doubt a start on this problem could be made by going back to the simplifying approximation $\Sigma(P,\beta) \approx \Sigma(0,\beta)$, which is likely to be quite reliable [see Eq. (2.54) of [3]]. We hope to return to this problem elsewhere.

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