Uniqueness of Amplitudes Satisfying the Mandelstam Representation*

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It is shown that within the framework of the Mandelstam representation, the scattering amplitudes, if they exist, are uniquely determined by the absorptive parts over the elastic region of one channel. It is illustrated in detail with pion-pion and pion-nucleon scattering.

I. INTRODUCTION

In a nonrelativistic theory it is, in general, necessary to have the potential fully given in order to determine the scattering amplitude completely. Similarly, in a relativistic theory the knowledge of absorptive parts and double-density functions which play the role here as potentials will determine the scattering amplitude. On the other hand, a relativistic scattering amplitude is also severely restricted by analyticity, crossing symmetry, and unitarity conditions. One expects that partial information on the absorptive parts or the double-density functions may already be sufficient to determine uniquely the scattering amplitude. This expectation has been borne out by detailed analysis. An example is the following, in the framework of the Mandelstam representation[1]: If the absorptive parts in one channel are given over their elastic region, then the amplitude is uniquely determined if it exists. This result is related to another excellent example given by Martin[2] who showed that for a neutral spinless particle, the amplitude is uniquely determined if the double-density functions $\rho_{st}(s,t)$ and $\rho_{su}(s,u)$ are given over the elastic region in the s channel, and if there is an elastic region in either the u or the t channel.

Probably the most remarkable fact about these results is the way elastic-unitarity conditions can fix the scattering amplitude when it is used jointly with analyticity. The full unitarity condition is not necessary. Even crossing symmetry is not much used although the necessary input information can often be lessened if crossing symmetry is fully utilized. Thus, for a neutral scalar particle satisfying the Mandelstam representation, there is only one independent double-density function if crossing symmetry is taken into account. Accordingly, knowing this one double-density function over the elastic region in one channel will uniquely determine the scattering amplitude.

We may also remark that a result of this kind is not an existence proof of a solution for an amplitude satisfying the Mandelstam representation but rather a uniqueness theorem. Whether there does exist a solution to the Mandelstam representation consistent with unitarity and crossing conditions is not indicated. But if such a solution does exist, then it is unique under the given premise.

The analysis we will consider is connected with the pioneer work of Martin[2] in the following sense. Assuming that the scattering amplitude satisfies the Mandelstam representation, the absorptive part $A_s(s,t)$ is determined up to subtraction terms by the double-density functions $\rho_{st}(s,t)$ and $\rho_{su}(s,u)$. To a large extent, the subtraction terms are not free parameters of the theory, but rather severely constrained by analyticity, unitarity, and crossing conditions. First of all, the Froissart-Martin[3] bound limits the number of subtraction terms to two. If there are two sets of amplitudes having in common either the same absorptive part $A_s(s,t)$ or the same double-density functions $\rho_{st}(s,t)$ and $\rho_{su}(s,u)$ in the elastic region of the s channel, then, as will be shown in the following, the two amplitudes can differ at most by the real part of their s wave. However, the real part of the s wave is determined up to a sign by its imaginary part in the elastic region. If the wrong sign is chosen, the s wave will have no left-hand cut and admit no inelastic contribution in the crossed channel and hence must be rejected. In the end we see that the subtraction terms are entirely fixed so that given the s-channel absorptive part $A_s(s,t)$, the double-density functions $\rho_{st}(s,t)$ and $\rho_{su}(s,u)$ will be determined, and vice versa.

Martin's analysis was done for a neutral scalar particle. But he remarked that difficulties associated with charge-exchange scattering may be overcome in some cases, and he showed that in pion-pion scattering, the Bose statistics can be utilized to make the argument work for the $I=2$ scattering amplitude. However, because of his reliance on Bose statistics, Martin's argument cannot be repeated with other isotopic-spin amplitudes of the pion-pion scattering. We will show that complications coming from the isotopic spin can be systematically handled. For illustration, we will consider pion-pion scattering in great detail but the method presented is directly generalizable to scalar particles with arbitrary isotopic spin.

Difficulties arising from the presence of spin are much more serious. This is due mainly to the vast increase of possible alternatives for the real part of the scattering amplitudes when the s-channel elastic imaginary part is given. For spinless particles there are four alternatives, for pion-nucleon scattering there are 16, and for nucleon-nucleon scattering the number of possibilities increases to 2^{10}. For this reason, it appears unlikely that one can treat the general case of arbitrary spin. However, for the important case of pion-nucleon scattering, there is a relationship between the spin-up and spin-down partial-wave amplitudes which rules out 12 otherwise possible alternatives for the real parts of the helicity amplitudes and thereby enables us to carry through the analysis.

Martin[4] has given yet another variation of the theorem, namely, for spinless particles, if the inelastic contributions to the absorptive part are given, the amplitude is again uniquely determined. This is so because within the elastic t region, the double-density function $\rho_{st}(s,t)$ has no contributions from the elastic absorptive part in s channel and is determined completely by the inelastic part of $A_s(s,t)$; $\rho_{ut}(u,t)$ in the t-channel elastic region will either be given by s-u crossing symmetry when there is one, or by assuming that the inelastic contributions to the u-channel absorptive part are given. These premises will then be equivalent to knowing $\rho_{st}(s,t)$ and $\rho_{ut}(u,t)$ over the elastic region in the t-channel from which it follows that the amplitude is uniquely determined. However, this is not applicable to scattering amplitudes when there is no elastic region in the t channel such as in pion-nucleon or kaon-kaon scattering.

II. UNIQUENESS OF PION-PION SCATTERING AMPLITUDES WHEN THE ABSORPTIVE PART IS GIVEN OVER THE ELASTIC REGION IN THE s CHANNEL

It is not difficult to see that the scattering amplitude for a spinless particle satisfying the Mandelstam representation is uniquely determined if the absorptive part is given over the elastic region in one channel. Under the given condition, the imaginary part for each partial wave is fully determined and through the elastic unitarity condition, the real part $\mathrm{Re} f_l$ will be determined up to a sign. *A priori*, this sign can be different for different l. If this were the case, no useful result would follow. However, from the fact that the amplitude satisfies an N-subtracted Mandelstam representation, all the even partial waves as well as all the odd ones can be analytically continued[5] into the complex l plane for $l>N$ and one finds that the choice of sign must be the same for all even $l>N$ and likewise for all odd $l>N$. As a consequence, up to a polynomial in $\cos\theta$ the real part of the full amplitude is limited to only four alternatives. The main task is to be able to pick just the right choice and eliminate all the other possibilities. The net result is that the amplitude itself is now determined up to a polynomial of order N in $\cos\theta$. In particular, all the double-density functions are now determined everywhere. In order to restrict further the subtraction terms, one must perform a crossing to another channel, and here complications from charge-exchange scattering threaten the simplicity of the ensuing end result that the scattering amplitude is, in fact, uniquely determined. On the other hand, we notice that when the scattering particles are spinless, the crossing matrices are momentum-independent numerical matrices. This means that we can proceed to analyze each scattering amplitude with a fixed isotopic spin. When we perform a crossing to another channel, it is true that an amplitude with a given isotopic spin will become a mixture of amplitudes of all isotopic spins. However, it is clear that if every amplitude of fixed isotopic spin has a common property, then each amplitude in the crossed channel likewise has the same property. Thus the Froissart-Martin bound in the crossed channel will limit the number of subtraction terms to two. Further restrictions from the unitarity condition and analyticity require in the end that the amplitudes are in fact uniquely determined. We see that in this way the case of two spinless particles having arbitrary isotopic spin can be systematically treated. In the following, we shall illustrate in detail the general method with pion-pion scattering.

The T matrix for the pion-pion scattering may be decomposed into the following[6]:

$$\langle\gamma\delta|T|\alpha\beta\rangle = A^1(s,t,u)\delta_{\alpha\beta}\delta_{\gamma\delta} + A^2(s,t,u)\delta_{\alpha\gamma}\delta_{\beta\delta} + A^3(s,t,u)\delta_{\alpha\delta}\delta_{\beta\gamma}, \tag{1}$$

where α, β, γ, δ are charge indices for the pion and take on the values 1, 2, 3. The invariant amplitudes $A^i(s,t,u)$ have the following simple crossing symmetries:

$$A^1(\underline{s},t,u) = A^1(\underline{s},u,t),$$

$$A^2(\underline{s},t,u) = A^3(\underline{s},u,t),$$

$$A^2(s,\underline{t},u) = A^2(u,\underline{t},s),$$

and (2)

$$A^1(s,\underline{t},u)=A^3(u,\underline{t},s),$$
$$A^3(s,t,\underline{u})=A^3(t,s,\underline{u}),$$
$$A^1(s,t,\underline{u})=A^2(t,s,\underline{u}),$$

where we have underlined a variable whenever it is held fixed. From these relations it is clear that there is, in fact, only one independent pion-pion scattering amplitude because given any one of the A's, the other two can be obtained simply from the above crossing relations.

In order to make use of the unitarity condition in a simple way, we must decompose the T matrix into scattering amplitudes of fixed isotopic spin,

$$\langle I,\gamma\delta|T|I,\alpha\beta\rangle = B^0(s,t,u)\delta_{I,0}+B^1(s,t,u)\delta_{I,1}+B^2(s,t,u)\delta_{I,2}. \tag{3}$$

Let us denote $B^{i,I}(s,t,u)$ as the invariant scattering amplitude with total isotopic spin I in the i channel with $i=s,t,u$; $B_j^{i,I}$, the absorptive part of the i-channel amplitude in crossing the j-channel physical cut; and $B_{xy}^{i,I}$ with xy taking on st, tu, and us, the i-channel double-density functions. It is assumed that like the $A^i(s,t,u)$, the $B^{i,I}$ satisfy an N-subtracted Mandelstam representation,

$$\begin{aligned}B^{s,I}(s,t,u)=&\frac{s^Nt^N}{\pi^2}\iint\frac{B_{st}^{s,I}(s',t')ds'dt'}{(s'-s)(t'-t)s'^Nt'^N}+\frac{t^Nu^N}{\pi^2}\iint\frac{B_{tu}^{s,I}(t',u')dt'du'}{(t'-t)(u'-u)t'^Nu'^N}+\frac{u^Ns^N}{\pi^2}\iint\frac{B_{us}^{s,I}(u',s')du'ds'}{(u'-u)(s'-s)u'^Ns'^N}\\&+\sum_{p=0}^{M<N}t^ps^M\frac{1}{\pi}\int\frac{B_{p,s}^{s,I}(s')ds'}{(s'-s)s'^M}+\sum_{p=0}^{M<N}u^pt^M\frac{1}{\pi}\int\frac{B_{p,t}^{s,I}(t')dt'}{(t'-t)t'^M}+\sum_{p=0}^{M<N}s^pu^M\frac{1}{\pi}\int\frac{B_{p,u}^{s,I}(u')du'}{(t'-t)t'^M}\\&+\sum_{p,q=0}^{N}t^ps^qB_{pq}^{ts,I}+\sum_{p,q=0}^{N}s^pu^qB_{pq}^{su,I}+\sum_{p,q=0}^{N}u^pt^qB_{pq}^{ut,I}.\end{aligned}\tag{4}$$

The A's and the B's as well as the corresponding density functions are related to each other by the following:

$$B^{i,I}(s,t,u)=\sum_{j=1}^{3}(\alpha^i)_j^IA^j(s,t,u), \tag{5}$$

and, correspondingly,

$$B_{xy}^{i,I}(s,t,u)=\sum_{j=1}^{3}(\alpha^i)_j^IA_{xy}^j(s,t,u), \tag{6}$$

where $I=0,1,2$ and i runs over s, t, u while xy stands for tu, us, and st. The α^i are given explicitly by

$$\alpha^s=\begin{pmatrix}3&1&1\\0&1&-1\\0&1&1\end{pmatrix},\quad \alpha^t=\begin{pmatrix}1&3&1\\-1&0&1\\1&0&1\end{pmatrix},\quad \alpha^u=\begin{pmatrix}1&1&3\\1&-1&0\\1&1&0\end{pmatrix}. \tag{7}$$

Because of the crossing relations given by Eq. (2), there are only two independent double-density functions $\sigma(s,y)$ and $\rho(s,y)$ in terms of which the $B_{xy}^{s,I}$, for instance, may be arranged in matrix form:

$$B_{xy}^{s,I}=\begin{pmatrix}B_{tu}^{s,0}&B_{us}^{s,0}&B_{st}^{s,0}\\B_{tu}^{s,1}&B_{us}^{s,1}&B_{st}^{s,1}\\B_{tu}^{s,2}&B_{us}^{s,2}&B_{st}^{s,2}\end{pmatrix}=\begin{pmatrix}3\sigma+2\rho&4\rho+\sigma&4\rho+\sigma\\0&\sigma-\rho&\rho-\sigma\\2\rho&\sigma+\rho&\rho+\sigma\end{pmatrix}. \tag{8}$$

$B_{xy}^{t,I}$ and $B_{xy}^{u,I}$ can be obtained directly from $B_{xy}^{s,I}$ by cyclic permutation on the columns of the above matrix.

The crossing relations for the B amplitudes may be obtained by first expressing the B's in terms of the A's, performing the necessary crossing on the A amplitudes according to Eq. (2), and then reexpressing the result in terms of the B's. This will give, for instance, the following s-u crossing relations:

$$\begin{aligned}B^{u,I=0}(s,t,u)&=B^{s,I=0}(u,t,s)=\tfrac{1}{3}[B^{s,0}(s,t,u)-3B^{s,1}(s,t,u)+5B^{s,2}(s,t,u)],\\B^{u,I=1}(s,t,u)&=B^{s,I=1}(u,t,s)=\tfrac{1}{6}[2B^{s,0}(s,t,u)-3B^{s,1}(s,t,u)-5B^{s,2}(s,t,u)],\\B^{u,I=2}(s,t,u)&=B^{s,I=2}(u,t,s)=\tfrac{1}{6}[2B^{s,0}(s,t,u)+3B^{s,1}(s,t,u)+B^{s,2}(s,t,u)].\end{aligned}\tag{9}$$

Now suppose the absorptive part is given over the elastic region in s channel. If there are two sets of amplitude B^I and B'^I having in common these absorptive parts, then in their partial-wave expansion we have

$$\mathrm{Im} f_l^I(s) = \mathrm{Im} f_l'^I(s) \quad \text{for all } l, \tag{10}$$

and from the elastic-unitarity condition we see that the real part of each partial wave inside the elastic region is determined up to a sign,

$$(\mathrm{Re} f_l^I)^2 = (\mathrm{Re} f_l'^I)^2$$

or

$$\mathrm{Re} f_l^I = \pm \mathrm{Re} f_l'^I \quad \text{for each } l. \tag{11}$$

In principle, this sign can be different for different l. If this is so, few useful results can be extracted from such a situation. However, since the amplitudes are assumed to satisfy the Mandelstam representation, $f_l^I(s)$ as well as $\mathrm{Re} f_l^I(s)$ can be analytically continued into the complex l plane[6] separately for l even and l odd provided $l > N$. The analytically continued partial waves $f_l^{I,+}(s)$ and $f_l^{I,-}(s)$ will coincide with the physical $f_l^I(s)$ for l even and l odd, respectively. Then from Carlson's theorem,[7] Eq. (11) may be generalized to the following two equations, valid with l complex and $\mathrm{Re}\, l > N$:

$$\begin{aligned}&(\mathrm{Re} f_l^{I,+} - \mathrm{Re} f_l'^{I,+})(\mathrm{Re} f_l^{I,+} + \mathrm{Re} f_l'^{I,+}) = 0,\\ &(\mathrm{Re} f_l^{I,-} - \mathrm{Re} f_l'^{I,-})(\mathrm{Re} f_l^{I,-} + \mathrm{Re} f_l'^{I,-}) = 0.\end{aligned} \tag{12}$$

From this it follows that

$$\mathrm{Re} f_l^{I,+} = \mathrm{Re} f_l'^{I,+}$$

or

$$\mathrm{Re} f_l^{I,+} = -\mathrm{Re} f_l'^{I,+} \quad \text{for all even } l > N, \tag{13}$$

and

$$\mathrm{Re} f_l^{I,-} = \mathrm{Re} f_l'^{I,-}$$

or

$$\mathrm{Re} f_l^{I,-} = -\mathrm{Re} f_l'^{I,-} \quad \text{for all odd } l > N. \tag{14}$$

The reason that these relations now hold for all l even or all l odd rather than for each l even or each l odd is that each factor in Eqs. (13) and (14) is an analytic function of l. The case that both factors in Eqs. (13) and (14) are zero will mean that

$$\mathrm{Re} f_l^{I,+} = \mathrm{Re} f_l'^{I,+} \equiv 0$$

or

$$\mathrm{Re} f_l^{I,-} = \mathrm{Re} f_l'^{I,-} \equiv 0.$$

We see that given the absorptive part over the elastic region in s channel, the elastic-unitarity condition and the l-plane analyticity will admit, up to a polynomial in $\cos\theta$, only four possibilities for the real part of the scattering amplitude.

In order to arrive at the desired result

$$\text{(a)}\quad \mathrm{Re} f_l^I = \mathrm{Re} f_l'^I \quad \text{for all } l > N, \tag{15}$$

we proceed to show that all the other three cases are physically inadmissible.

(b) Suppose $\mathrm{Re} f_l^I = -\mathrm{Re} f_l'^I$ for all $l > N$. Then we have

$$B^{s,I}(s,\cos\theta) + B'^{s,I}(s,\cos\theta) = \frac{\sqrt{s}}{q_s}\sum_0^N (2l+1)(\mathrm{Re} f_l^I + \mathrm{Re} f_l'^I) P_l(\cos\theta) + 2iB_s^{s,I}(s,t). \tag{16}$$

Taking the discontinuity across the t cut for both sides of the above equation, we get

$$\frac{s^N}{\pi}\int \frac{B_{st}^{s,I}(s',t) + B_{st}'^{s,I}(s',t)}{(s'-s)s'^N}\,ds' + \frac{u^N}{\pi}\int \frac{B_{tu}^{s,I}(tu') + B_{tu}'^{s,I}(tu')}{(u'-u)u'^N}\,du' + \sum_0^N C_n(t)s^n = 2iB_{st}^{s,I}(s,t). \tag{17}$$

Now for a given $t = t_0$, the right-hand side vanishes before the first Landau curve is reached,[8] i.e., for

$$4\mu^2 < s < s_1(t_0) \text{ where } (s_1 - 4\mu^2)(t_0 - 16\mu^2) = 64\mu^4.$$

On the other hand, the left-hand side for any fixed t is an analytic function of s with two nonoverlapping cuts. The fact that it vanishes over a finite segment $4\mu^2 < s < s_1(t_0)$ inside its analyticity domain leads to the result that it vanishes identically, and hence $B_{st}^{sI}(s,t)$ vanishes identically. This in turn will imply that there is no scattering at all.[9] The premise is therefore unacceptable.

(c) Suppose we have

$$\mathrm{Re} f_l^{I,-} = \mathrm{Re} f_l'^{I,-},$$

$$\mathrm{Re} f_l^{I,+} = -\mathrm{Re} f_l'^{I,+}.$$

If $B^{s,I}(s,t,u)$ is symmetric with respect to an exchange of t and u, then $\mathrm{Re} f_l^{I,-} = 0$, and it reduces to case (b). On the other hand, if $B^{s,I}(s,t,u)$ is antisymmetric with respect to an exchange of t and u, then $\mathrm{Re} f_l^{I,+} = 0$ and it reduces to case (a). If $B^{s,I}(s,t,u)$ has no symmetry with respect to an exchange of t and u as it is the case for pion-pion scattering, we can form the following symmetrized amplitude:

$$\tfrac{1}{2}[B^{s,I}(s,t,u) + B^{s,I}(s,u,t)],$$

where $B^{s,I}(s,u,t)$ for a given I will be a linear combination of all $B^{s,I'}(s,t,u)$ with $I'=0,1,2$. However, at this point we are not concerned with reexpressing $B^{s,I}(s,u,t)$ in terms of $B^{s,I}(s,t,u)$ through the crossing relations. We merely want the $B^{s,I}(s,u,t)$ as obtained, for instance, from the partial-wave expansion of $B^{s,I}(s,t,u)$ by replacing $\cos\theta$ with $-\cos\theta$ and add the series so obtained to the original one. The resultant partial-wave expansion now contains only terms of even l, but

$$\mathrm{Re} f_l^{I,+} = -\mathrm{Re} f_l'^{I,+}$$

by premise, so that the reasoning given in case (b) can be used here to obtain the relation that

$$B_{st}^{s,I}(s,t) + B_{su}^{s,I}(s,u) = 0.$$

This equation implies through the Mandelstam representation that the absorptive part of the symmetrized amplitude can only be a polynomial of degree N in $\cos\theta$,

$$\tfrac{1}{2}[B_s^{s,I}(s,t,u) + B_s^{s,I}(s,u,t)] = \sum_{p=0}^{N} B_{p,s}^{s,I}(t^p + u^p),$$

so that for all even $l > N$,

$$\mathrm{Im} f_l^{I,+} = \mathrm{Im} f_l'^{I,+} \equiv 0,$$

and hence, from the elastic-unitarity condition,

$$\mathrm{Re} f_l^{I,+} = \mathrm{Re} f_l'^{I,+} \equiv 0.$$

This, taken together with $\mathrm{Re} f_l^{I,-} = \mathrm{Re} f_l'^{I,-}$, reduces the situation under consideration to case (a).

Finally, let us suppose

(d) $\mathrm{Re} f_l^{I,+} = \mathrm{Re} f_l'^{I,+}$,

$\mathrm{Re} f_l^{I,-} = -\mathrm{Re} f_l'^{I,-}$.

The reasoning for this case is very similar to case (c) if we consider the amplitude with a given isotopic spin but antisymmetrized with respect to t and u,

$$\tfrac{1}{2}[B^{s,I}(s,t,u) - B^{s,I}(s,u,t)].$$

In the end, we come to the conclusion that for all possible cases, we have

$$\mathrm{Im} f_l^I = \mathrm{Im} f_l'^I,$$

$$\mathrm{Re} f_l^I = \mathrm{Re} f_l'^I \quad \text{for all } l > N.$$

So far the I is just an index and the above result holds separately for $I = 0,1,2$. Hence, the two sets of amplitudes with fixed isotopic spin can differ at most by a polynomial of degree N in $\cos\theta$,

$$B^{s,I}(s,t,u) - B'^{s,I}(s,t,u) = \sum_{n=0}^{N} \alpha_n^I(s) u^n. \qquad (18)$$

This implies in particular that all the double-density functions for the primed and unprimed amplitudes must be identically the same and not just over the elastic region of s-channel because if we take the discontinuity across the u cut for both sides of Eq. (18), for example, we get

$$\frac{s^N}{\pi}\int \frac{B_{su}^{s,I}(s',u) - B_{su}'^{s,I}(s',u)}{(s'-s)s'^N}\,ds' + \frac{t^N}{\pi}\int \frac{B_{tu}^{s,I}(t',u) - B_{tu}'^{s,I}(t',u)}{(t'-t)t'^N}\,dt' + \sum_{p=0}^{N} t^p [B_{p,u}^{s,I}(u) - B_{pu}'^{s,I}(u)] = 0. \qquad (19)$$

The left-hand side of this equation is an analytic function of s with two nonoverlapping cuts. Since it vanishes over a finite segment inside its analyticity domain, $s_0 < s < s_1$, where s_1 is the first inelastic threshold, it vanishes identically, and we find that throughout the s-t complex domain,

$$B_{su}^{s,I}(s,u) = B_{su}'^{s,I}(s,u),$$

$$B_{tu}^{s,I}(t,u) = B_{tu}'^{s,I}(t,u), \qquad (20)$$

and

$$B_{p,u}^{s,I} = B_{p,u}'^{s,I}.$$

Similarly, if we consider the discontinuity over the t cut of Eq. (18), we will obtain

$$B_{st}^{s,I}(s,t) = B_{st}'^{s,I}(s,t)$$

and (21)

$$B_{p,t}^{s,I}(t) = B_{p,t}'^{s,I}(t)$$

for all s and t.

Returning to Eq. (18), if we now perform a crossing to the u channel, each $B^{s,I}(u,t,s)$ and $B'^{s,I}(u,t,s)$ will be a linear combination of $B^{s,I'}(s,t,u)$ and $B'^{s,I'}(s,t,u)$, respectively, as given by Eq. (9). However, since each and every amplitude in the u channel is bounded by the Froissart-Martin bound at high u, their linear combinations will again be bounded by the Froissart-Martin bound. Consequently, $N \le 1$, and we have

$$B^{s,I}(s,t,u) - B'^{s,I}(s,t,u) = \beta_0^I(s) + u\beta_1^I(s). \qquad (22)$$

Since all the double-density functions for the primed and unprimed amplitudes have been shown to be the same, the u-channel absorptive part will also be determined up to a polynomial in $\cos\theta_u$ and we can *a posteriori* carry out a similar analysis in the u channel to obtain

$$B^{u,I}(s,t,u) - B'^{u,I}(s,t,u) = \gamma_0^I(u) + s\gamma_1^I(u), \qquad (23)$$

where the u-channel unitarity condition further requires that

$$\gamma_0^I < \text{const}\,,$$
$$\gamma_0^I < \text{const}/u \quad \text{at large } u. \tag{24}$$

Let us express, for instance, the u-channel $I = 2$ amplitude in terms of s-channel amplitudes. As given by Eq. (10), we have

$$B^{u,I=2}(s,t,u)=\tfrac{1}{6}\left[2B^{s,I=0}(s,t,u)-3B^{s,I=1}(s,t,u)+B^{s2}(s,t,u)\right].$$

Hence,

$$B^{u,I=2}(s,t,u)-B'^{u,I=2}(s,t,u)=\tfrac{1}{6}\{2\beta_0^0(s)-3\beta_0^1(s)+\beta_0^2(s)+u[2\beta_1^0(s)-3\beta_1^1(s)+\beta_1^2(s)]\} \tag{25}$$

Comparing this with Eq. (23), we see that γ_0^2 and γ_1^2 can only be linear functions of u. Quite generally, we may put

$$\gamma_0^I = a^I + b^I u,$$
$$\gamma_1^I = c^I + d^I u. \tag{26}$$

Then the u-channel unitarity bound given by Eq. (24) demands that

$$b^I = c^I = d^I = 0 \quad \text{for } I = 0, 1, 2.$$

Furthermore, a^I for a given I is a real constant since $B^{u,I}$ and $B'^{u,I}$ are both real for $-t<u<4\mu^2$. As a result, only the real part of the s wave can possibly differ,

$$\text{Im} f_l^I = \text{Im} f_l'^I, \quad l \geqslant 0$$
$$\text{Re} f_l^I = \text{Re} f_l'^I, \quad l \geqslant 1$$

and

$$\text{Re} f_0^I = \pm\, \text{Re} f_0'^I\,.$$

If $\text{Re} f_0^I = -\text{Re} f_0'^I$, we would have

$$\text{Re} f_0^I = (q_s/\sqrt{s}\,) C^I, \tag{27}$$

where C^I is a constant. This is physically inadmissible because it has no left-hand cut, in violation with crossing symmetry and because by analytic continuation it gives no inelastic contributions in crossed channels. We are left with

$$\text{Re} f_0^I = \text{Re} f_0'^I, \tag{28}$$

and hence,

$$F^I(s,t,u) \equiv F'^I(s,t,u). \tag{29}$$

We have reached the final conclusion that the scattering amplitudes for the pion-pion scattering, if they exist, are uniquely determined by the absorptive parts within the elastic region in the s channel.

From our earlier discussion, it is clear that the same result follows if we are given instead the double-density functions $B_{st}^{s,I}$ and $B_{su}^{s,I}$ over the s-channel elastic region. This is a generalization to Martin's analysis with neutral scalar particles. Also, from our presentation it is clear that the method is directly applicable to spinless particles with arbitrary isotopic spins.

III. COMPLICATIONS FROM UNEQUAL-MASS KINEMATICS

In considering cases (c) and (d) in the previous section, we had to symmetrize or antisymmetrize the scattering amplitude with respect to t and u in order to eliminate the odd or even partial waves, as the case may be. Of course, what was really being symmetrized or antisymmetrized was the scattering amplitude with respect to the cosine of the scattering angle. For equal-mass scattering, this is equivalent to symmetrizing or antisymmetrizing with respect to an exchange of t and u. This simple correspondence is no longer true for unequal-mass scattering. However, it is quite clear how such kinematic complications may be handled, namely, by employing a new set of invariant variables s, $\bar{t}$, $\bar{u}$ for which the above simple correspondence is restored. Consider, for instance, two neutral particles of masses μ and M, in the center-of-mass system in s channel, we have

$$4q_s{}^2 = s - 2(M^2+\mu^2) + (M^2-\mu^2)^2/s\,,$$
$$t = -2q_s{}^2(1-\cos\theta)\,, \tag{30}$$
$$u = -2q_s(1+\cos\theta) + (M^2-\mu^2)^2/s\,.$$

For kinematic relations, we see that an exchange of t and u does not correspond to $\cos\theta \leftrightarrow -\cos\theta$. Now if we change variables to

$$s \to \bar{s} = s\,,$$
$$t \to \bar{t} = t + (M^2-\mu^2)^2/2s\,, \tag{31}$$
$$u \to \bar{u} = u - (M^2-\mu^2)^2/2s\,,$$

then an exchange of $\bar{t}$ and $\bar{u}$ does correspond to $\cos\theta \leftrightarrow -\cos\theta$. Under these changes of variables, the Mandelstam representation for the scattering amplitude for two neutral particles may be rewritten as

$$G(s,\bar{t},\bar{u})=F(s,t,u)$$

$$=\frac{s^N\bar{t}^N}{\pi^2}\int\int\frac{\rho_{st}(s',t'-(M^2-\mu^2)^2/s)ds'dt'}{s'^Nt'^N(t'-\bar{t})(s'-s)}$$

$$+\frac{\bar{t}^N\bar{u}^N}{\pi^2}\int\int\frac{\rho_{tu}(t'-(M^2-\mu^2)^2/s,u'+(M^2-\mu^2)^2/s)du'dt'}{u'^Nt'^N(u'-\bar{u})(t'-\bar{t})}$$

$$+\frac{s^N\bar{u}^N}{\pi^2}\int\int\frac{\rho_{us}(u'+(M^2-\mu^2)^2/s,s')ds'du'}{s'^Nu'^N(u'-\bar{u})(s'-s)}+\sum_{p=0}^{M<N}\bar{t}^ss^M\frac{1}{\pi}\int\frac{\rho_{p,s}(s')ds'}{(s'-s)s'^M}$$

$$+\sum_{p=0}^{M<N}s^p\bar{u}^M\frac{1}{\pi}\int\frac{\rho_{p,u}(u'+(M^2-\mu^2)^2/s)du'}{(u'-\bar{u})u'^M}+\sum_{p=0}^{M<N}\bar{u}^p\bar{t}^M\frac{1}{\pi}\int\frac{\rho_{p,t}(t'-(M^2-\mu^2)^2/s)dt'}{(t'-\bar{t})t'^M}$$

$$+\sum_{p,q=0}^{N}\bar{t}^ps^q\rho_{pq}^{ts}+\sum_{p,q=0}^{N}s^p\bar{u}^q\rho_{pq}^{su}+\sum_{p,q=0}^{N}\bar{u}^p\bar{t}^q\rho_{pq}^{tu}. \quad (32)$$

An analysis similar to the one made in Sec. II may now be made on $G(s,\bar{t},\bar{u})$ replacing t and u wherever they appear in the argument by $\bar{t}$ and $\bar{u}$ until we come to the result that the primed and unprimed G's and hence F's can differ at most by a polynomial in $\bar{u}$ or equivalently in u. Similarly, when we have to go over to the u channel, we can change s,t variables into $\bar{s}$ and $\bar{t}$ in such a way that an exchange of $\bar{s}\leftrightarrow\bar{t}$ corresponds to $\cos\theta_u\leftrightarrow-\cos\theta_u$. In the end, one will arrive at the same conclusion as in the equal-mass case. Thus the unequal-mass kinematics give rise to no new difficulties other than making the argument more inconvenient. For this reason, we will continue to employ equal-mass kinematics, assuming the scattering particles always have equal masses.

IV. UNIQUENESS OF MANDELSTAM AMPLITUDES FOR PION-NUCLEON SCATTERING

When the scattering particles carry spin, there will be $2s+1$ possible total angular momenta for each orbital angular momentum l, due to different orientation of the total spin s. Since the analysis must be carried out in partial waves, this will mean that the number of possible alternatives for the real part of the scattering amplitudes when the imaginary part is given over the elastic region of one channel will greatly increase with the presence of spin. Thus, for pion-nucleon scattering, which is the simplest possible case as far as spin complications are concerned, there are 16 possibilities as compared to four for the spinless case, while for nucleon-nucleon scattering there will be 2^{10} cases. It is not at all clear how one can eliminate these extra possibilities, and it appears unlikely that we can find a general treatment for arbitrary spin. For pion-nucleon scattering, there is a fortunate coincidence that the analytically continued spin-up and spin-down partial waves have a special symmetry between them and the argument can be carried out after this is fully taken into account.

The spin and isotopic-spin dependence in the T matrix of the pion-nucleon scattering can be separated out by the following decomposition:

$$T=\delta_{\alpha\beta}[-A^+(s,t,u)+\tfrac{1}{2}i\gamma\cdot(q+q')B^+(s,t,u)]$$
$$+\tfrac{1}{2}[\tau_\beta,\tau_\alpha][-A^-(s,t,u)+\tfrac{1}{2}i\gamma\cdot(q+q')B^-(s,t,u)], \quad (33)$$

where, as usual, α and β are charge indices of the pions; q and q' are their incoming and outgoing momenta. The covariant scattering amplitudes $A^\pm$ and $B^\pm$ have the following simple crossing symmetry:

$$A^\pm(s,t,u)=\pm A^\pm(u,t,s),$$
$$B^\pm(s,t,u)=\mp B^\pm(u,t,s). \quad (34)$$

If we write down an N-subtracted Mandelstam representation for $A^\pm$ and $B^\pm$, these relations will imply the following crossing relations for the double-density functions:

$$A^\pm_{st}(s,t)=\pm A^\pm_{ut}(u,t),$$
$$A^\pm_{su}(s,u)=\pm A^\pm_{su}(u,s),$$
$$B^\pm_{st}(s,t)=\mp B^\pm_{ut}(u,t),$$
$$B^\pm_{su}(s,u)=\mp B^\pm_{su}(u,s). \quad (35)$$

We note that there is no crossing symmetry in exchanging s and t or u and t because the t channel corresponds to the annihilation process $\pi\pi\rightarrow N\bar{N}$, which is entirely different from the pion-nucleon scattering. We should also notice that there is no elastic scattering region in the t channel since the annihilation process itself corresponds to an inelastic scattering.

The invariant amplitudes with fixed total isotopic spin A^I and B^I are related to $A^\pm$ and $B^\pm$ as follows:

$$A^{1/2}=A^++2A^-,$$
$$A^{3/2}=A^+-A^-$$

and

$$B^{1/2} = B^+ + 2B^-,$$
$$B^{3/2} = B^+ - B^-. \tag{36}$$

The A^I and B^I do not have simple crossing symmetry. However, their crossing relations in exchanging s and u can be obtained from those for $A^\pm$ and $B^\pm$ through Eqs. (34) and (36),

$$\begin{aligned}
A^{1/2}(u,t,s) &= \tfrac{1}{3}[-A^{1/2}(s,t,u) + 4A^{3/2}(s,t,u)],\\
A^{3/2}(u,t,s) &= \tfrac{1}{3}[2A^{1/2}(s,t,u) + A^{3/2}(s,t,u)],\\
B^{1/2}(u,t,s) &= \tfrac{1}{3}[B^{1/2}(s,t,u) - 4B^{3/2}(s,t,u)],\\
B^{3/2}(u,t,s) &= \tfrac{1}{3}[-2B^{1/2}(s,t,u) - B^{3/2}(s,t,u)].
\end{aligned} \tag{37}$$

Since we have to use the partial-wave expansion in the s channel and analytically continue the partial waves into the complex l plane, we need to employ also the helicity amplitudes as defined by the following:

$$\begin{aligned}
f_1^I(W,t) &= \frac{E+M}{8\pi W}[A^I(W,t) + (W-M)B^I(W,t)],\\
f_2^I(W,t) &= -\frac{E-M}{8\pi W}[A^I(W,t) - (W+M)B^I(W,t)],
\end{aligned} \tag{38}$$

where $W=\sqrt{s}$, E is the c.m. energy of the nucleon in s channel, and M is the nucleon mass. f_1^I and f_2^I have the following partial-wave expansions:

$$\begin{aligned}
f_1^I(s,\cos\theta) &= \frac{\sqrt{s}}{q}\sum_{l=0}^{\infty}[a_{l+}^I P_{l+1}'(\cos\theta) - a_{l-}^I P_{l-1}'(\cos\theta)],\\
f_2^I(s,\cos\theta) &= \frac{\sqrt{s}}{q}\sum_{l=0}^{\infty}(a_{l-}^I - a_{l+}^I)P_l'(\cos\theta),
\end{aligned} \tag{39}$$

where a prime on a Legendre function means taking derivatives with respect to $\cos\theta$. a_{l+} is the lth partial wave with total angular momentum $l+\frac{1}{2}$ and a_{l-} is the partial wave with total angular momentum $l-\frac{1}{2}$. The inversion formulas for these partial waves are given by

$$\begin{aligned}
a_{l-}^I(s) &= \tfrac{1}{2}\int_{-1}^{1} dx[f_1^I(W,x)P_l(x) + f_2^I(W,x)P_{l-1}(x)],\\
a_{l+}^I(s) &= \tfrac{1}{2}\int_{-1}^{1} dx[f_1^I(W,x)P_l(x) + f_2^I(W,x)P_{l+1}(x)].
\end{aligned} \tag{40}$$

In the s-channel elastic region, we have

$$\begin{aligned}
\mathrm{Im}a_{l-}(s) &= |a_{l-}(s)|^2,\\
\mathrm{Im}a_{l+}(s) &= |a_{l+}(s)|^2.
\end{aligned} \tag{41}$$

We assume that $A^I(s,t,u)$ and $B^I(s,t,u)$ satisfy an N-subtracted Mandelstam representation. They are of the form given by Eq. (4) and to save space, we will not write them down explicitly. For simplicity, we further take the pion and nucleon to have equal mass.

Now suppose the absorptive parts of the scattering amplitudes are given over the elastic region in the s channel; then the helicity amplitudes $f_1^I, f_1'^I$ and $f_2^I, f_2'^I$ will also have equal absorptive parts inside the elastic region in the s channel. From the inversion formula for $a_{l\pm}$, it follows that

$$\mathrm{Im}a_{l\pm}^I = \mathrm{Im}a_{l\pm}'^I \quad \text{for all } l, \tag{42}$$

and the s-channel elastic-unitarity condition implies that

$$(\mathrm{Re}a_{l\pm}^I)^2 = (\mathrm{Re}a_{l\pm}'^I)^2 \quad \text{for all } l. \tag{43}$$

If we now analytically continue[10] $a_{l\pm}^I$ as well as $\mathrm{Re}a_{l\pm}^I$ for $l>N$ into the complex l plane, then from Carlson's theorem the above equation can be generalized to the following set of equations valid for complex l, with $\mathrm{Re}\,l>N$:

$$\begin{aligned}
(\mathrm{Re}a_{l+}^+ + \mathrm{Re}a_{l+}'^+)(\mathrm{Re}a_{l+}^+ - \mathrm{Re}a_{l+}'^+) &= 0,\\
(\mathrm{Re}a_{l+}^- + \mathrm{Re}a_{l+}'^-)(\mathrm{Re}a_{l+}^- - \mathrm{Re}a_{l+}'^-) &= 0,\\
(\mathrm{Re}a_{l-}^+ + \mathrm{Re}a_{l-}'^+)(\mathrm{Re}a_{l-}^+ - \mathrm{Re}a_{l-}'^+) &= 0,\\
(\mathrm{Re}a_{l-}^- + \mathrm{Re}a_{l-}'^-)(\mathrm{Re}a_{l-}^+ - \mathrm{Re}a_{l-}'^-) &= 0,
\end{aligned} \tag{44}$$

where $a_{l\pm}^+$ and $a_{l\pm}^-$ coincide with the physical $a_{l\pm}$ for l even and l odd, respectively. Since each factor in the above equation is an analytic function of l, it follows that

$$\begin{aligned}
&\mathrm{Re}a_{l+}^+ = \mathrm{Re}a_{l+}'^+,\\
\text{or}\quad &\mathrm{Re}a_{l+}^+ = -\mathrm{Re}a_{l+}'^+ \quad \text{for all even } l>N;\\
&\mathrm{Re}a_{l-}^+ = \mathrm{Re}a_{l-}'^+,\\
\text{or}\quad &\mathrm{Re}a_{l-}^+ = -\mathrm{Re}a_{l-}'^+ \quad \text{for all even } l>N;\\
&\mathrm{Re}a_{l+}^- = \mathrm{Re}a_{l+}'^-,\\
\text{or}\quad &\mathrm{Re}a_{l+}^- = -\mathrm{Re}a_{l+}'^- \quad \text{for all odd } l>N;\\
\text{and}\quad &\mathrm{Re}a_{l-}^- = \mathrm{Re}a_{l-}'^-,\\
\text{or}\quad &\mathrm{Re}a_{l-}^- = -\mathrm{Re}a_{l-}'^- \quad \text{for all odd } l>N.
\end{aligned} \tag{45}$$

We immediately see the vast complication coming from the nucleon spin because we are now faced with 16 possible alternatives instead of only four as in the case without spin. Of these 16 cases, the following four are the same as in the spinless case:

$$\begin{aligned}
&\text{(a)}\ \ \mathrm{Re}a^{\pm}_{l\pm}=\mathrm{Re}a'^{\pm}_{l\pm}\,,\\
&\text{(b)}\ \ \mathrm{Re}a^{\pm}_{l\pm}=-\mathrm{Re}a'^{\pm}_{l\pm}\,,\\
&\text{(c)}\ \ \mathrm{Re}a^{+}_{l\pm}=-\mathrm{Re}a'^{+}_{l\pm}\,,\\
&\qquad \mathrm{Re}a^{-}_{l\pm}=\mathrm{Re}a'^{-}_{l\pm}\,,\\
&\text{(d)}\ \ \mathrm{Re}a^{+}_{l\pm}=\mathrm{Re}a'^{+}_{l\pm}\,,\\
&\qquad \mathrm{Re}a^{-}_{l\pm}=-\mathrm{Re}a'^{-}_{l\pm}\,.
\end{aligned} \tag{46}$$

Take, for instance, case (c). If we symmetrize the helicity-nonflip amplitudes,

$$\begin{aligned}
&\tfrac{1}{2}[f_1^I(s,t,u)+f_1^I(s,u,t)]\,,\\
&\tfrac{1}{2}[f_1'^I(s,t,u)+f_1'^I(s,u,t)]\,,
\end{aligned} \tag{47}$$

and antisymmetrize the spin-flip amplitudes,

$$\begin{aligned}
&\tfrac{1}{2}[f_2^I(s,t,u)-f_2^I(s,u,t)]\,,\\
&\tfrac{1}{2}[f_2'^I(s,t,u)-f_2'^I(s,u,t)]\,,
\end{aligned} \tag{48}$$

then from the inversion formula for $a_{l\pm}$, we see that only $a_{l\pm}$ and $a'_{l\pm}$ with l even will appear in the Legendre expansion of these expansions. Now from the Mandelstam representation for A^I and B^I we can write down those for f_1^I and f_2^I and following the previous line of argument, we can obtain

$$f^I_{1,st}(s,t)+f^I_{1,su}(s,u)=0$$

and

$$f^I_{2,st}(s,t)-f^I_{2,su}(s,u)=0\,,$$

where $f^I_{1,st}$, for example, is the s-t double-density function for f_1 with total isotopic spin I. From these equations, it follows through the Mandelstam representation that the s-channel absorptive part of the amplitudes given by Eqs. (47) and (48) can, at most, be polynomial in $\cos\theta_s$ of order N, so that for $l>N$

$$\mathrm{Im}a^{+}_{l\pm}=0$$

and from the elastic-unitarity condition,

$$\mathrm{Re}a^{+}_{l\pm}=\mathrm{Re}a'^{+}_{l\pm}=0 \quad \text{for all } l>N.$$

Since $\mathrm{Re}a^{-}_{l\pm}=\mathrm{Re}a'^{-}_{l\pm}$, $l>N$ by premise, we conclude that $f_1^I-f_1'^I$ and $f_2^I-f_2'^I$ can, at most, be polynomials in $\cos\theta_s$ of order N.

Similar considerations may be given to cases (b) and (d). However, such reasoning will be insufficient to deal with the other 12 cases, such as

$$\begin{aligned}
&\mathrm{Re}a^{\pm}_{l+}=\mathrm{Re}a'^{\pm}_{l+}\,,\\
&\mathrm{Re}a^{\pm}_{l-}=-\mathrm{Re}a'^{\pm}_{l-}\,.
\end{aligned} \tag{49}$$

To be able to treat a case like this by the above method, we need to find a linear combination of f_1^I and f_2^I and their derivatives in such a way that its Legendre expansion involves only a_{l+} or a_{l-}. However, a careful look at the inversion formula for $a_{l\pm}$ will reveal that this is not possible. The closest one can get along this direction is to make use of the relation

$$P_{l+1}(x)=xP_l(x)-\frac{1-x^2}{l+1}\,P_l'(x)$$

and reexpress a_{l+} as

$$a_{l+}=\tfrac{1}{2}\int_{-1}^{1}P_l(x)\left[f_1^I+xf_2^I+\frac{d}{dx}\left(\frac{1-x^2}{l+1}\,f_2^I\right)\right].$$

Were it not for the factor $1/(l+1)$ in the last term of the integrand, we would have found a simple function of f_1^I and f_2^I and their derivatives, for which only a_{l+} appears in its Legendre expansion. Then by symmetrizing or antisymmetrizing this expression, we may eventually deduce that the primed and unprimed amplitudes can differ at most by a polynomial in $\cos\theta_s$. Since this is not possible, we have to look for other means. Fortunately, for pion-nucleon scattering there is a relationship[11] between $a^{\pm}_{l+}$ and $a^{\pm}_{l-}$, namely,

$$\begin{aligned}
&a^{+}_{l+}(W)=-a^{-}_{(l+1)-}(-W)\,,\\
&a^{-}_{l+}(W)=-a^{+}_{(l+1)-}(-W)\,.
\end{aligned} \tag{50}$$

These relations may be obtained directly from the definition of $a_{l\pm}(W)$,

$$a_{l\pm}(W)=\frac{1}{32\pi W^2}\{[(W+M)^2-\mu^2][A_l(W^2)+(W-M)B_l(W^2)]+[(W-M)^2-\mu^2][-A_{l\pm1}(W^2)+(W+M)B_{l\pm1}(W^2)]\}\,, \tag{51}$$

where

$$\begin{aligned}
&A_l(s)=\tfrac{1}{2}\int_{-1}^{1}P_l(x)A(s,t,u)dx\,,\\
&B_l(s)=\tfrac{1}{2}\int_{-1}^{1}P_l(x)B(s,t,u)dx\,.
\end{aligned} \tag{52}$$

$A_l(s)$ and $B_l(s)$ depend only on $s=W^2$. They are analytic in the whole s plane except for poles and cuts on the real axis. In the W plane, $A_l(W)$ and $B_l(W)$ have cuts along the whole imaginary axis, along the circle $|W|=(M^2-\mu^2)^{1/2}$ as well as cuts along the real axis from $-\infty$ to $-M-\mu$, from $-(M^2+2\mu^2)^{1/2}$ to $-[(1-\mu^2/M^2)(M^2-\mu^2)]^{1/2}$, from $-M+\mu$ to $M-\mu$, from $[(1-\mu^2/M^2)(M^2-\mu^2)]^{1/2}$ to $(M^2+2\mu^2)^{1/2}$, and from $M+\mu$ to ∞. The complication from these kinematic cuts can be avoided if we use the reduced partial waves

$$b_{l+}^{\pm}(W) = (k^2)^{-l} a_{l+}^{\pm}(W)\,, \qquad (53)$$

$$b_{l-}^{\pm}(W) = (k^2)^{-l} a_{(l+1)-}^{\pm}(W) \qquad (54)$$

so that the corresponding MacDowell reflection relations for $b_{l\pm}^{\pm}(W)$ may be viewed upon as relations obtained by analytic continuation of $b_{l\pm}^{\pm}(W)$ from W to $-W$. One can then show[12] that the relations

$$a_{l+}^{+}(W) = a_{l+}^{\prime +}(W)\,, \qquad a_{l-}^{-}(W) = -a_{l-}^{-}(W) \qquad (55)$$

are incompatible with the nature of the singularities of partial waves in the variable l and s, namely, with the existence of moving singularities. As a result, the case given by Eq. (49) must be ruled out because it is not consistent with the MacDowell reflection symmetry of $a_{l\pm}$ in Eq. (50). All the other cases not yet considered are of this type and hence can be ruled out by the same symmetry argument. We finally arrive at the result that f_i^I and $f_i'^I$ for $i = 1, 2$ can differ at most by a polynomial of order N in $\cos\theta_s$,

$$f_1^I - f_1'^I = \alpha_0^I(s) + u\alpha_1^I(s)\,, \qquad (56)$$

$$f_2^I - f_2'^I = \beta_0^I(s)\,, \qquad (57)$$

where all higher terms have been put equal to zero in accordance with the Froissart-Martin bound in the u channel. As can be seen from Eq. (37), an amplitude with a given I will become a linear combination of both $I = \frac{1}{2}$ and $\frac{3}{2}$ amplitudes when it is crossed over to the u channel. However, since each amplitude satisfies the Froissart-Martin bound, so will be their linear combination, giving us Eqs. (56) and (57). The coefficients α_0^I, α_1^I, and β_0^I are further restricted by the partial-wave unitarity bound in the s channel,

$$\alpha_0^I < \text{const}\,, \quad \alpha_1^I < \text{const}/s\,, \quad \text{and} \quad \beta_0^I < \text{const for large } s\,. \qquad (58)$$

Just as in pion-pion scattering, Eqs. (56) and (57) imply in particular that all double-density functions must be identically the same everywhere for the primed and unprimed amplitudes, so that the u-channel absorptive parts are also determined up to a polynomial in $\cos\theta_u$. If we now make a similar analysis in the u channel we obtain

$$f_1^I(u,t,s) - f_1'^I(u,t,s) = \gamma_0^I(u) + s\gamma_1^I(u)\,, \qquad f_2^I(u,t,s) - f_2^I(u,t,s) = \delta_0^I(u)\,, \qquad (59)$$

where from the u-channel unitarity condition,

$$\gamma_0^I < \text{const}\,, \quad \gamma_1^I < \text{const}/u\,, \quad \delta_0^I < \text{const for large } u. \qquad (60)$$

If we perform a crossing on Eq. (59) back to the s channel and compare the result with Eq. (56), we find that γ_0^I, γ_1^I, and δ_0^I can depend on u only linearly:

$$\gamma_0^I = a^I + b^I u\,,$$

$$\gamma_0^I = c^I + d^I u\,,$$

and

$$\delta_0^I = p^I + q^I u\,,$$

where a^I, b^I, c^I, d^I, p^I, and q^I are just constants. From the unitarity bound in the s channel Eq. (57), we must have

$$b^I = c^I = d^I = q^I = 0\,.$$

Once again we come to conclude that the primed and unprimed amplitudes for pion-nucleon scattering can differ at most by a constant, which must be real since both sets of amplitudes are real below the elastic threshold.

Now from the partial-wave expansions for f_1^I and f_2^I

$$f_1^I = (\sqrt{s}/q_s)[a_{0+}^I + 3a_{1+}^I \cos\theta + \cdots - a_{2-}^I - \cdots]\,,$$

$$f_2^I = (\sqrt{s}/q_s)[(a_{10}^I - a_{1+}^I) + 3(a_{20}^I - a_{2+}^I)\cos\theta_s + \cdots]\,,$$

where all terms not explicitly written must be the same for primed and unprimed amplitudes. If $a_{a-}^I \neq a_{2-}'^I$, then $f_2^I - f_2'^I$ will not just be a real constant. Also, if $a_{1-}^I \neq a_1'^I$, then $f_1^I - f_1'^I$ will not just be a real constant. We see that all the partial waves in the s channel must be the same for the primed and unprimed amplitudes except possibly for the real part of a_{0+}^I. In particular, we have

$$f_2^I = f_2'^I \quad \text{for } I = \tfrac{1}{2}, \tfrac{3}{2}\,. \qquad (61)$$

As for $\text{Re}a_{0+}^I$, since $\text{Im}a_{0+}^I = \text{Im}a_{0+}'^I$, we are limited to only two final possibilities due to the s-channel elastic-unitarity condition,

$$\text{Re}a_{0+}^I = \pm\text{Re}a_{0+}'^I\,. \qquad (62)$$

If

$$\text{Re}a_{0+}^I = -\text{Re}a_{0+}'^I\,,$$

we get

$$\text{Re}a_{0+}^I = (q_s/\sqrt{s})h^I\,, \qquad (63)$$

where h^I is a constant. But such an s wave has no left-hand cut in the s plane; it allows no inelastic contributions in the u channel and leads to the result of no scattering.[9] Hence, it must be discarded and we arrive at the final conclusion that

$$f_1^I(s,t,u) = f_1'^I(s,t,u) \quad \text{for } I=\tfrac{1}{2} \text{ or } \tfrac{3}{2}, \tag{64}$$

which, together with Eq. (61), says that the scattering amplitudes for pion-nucleon scattering are uniquely determined by the absorptive parts over the elastic region in the s channel.

V. CONCLUSION

Quite generally, an amplitude satisying the Mandelstam representation is so severely restricted by the elastic-unitarity condition that it is uniquely determined if its absorptive part is given over the elastic region in one channel. Under the given premise, the real part of each partial wave will be determined up to sign. *A priori*, this sign may be different for different partial waves. For the argument to go through, it is crucial that partial waves can be analytically continued into the complex l plane. Since so far this is known to be possible only for amplitudes satisfying the Mandelstam representation, the latter remains a necessary assumption in the argument. The result has been shown to hold in general for scattering of two spinless particles with arbitrary isotopic spins. In particular, it holds for pion-pion scattering, which was considered in great detail. We have also seen that vast complications occur when spins are present. However, at least for pion-nucleon scattering, there is the MacDowell reflection symmetry relating the spin-up and spin-down partial waves and the analysis can be carried out when this is fully taken into account. If enough similar symmetry relations can be found for scattering of particles with higher spins, it may be possible that the analysis can be extended to the general case of arbitrary spin.

We have seen how the elastic-unitarity condition and analyticity can uniquely determine the scattering amplitude. The full unitarity condition is not required and crossing symmetry is only sparingly used. It is not known that an amplitude so determined can always be made consistent with the full unitarity condition and all the crossing symmetries. In other words, it may be possible that the Mandelstam analyticity, unitarity, and crossing conditions are so restrictive that no solution satisfying them exists. What has been shown is that if such a solution exists, it is also unique under the given premise.

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[5]See, for example, V. N. Gribov, Zh. Eksperim. i Teor. Fiz. 41, 1962 (1962) [Soviet Phys. JETP 14, 1395 (1962)].

[6]For notation see G. Chew, *S-Matrix Theory of Strong Interactions* (Benjamin, New York, 1961), p. 10.

[7]See, for example, E. C. Titchmarsh, *Theory of Functions* (Oxford Univ. Press, New York, 1960), p. 186.

[8]L. D. Landau, Nucl. Phys. 13, 186 (1959); W. Zimmermann, Nuovo Cimento 21, 249 (1961).

[9]F. F. K. Cheung, Phys. Letters 27B, 302 (1968).

[10]See, for example, E. J. Squires, *Complex Angular Momenta and Particle Physics* (Benjamin, New York, 1960), p. 48.

[11]S. W. MacDowell, Phys. Rev. 116, 774 (1959) and Ref. 1.

[12]S. W. MacDowell (private communication). We thank Dr. MacDowell for pointing out the appropriate argument.