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# Self-Consistent Pomeranchukon Singularities. II*

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We assume that the Pomeranchukon poles lie on Schwarz trajectories: $\alpha_{\pm}(t)=1\pm\gamma t^{1/2}+g(t)$. We develop a functional self-consistency equation for $g(t)$ based on $t$-channel unitarity, and find that near $t=0$, $g(t)$ has the behavior $(\text{const})^2 t^{3/2}(-\ln t)^{1/2}$. Finally, we obtain an expression for the $t$-channel partial wave and determine the high-energy limit of the scattering amplitude in the $s$ channel.

## I. INTRODUCTION

This paper is the second in a series devoted to self-consistency conditions on Pomeranchukon Regge singularities. The consistency conditions are derived from unitarity in the channel of the singularities, which we take to be the $t$ channel. This origin of the conditions determines which properties of the Pomeranchukon can be chosen arbitrarily, and which are determined self-consistently. The Froissart condition,[1] $\alpha(0)\leqslant 1$, follows from $s$-channel unitarity, not $t$-channel unitarity, so in these papers $\alpha(0)$ is a free parameter. We will choose the experimental value $\alpha(0)=1$ in the present paper. On the other hand, once $\alpha(0)$ and some other properties of the Pomeranchukon singularities are specified, $t$-channel unitarity determines the $t$-channel partial-wave amplitude in terms of a few parameters. Most important, the strengths of the Regge cuts generated from the Pomeranchukon are determined.

In Paper I of this series,[2] we constructed $t$-channel partial-wave amplitudes under the assumption that the leading Pomeranchukon singularity for $t>0$ is a pole, and that the two-Pomeranchukon cut contributes negatively to total cross sections in the $s$ channel. In order to achieve the negative sign, which is required by Mandelstam[3] and the absorption model,[4] we had to assume the presence of moving poles, similar to Castillejo-Dalitz-Dyson poles, in certain meromorphic functions. At the same time, this assumption led to a three-Pomeranchukon coupling which vanishes when the Pomeranchukons have $j=1$ and $t=0$, a property that Gribov and Migdal termed "quasistability" of the Pomeranchukon.[5]

The results of Paper I depended upon the behavior of multi-Pomeranchukon phase space near the tip of a multi-Pomeranchukon cut in the angular momentum plane. Because the phase-space volume grows more slowly when more Pomeranchukons are present, we could argue that the multi-Pomeranchukon cuts do not enter in the self-consistent determination of $\alpha(t)$ near $t=0$; nor do they contribute importantly to asymptotic expressions in the crossed channel when $-t\ln s$ is less than some constant.

In the present paper we construct self-consistent amplitudes beginning with a somewhat different assumption, namely, that there is a pair of Pomeranchukon singularities on Schwarz trajectories[6]:

$$\alpha_{\pm}(t)=1\pm\gamma t^{1/2}+g(t),\quad g(0)=0\,. \tag{1.1}$$

Both Schwarz trajectories must be present to avoid a spurious singularity at $t=0$. We pointed out in Paper I that the behavior of multi-Pomeranchukon phase space changes for the particular case of Schwarz singularities, so the separate discussion given here is required for the sake of completeness. However, a special status for Schwarz singularities is suggested by their presence in several investigations. Among these are the following.

(i) Schwarz[6] pointed out that the "exact" Schwarz trajectory with $g(t)\equiv 0$ satisfies the functional equation $\alpha(t)=\alpha_n(t)$, where $\alpha_n(t)$ is the $n$-Pomeranchukon trajectory

$$\alpha_n(t)=n\alpha(t/n^2)-n+1\,. \tag{1.2}$$

This property opens the possibility that the singularity bootstraps itself.[7] In addition, $\alpha_n(t)=\alpha(t)$ means there are only two cuts in the angular momentum plane condensing on $j=1$ at $t=0$, instead of the infinite number present in Paper I. As we pointed out there, the full infinite series must be retained in the diffraction region $t<0$, since the cuts with larger $n$ lead those with smaller $n$.

Therefore, Schwarz trajectories simplify the Regge description of diffraction scattering.

(ii) Schwarz trajectories occur in discussions of violations of Pomeranchuk's theorem.[8]

(iii) Schwarz trajectories appear in the eikonal model.[9] Eikonalization renders an amplitude consistent with $s$-channel unitarity and the Froissart bound. When the full amplitude corresponding to a linear Regge-pole trajectory $b+ct$, $b>1$, is eikonalized, one obtains a $t$-channel partial-wave amplitude corresponding to a pair of Schwarz cuts: $f(t,j)=a[(j-1)^2-\gamma^2 t]^{-3/2}$. At $t=0$ the cuts collapse to a third-order pole, which just saturates the Froissart bound in the crossed channel.

In Sec. II we review the constraints imposed by a very crude form of $t$-channel unitarity, elastic unitarity. This does not lead to self-consistency conditions, nor does it involve the cuts that accompany the Pomeranchukon, but it does restrict the general form of the amplitude. In Sec. III we use multi-particle $t$-channel unitarity to obtain an expression for the contribution of the two-Pomeranchukon cuts. Here our work resembles that of Gribov, Pomeranchuk, and Ter-Martirosyan, whose results were used in Paper I.[10] The expression that we find is different from that of Gribov, *et al.* because, as we have stressed, two-Pomeranchukon phase space is radically different for ordinary Regge trajectories and Schwarz trajectories. In Sec. IV we obtain a self-consistent $t$-channel partial-wave amplitude in which the Pomeranchukons are Schwarz poles. The amplitude has the Mandelstam sign for the contribution of the two-Pomeranchukon cuts, and it leads to constant total cross sections at high energy in the $s$ channel. As in Paper I, moving poles are required in certain meromorphic functions to produce these desirable properties, and they lead to quasistability of the Pomeranchukon poles.

## II. ELASTIC $t$-CHANNEL UNITARITY

The constraints imposed on Regge cuts by elastic $t$-channel unitarity have been discussed generally,[11] and for Schwarz cuts specifically.[12,13] The results of these papers can be stated in terms of the family of elastic $t$-channel partial-wave amplitudes

$$f(t,j)=a\{[j-1-g(t)]^2-\gamma^2 t\}^p . \tag{2.1}$$

The Froissart bound requires $p\geq -\frac{3}{2}$ if $a$ is nonvanishing at $j=1$.

For $0>p\geq-\frac{3}{2}$, $p\neq -1$, we have Schwarz cuts with discontinuities that are infinite at threshold. This behavior is inconsistent with elastic $t$-channel unitarity, and there must be either fixed or moving cuts in the angular momentum plane that collide with the Schwarz cuts at $t=4\mu^2$, $j=\alpha_\pm(4\mu^2)$, where $\mu$ is the mass of the external particles. These additional cuts are presumably as important for diffraction scattering as the Schwarz cuts. We will not attempt to construct amplitudes of this sort because of the large number of cuts involved.

For $p=-1$ the Regge singularities are a pair of Schwarz poles. Here elastic unitarity requires only that $g(t)$ have a cut starting at $t=4\mu^2$.

For $p>0$ there are difficulties not related to elastic unitarity. The diffraction pattern in the $s$ channel is now anomalous in that the secondary, tertiary, and subsequent diffraction maxima grow in amplitude.[13] The scattering is not forward-peaked. This possibility arises because $\mathrm{Re}\,\alpha_\pm(t)\approx 1$ for $t<0$, which is a consequence of the Schwarz trajectory.

The only amplitude in our family that is attractive for further study is the amplitude with $p=-1$, corresponding to Schwarz poles. Since $g(t)$ must have a branch cut, it cannot vanish identically, the Schwarz trajectories are not "exact," and the multi-Pomeranchukon cuts are degenerate with the poles only near $t=0$. We wish to have a total cross section in the crossed channel which approaches a constant at high energy, so in Eq. (2.1) we make the factor $a$ vanish linearly at $j=1$. In Sec. III we compute the Regge cuts generated by the resulting amplitude

$$f(t,j)=\frac{-a(j-1)}{[j-1-g(t)]^2-\gamma^2 t} . \tag{2.2}$$

This will lead to a corrected representation for $f(t,j)$ which incorporates Pomeranchukon cuts as well as poles. Note that Eq. (2.2) leads to finite pole residues at $t=0$.

## III. THE TWO-POMERANCHUKON CUTS

The work of Gribov *et al.*[10] on the two-Pomeranchukon cuts leads to the representation

$$f(t,j)=\frac{A(t,j)}{B(t,j)+S(t,j)} , \tag{3.1}$$

where $A(t,j)$ and $B(t,j)$ have no branch point at the two-Pomeranchukon branch points, and $S(t,j)$ has the same discontinuity across the two-Pomeranchukon cuts as the integrals

$$I_{12}(t^{1/2},j)=4i\int_{C_2}\frac{dt_2}{2i}\int_{C_1}\frac{dt_1}{2i}\,\frac{1}{2t^{1/2}p(t;t_1,t_2)[j+1-\alpha_1(t_1)-\alpha_2(t_2)]}\,. \tag{3.2}$$

Here $\alpha_1$ and $\alpha_2$ are one of the Schwarz trajectories $\alpha_\pm$, $2t^{1/2}p(t;t_1,t_2)$ is $[t-(t_1^{1/2}+t_2^{1/2})^2]^{1/2}\times[t-(t_1^{1/2}-t_2^{1/2})^2]^{1/2}$, and the contours $C_1$ and $C_2$ are depicted in Fig. 1 as they appear for $t>16\mu^2$. The two-Pomeranchukon cuts that are on the physical sheet in the diffraction region $t\leq 0$ are generated by pinches on the upper arcs of the contours. It is convenient to write Eq. (3.2) in terms of the variables

$$x=t_1^{1/2}+t_2^{1/2},\quad y=t_1^{1/2}-t_2^{1/2}\,. \tag{3.3}$$

We find

$$I_{12}(t^{1/2},j)=\frac{i}{2}\int_{4\mu}^{t^{1/2}}\frac{dx}{(t-x^2)^{1/2}}\oint\frac{dy}{(t-y^2)^{1/2}}\,\frac{x^2-y^2}{j+1-\alpha_1(\frac{1}{4}(x+y)^2)-\alpha_2(\frac{1}{4}(x-y)^2)}\,. \tag{3.4}$$

The contour in the $y$ plane is the double loop indicated in Fig. 2; the points $\pm(x-4\mu)$ are the elastic-unitarity thresholds of $\alpha_{1,2}$, and the four arcs correspond to the four arcs of $C_1$ and $C_2$.

It can be shown that in the diffraction region only Regge cuts generated by $I_{++}$ and $I_{--}$ remain on the physical sheet.[6,14] We call these $I_+$ and $I_-$, and substitute from Eq. (1.1):

$$I_\pm(t^{1/2},j)=\frac{i}{2}\int_{4\mu}^{t^{1/2}}\frac{dx}{(t-x^2)^{1/2}}\oint\frac{dy}{(t-y^2)^{1/2}}\,\frac{x^2-y^2}{j-1\mp\gamma x-g(\frac{1}{4}(x+y)^2)-g(\frac{1}{4}(x-y)^2)}\,. \tag{3.5}$$

The singularities of these integrals are generated by pinches at $y=0$, and end-point singularities at $x=t^{1/2}$, so it is appropriate to expand $g$ about the argument $\frac{1}{4}t$. We retain the first two terms in the expansion; the second term in the denominator of Eq. (3.5) becomes

$$\tfrac{1}{2}g'(\tfrac{1}{4}t)(\lambda_\pm-y^2)\,, \tag{3.6}$$

$$\lambda_\pm=2[j-1-2g(\tfrac{1}{4}t)\mp\gamma x-\tfrac{1}{2}(x^2-t)g'(\tfrac{1}{4}t)]/g'(\tfrac{1}{4}t)\,.$$

$g'(t)$ must be positive in the diffraction region in order to satisfy the condition $\mathrm{Re}\,\alpha_\pm(t)\leq 1$ there.

The integral over $y$ in Eq. (3.5) has the form[15]

$$I_0(\lambda)=\frac{2}{g'(\frac{1}{4}t)}\int_{-\delta}^{\delta}\frac{dy(x^2-y^2)}{(t-y^2)^{1/2}(\lambda-y^2)}\,. \tag{3.7}$$

The contour is pinched at $\lambda=0$ by two poles which approach opposite sides of the contour at $y=0$. The resulting cut in the $\lambda$ plane is dressed to the right, with discontinuity

$$I_0(\lambda+i\epsilon)-I_0(\lambda-i\epsilon)=-\frac{4i\pi(x^2-\lambda)}{g'(\frac{1}{4}t)[\lambda(t-\lambda)]^{1/2}}\,. \tag{3.8}$$

In writing this equation we have used the fact that the lower arc in Fig. 2 is relevant in our case. In the $x$ plane, $I_0(\lambda_\pm)$ has two branch points, which are the solutions of $0=\lambda_\pm$. These are

$$x=\mp\frac{\gamma}{g'(\frac{1}{4}t)}+\epsilon\left[\left(\frac{\gamma}{g'(\frac{1}{4}t)}\right)^2+\frac{2}{g'(\frac{1}{4}t)}[j-1-2g(\tfrac{1}{4}t)+\tfrac{1}{2}tg'(\tfrac{1}{4}t)]\right]^{1/2}. \tag{3.9}$$

Here the signs $\mp$ are to be used for $I_0(\lambda_\pm)$, and $\epsilon$ takes on the two values $\pm1$ for the two branch points of $I_0$. $\lambda_\pm$ is positive between the two roots.

The remaining integration over $x$,

$$I_\pm(t^{1/2},j)=\frac{i}{2}\int_{4\mu}^{t^{1/2}}\frac{dx\,I_0(\lambda_\pm)}{(t-x^2)^{1/2}}\,, \tag{3.10}$$

produces functions that are singular when the branch points of $I_0$ collide with the end points of the integration. The collisions that produce the two-Pomeranchukon cuts occur in $I_+$ ($I_-$) when the roots in Eq. (3.9) with $\epsilon=+1$ ($-1$) take on the value $t^{1/2}$. The branch points occur at the solutions of

$$t^{1/2}=x_\pm=\mp\frac{\gamma}{g'(\frac{1}{4}t)}\pm\left[\left(\frac{\gamma}{g'(\frac{1}{4}t)}\right)^2+\frac{2}{g'(\frac{1}{4}t)}[j-1-2g(\tfrac{1}{4}t)+\tfrac{1}{2}tg'(\tfrac{1}{4}t)]\right]^{1/2}, \tag{3.11}$$

which are $j=1\pm\gamma t^{1/2}+2g(\frac{1}{4}t)=2\alpha_\pm(\frac{1}{4}t)-1$. This is

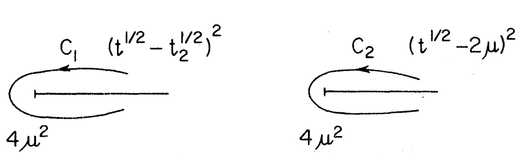


FIG. 1. Contours for the integral $I_{12}(t^{1/2},j)$.

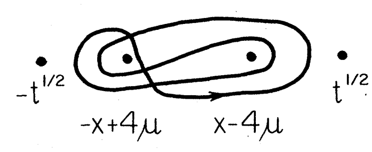


FIG. 2. Contour in the $y$ plane for Eq. (3.3).

the expected result. It will be convenient to regard the two-Pomeranchukon cuts as two cuts that move with $j$ in the complex $E=t^{1/2}$ plane, rather than as cuts that move with $t$ in the angular momentum plane. The integrals $I_\pm$ have cuts starting at $E=\pm E(j)$, where $E(j)$ satisfies the functional equation

$$\gamma E(j)=j-1-2g(\tfrac{1}{4}E^2(j)) \ . \tag{3.12}$$

The asymmetry in these cuts confirms that $I_\pm$ should be regarded as functions of $E$ rather than $t$.

In order to compute the discontinuity across the cut of $I_+$, we must study the motion of $x_+$ as $E$ passes above and below $E(j)$. This is shown in Fig. 3, together with the distortion of the integral over $x$ in Eq. (3.10). The discontinuity is

$$I_+(E+i\epsilon,j)-I_+(E-i\epsilon,j)$$
$$=\frac{2i\pi}{g'(\frac{1}{4}t)}\int_{x_+}^{t^{1/2}}\frac{dx(x^2-\lambda_+)}{[(t-x^2)(t-\lambda_+)(-\lambda_+)]^{1/2}} \ . \tag{3.13}$$

The integral $I_-$ is most easily handled by continuing Eq. (3.5) from $E$ to $-E$. We find

$$I_\pm(-E,j)=I_\mp(E,j)+I_1(E^2,j)\ , \tag{3.14}$$

where $I_1$ has no cuts that need be considered. The sum $I_+(E,j)+I_-(E,j)$ is an even function of $E$, so this linear combination has no branch point at $t=0$. The only cut of $I_++I_-$ in the $t$ plane begins at $t_c(j)\equiv E^2(j)$, with discontinuity given by Eq. (3.13). Thus, in the sum, both two-Pomeranchukon cuts in the angular momentum plane are represented by a single moving cut in the $t$ plane. The unitarity study of Gribov *et al.* does not determine the relative weight to be given to the two integrals $I_+$ and $I_-$ in the function $S$ of Eq. (3.1). The correct combination, the sum, is dictated by the requirement that $f(t,j)$ be analytic at $t=0$.

The integral in Eq. (3.13) cannot be evaluated in terms of elementary functions. However, a simple expression emerges if we regard $t^{1/2}g'(\frac{1}{4}t)$ as a small parameter, and evaluate the integral to lowest order in the parameter. It is natural to look for self-consistent solutions for values of $t$ where the parameter is small, because the condition $|t^{1/2}g'(\frac{1}{4}t)|\ll c$ can be integrated to read $|g(t)|\ll c\,|t^{1/2}|$. The parameter is small where the Schwarz trajectories are nearly exact. To proceed with this program, we note that to lowest order in the parameter, $x_+$ is independent of $t$. This means that in Eq. (3.13) it is sufficient to make an approximation to $\lambda_+$ that is uniformly accurate to lowest order in the parameter for $x$ in the interval $-t^{1/2}\leqslant x\leqslant t^{1/2}$. Such an approxima-

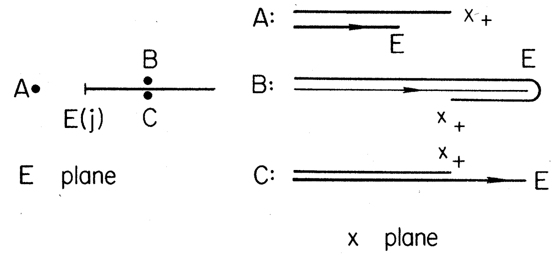


FIG. 3. Distortion of the integration contour in Eq. (3.10) near the branch point at $E=E(j)$. The discontinuity does not change if the branch line of $I_0(\lambda_+)$ is dressed under the integration contour.

tion is achieved by dropping the term proportional to $x^2-t$ in Eq. (3.6). We can also drop $x^2$ and $t$ relative to $\lambda_+$ in Eq. (3.13). The final result is

$$S(t+i\epsilon,j)-S(t-i\epsilon,j)$$
$$=\frac{2i\pi}{g'(\frac{1}{4}t)}\int_{x_+}^{t^{1/2}}\frac{dx}{(t-x^2)^{1/2}}$$
$$=\frac{2i\pi}{g'(\frac{1}{4}t)}\sin^{-1}\left(\frac{\gamma^2t-[j-1-2g(\frac{1}{4}t)]^2}{\gamma^2t}\right)^{1/2} . \tag{3.15}$$

We can simplify Eq. (3.15) still further by evaluating $g'(\frac{1}{4}t)$ at $t=t_c(j)$, and by expanding the argument of the inverse sine about $t_c(j)$. Both of these approximations are consistent with the paper of Gribov *et al.*, where only leading terms at the Regge-cut threshold are retained. We write the inverse to the Schwarz-pole trajectories as

$$t(j)=\frac{(j-1)^2}{\gamma^2}[1+(j-1)h(j)]^2\ , \tag{3.16}$$

where $t(j)$ satisfies the functional equation

$$\gamma^2t(j)=[j-1-g(t(j))]^2\ . \tag{3.17}$$

The two-Pomeranchukon branch point in the $t$ plane is related to the inverse pole trajectory by $t_c(j)=4t[\frac{1}{2}(j+1)]$. In terms of the inverse trajectory, the small parameter is $(j-1)h(j)$. The forementioned approximations are

$$g'(\tfrac{1}{4}t_c(j))=-\gamma^2[h(\tfrac{1}{2}(j+1))+\tfrac{1}{4}(j-1)h'(\tfrac{1}{2}(j+1))]\ , \tag{3.18}$$
$$\gamma^2t-[j-1-2g(\tfrac{1}{4}t)]^2\approx\gamma^2[t-t_c(j)]\ .$$

Using these approximations, the function $S(t,j)$ becomes

$$S(t+i\epsilon,j)-S(t-i\epsilon,j)=\frac{2i\pi}{g'(\frac{1}{4}t_c(j))}\sin^{-1}\left(1-\frac{t_c(j)}{t}\right)^{1/2}, \tag{3.19}$$
$$S(t,j)=\frac{-\pi}{g'(\frac{1}{4}t_c(j))}\ln\{t_c^{1/2}(j)+[t_c(j)-t]^{1/2}\}\ .$$

The discontinuity of $S$ vanishes at $t_c(j)$, in contrast to the findings of Gribov *et al.* This is a reflection of the modification of two-Pomeranchukon phase space that occurs for Schwarz trajectories. An even more dramatic change occurs for the "exact" case $g\equiv 0$: $S$ becomes infinite. As might be expected, this behavior is due to a change in the character of the discontinuity. For $g\equiv 0$, we can easily evaluate the integral in Eq. (3.5), and $I_+$ turns out to have the singularity

$$\frac{i\pi}{\gamma^2[(j-1)^2-\gamma^2]^{1/2}}\oint\frac{dy[(j-1)^2-\gamma^2 y]}{(t-y^2)^{1/2}}. \quad (3.20)$$

The contour integral is zero, but expression (3.20) is still very suggestive. There is no longer a pinch at $y=0$; instead, the singularity occurs at $t=t_c(j)$ for all values of $y$. This accounts for the change in the character of the discontinuity. There are two problems with the exact case. The fact that $y$ is not pinched means that Gribov's Reggeon production amplitude is a function of $y$ as well as $t$, and the derivation of Eq. (3.1) fails. The fact that $\alpha_\pm(t)$ have no elastic branch points means that the cut discontinuity of $f(t,j)$ cannot be separated from that of $f^{(4)}(t,j)$; $f^{(4)}$ is the continuation of $f$ around the four-particle unitarity branch point. For these reasons, and those discussed in Sec. II, we must exclude the case $g=0$.

## IV. THE SELF-CONSISTENT AMPLITUDE AND CROSSED-CHANNEL SCATTERING

Self-consistency requires that the partial-wave amplitude given by Eqs. (3.1), (3.18), and (3.19) have a pole at $t=t(j)$. Equation (2.2) indicates that the residue should vanish linearly at $j=1$. By elimination of simpler cases, we have found that the simplest choices for the meromorphic functions $A$ and $B$ in Eq. (3.1) are the rational functions

$$\begin{aligned}A(t,j)&=(a+\cdots)[c_1t+c_2(j-1)^2+\cdots]^{-2},\\ B(t,j)&=[\gamma^2t-(j-1)^2+\cdots](j-1)^{-1}\\ &\quad\times[c_1t+c_2(j-1)^2+\cdots]^{-2}.\end{aligned} \quad (4.1)$$

Here we have ignored multi-Pomeranchukon cuts and possible fixed $j$ cuts. As we emphasized in Paper I, the moving pole in $A$ must be second order, because the amplitude for two particles → two Pomeranchukons is $A^{1/2}(B+S)^{-1}$, and we do not want there to be a kinematic moving cut in this amplitude. Two changes have been made in taking over the functions $A$ and $B$ from Paper I: The linear factors in $j-1$ have been omitted from the moving pole and the numerator of $B$, and an extra factor $j-1$ has been inserted in the denominator of $B$. These changes are made so that the two-Pomeranchukon cut has the Mandelstam sign, and so that when we set $S=0$ we recover Eq. (2.2).

The elastic scattering amplitude may be written

$$\begin{aligned}f(t,j)&=-(a+\cdots)(j-1)D^{-1}(t,j),\\ D(t,j)&=(j-1)^2-\gamma^2t\\ &\quad-[c_1t+c_2(j-1)^2+\cdots]^2(j-1)S(t,j).\end{aligned} \quad (4.2)$$

The self-consistency equation for the Pomeranchukon poles in the presence of the two-Pomeranchukon cuts is $D(t(j),j)=0$. Using Eqs. (3.16), (3.19), and (4.2), near $j=1$ the self-consistency equation takes the form

$$0=-2(j-1)^3h(j)-\frac{\pi}{\gamma^2}\left(\frac{c_1}{\gamma^2}+c_2\right)^2\frac{(j-1)^5\ln(j-1)}{h(\frac{1}{2}(j+1))+\frac{1}{4}(j-1)h'(\frac{1}{2}(j+1))}. \quad (4.3)$$

Here we have omitted terms in the argument of the logarithm that we believe (and can later verify) to be unimportant near $j=1$. The leading behavior of $h(j)$ near $j=1$ is determined by Eq. (4.3) to be

$$\begin{aligned}h(j)&=-\rho(j-1)[-\ln(j-1)]^{1/2},\\ \rho&=(2\pi/3\gamma^2)^{1/2}\,|c_2/\gamma^2+c_1|.\end{aligned} \quad (4.4)$$

We have chosen the negative root so that we satisfy the requirement $\mathrm{Re}\,\alpha_\pm(t)\le 1$ for $t\le 0$. Equation (3.16) can now be inverted near $t=0$, and the leading behavior of the Schwarz trajectories is

$$\alpha_\pm(t)=1\pm\gamma t^{1/2}+2\rho(\tfrac{1}{2}\gamma^2t)^{3/2}(-\ln t)^{1/2}. \quad (4.5)$$

Equation (4.5) confirms the smallness of the parameter $t^{1/2}g'(\frac{1}{4}t)$; in fact, the parameter vanishes at $t=0$, which is the point of greatest interest.

The multi-Pomeranchukon cuts have been omitted from the self-consistency equation, but they should not be important near $t=0$ because, as in Gribov's original unitarity calculation, multi-Pomeranchukon phase space rises more slowly from threshold when more Pomeranchukon poles are added. This trend should not be modified because we are dealing with Schwarz trajectories. It is based on the fact that a three-Pomeranchukon cut can be regarded as being composed of a pole and a two-Pomeranchukon cut. Since the cut is weaker near threshold than the pole, the three-Pomeranchukon cut must be weaker than the two-Pomeranchukon cut. For the same reason, the fixed $j$ cut introduced via $h(j)$ can be ignored.

The denominator function at $t=0$ is

$$\begin{aligned}D(0,j)&=(j-1)^2\{1-\tau(j-1)^2[-\ln(j-1)]^{1/2}\},\\ \tau&=4\pi c_2^2/3\gamma^2\rho.\end{aligned} \quad (4.6)$$

The Sommerfeld-Watson integral leads to the crossed-channel behavior

$$\sigma_{\rm tot} \underset{s\to\infty}{\sim} \sigma_{\rm tot}(\infty)\left(1-\frac{\tau}{2[\ln(s/s_0)]^2[\ln\ln(s/s_0)]^{1/2}}\right),$$

$$\frac{\mathrm{Re}F(s,0)}{\mathrm{Im}F(s,0)} \underset{s\to\infty}{\sim} \frac{\tau\pi}{2[\ln(s/s_0)]^3[\ln\ln(s/s_0)]^{1/2}}, \qquad (4.7)$$

where $F(s,t)$ is the full amplitude. The two-Pomeranchukon cut contributes negatively to the total cross section, as required by Mandelstam and the absorption model.[3,4] The sign is not adjustable once we have committed $g(t)$ to have the sign indicated in Eq. (4.5).

The diffraction oscillations of $F(s,t)$ will be determined principally by the Pomeranchukon poles for the range of $t$ over which $|g(t)|$ is smaller than $|t|^{1/2}$. In this range the Pomeranchukon cuts do not lead the poles as they did in Paper I; rather, the cuts and poles are coincident, and the greater strength of the poles is decisive. Since $\mathrm{Re}\,\alpha_\pm(t)\approx 1$ in this range, the fixed $j$ cut also can be neglected. We are able to give $F$ over a finite range of $t$ at all energies, and not just in a range $-t\ln(s/s_0) <$ const, as in Paper I. The simplest approximation to $F(s,t)$ is obtained by using Eq. (2.2), ignoring $g(t)$; it is

$$F(s,t)\to(\mathrm{const})is\{\cos[\gamma(-t)^{1/2}\ln s] + i\tanh[\tfrac{1}{2}\pi\gamma(-t)^{1/2}]\sin[\gamma(-t)^{1/2}\ln s]\}. \qquad (4.8)$$

This amplitude shows no tendency to form a forward peak, but this is only because we have ignored $g(t)$. What Eq. (4.8) shows is that at high energy there is an increasingly rapid oscillation of the differential cross section under the diffraction envelope. This is characteristic of the diffraction process, and probably would occur under the assumptions made in Paper I if we could follow the amplitude over a finite range of $t$ at all energies.

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[1]M. Froissart, Phys. Rev. 123, 1053 (1961).

[2]J. B. Bronzan, Phys. Rev. D 4, 1097 (1971).

[3]S. Mandelstam, Nuovo Cimento 30, 1127 (1963); 30, 1148 (1963).

[4]J. Finkelstein and M. Jacob, Nuovo Cimento 56A, 681 (1968).

[5]V. N. Gribov and A. A. Migdal, Yadern. Fiz. 8, 1002 (1968) [Soviet J. Nucl. Phys. 8, 583 (1969)].

[6]J. H. Schwarz, Phys. Rev. 167, 1342 (1968); T. Sawada, *ibid.* 165, 1848 (1968); R. E. Mickens, Nuovo Cimento 56A, 799 (1968).

[7]H. Fujisaki, Rikkyo University Report No. RUP-71-1 (unpublished); J. Finkelstein and F. Zachariasen, Phys. Letters 34B, 631 (1971); J. R. Fulco and R. L. Sugar, Phys. Rev. D 4, 1919 (1971).

[8]A. A. Anselm, G. S. Danilov, I. T. Dyatlov, and E. M. Levin, Yadern. Fiz. 11, 896 (1970) [Soviet J. Nucl. Phys. 11, 500 (1970)]; J. Arafine and H. Sugawara, Phys. Rev. Letters 25, 1516 (1971); V. N. Gribov, I. Yu. Kobsarev, V. D. Mur, L. B. Okun, and V. S. Popov, Phys. Letters 32B, 129 (1970); J. Finkelstein, Phys. Rev. Letters 24, 172 (1970); R. Oehme, Phys. Rev. D 3, 3217 (1971); 4, 1485 (1971).

[9]H. Cheng and T. T. Wu, Phys. Rev. Letters 24, 1456 (1970); S.-J. Chang and T. M. Yau, *ibid.* 25, 1586 (1970).

[10]V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan, Phys. Rev. 139, B184 (1965).

[11]J. B. Bronzan and C. E. Jones, Phys. Rev. 160, 1494 (1967).

[12]R. Oehme, Phys. Rev. D 4, 1485 (1971).

[13]J. B. Bronzan, Phys. Rev. D 4, 2569 (1971).

[14]Yu. A. Simonov, Zh. Eksperim. i Teor. Phys. 48, 242 (1965) [Soviet Phys. JETP 21, 160 (1965)]; J. H. Schwarz, Phys. Rev. 162, 1671 (1967).

[15]Singularities of $I_0(\lambda)$ at $\lambda=t$, produced by pinches at $y=\pm t^{1/2}$, are to be ignored. This is evident in Fig. 2, where the points $y=\pm(x-4\mu)$ lie inside $y=\pm t^{1/2}$. One must make this observation before adopting the expansion of Eq. (3.6), because once the expansion is made the threshold singularities of $g(\frac{1}{4}(x\pm y)^2)$ are removed from the $y$ plane. Since only the pinch at $y=0$ is significant, we have set the limits in Eq. (3.7) at $\pm\delta$.