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*Harkness fellow.

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$$|\Sigma^+\rangle = -|\underline{8};\ Y=0,\ I=I_3=1\rangle;$$

$$|\Xi^-\rangle = -|\underline{8};\ Y=-1,\ I=\tfrac{1}{2},\ I_3=-\tfrac{1}{2}\rangle;$$

all other particle states are related to the corresponding octet states with a positive sign. The charges V^i and A^i are related to the tensor operators in exactly the same way. In the decuplet, Δ^{++}, Y_1^{*-}, Y_1^{*0}, and Ξ^{*0} all have positive signs; the remaining states have negative signs.

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Comments on Statistical Bootstrap Models of Hadrons

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The statistical bootstrap models of Hagedorn and Frautschi, modified so that the volume of a hadron is allowed to vary with the temperature, are considered. It is shown that a large class of polynomial solutions for the level density of hadrons is possible. A feature common to polynomial spectra is that the volume of a hadron must vanish as the temperature approaches infinity. The requirement that hadrons have a finite size implies both a maximum temperature and an exponential hadron mass spectrum. Also, the recent formulation in terms of quasiparticles demands an exponential hadron mass spectrum without requiring an asymptotic bootstrap condition. A unique solution, $\rho(m) \sim m^{-5/2} e^{m\beta}$ as $m \to \infty$, is obtained if one assumes the *asymptotic* bootstrap condition of Hagedorn.

I. INTRODUCTION

Recently there have been several attempts at understanding the dynamics of strongly interacting particles from the statistical point of view.[1-3] Although the different approaches lead to similar results, the underlying features of the models are quite different.

The thermodynamical model of strong interactions and a systematic comparison of theoretical predictions with experiments were started by Hagedorn in 1965. The main success of this program has been to introduce a bootstrap condition in statistical theories of hadrons leading to an exponential hadron mass spectrum with a universal highest temperature.

More recently, Frautschi[2] developed a statistical bootstrap model of hadrons closely related to that of Hagedorn. However, Frautschi opts to work in terms of phase space with explicit momentum conservation. Also, zero- and one-particle states are excluded. The results Frautschi obtains are analogous to those of Hagedorn.

The work of Ref. 3 is motivated somewhat differently. Its main effort is to describe an interacting system of particles which is localized in space, this being the main feature of a physical particle. This is accomplished by introducing distinguishable quasiparticles as constituents of a hadron. If the quasiparticles are treated quantum-mechanically the hadron mass spectrum rises faster than exponentially but less than or equal to $\exp(\text{const}\, m \ln m)$. On the other hand, if the quasiparticles are treated classically one obtains an exponential hadron mass spectrum as in previous works.

In this note the statistical bootstrap models of Hagedorn and Frautschi are examined mathematically under the condition of a temperature-dependent volume. Such freedom gives a physical insight for the existence of a maximum temperature. It is shown that if one requires particles to have a finite nonzero size then a maximum temperature necessarily emerges. This connection was first made in the work of Ref. 3 and serves to relate mathematically the values of the maximum temperature, the volume of a hadron, and the lowest mass of the system. [See Eq. (12) of Ref. 3.] Also, one finds the rather interesting result that the approach of Ref. 3 when extended to finite-width particles leads to an exponential hadron mass spectrum without requiring a bootstrap condition. Recall that exponential spectra were obtained in Ref. 3 only if an asymptotic bootstrap condition was assumed. If one still insists on an *asymptotic* bootstrap condition then a unique spectrum, $\rho(m) \sim m^{-5/2} e^{m\beta}$ as $m \to \infty$, is obtained.

II. STATISTICAL MODEL OF FRAUTSCHI

The work of Ref. 2 is based on the phase-space integral

$$\rho_{\text{out}}(m)=\sum_{n=2}^{\infty}\left(\frac{V}{h^3}\right)^{n-1}\frac{1}{n!}\prod_{i=1}^{n}\int dm_i \rho_{\text{in}}(m_i)\int d^3p_i\,\delta\Big(m-\sum_{i=1}^{n}E_i\Big)\delta^3\Big(\sum_{i=1}^{n}\vec{p}_i\Big), \tag{1}$$

with solutions satisfying the asymptotic bootstrap condition

$$\rho_{\text{out}}(m)\sim\rho_{\text{in}}(m) \quad \text{as} \quad m\to\infty, \tag{2}$$

where $\rho_{\text{in}}(m)$ is the single-particle density.

Frautschi finds that any solution must grow essentially exponentially as $m\to\infty$. We shall relax the condition that the volume V is a constant and allow it to vary with the temperature. Define

$$\rho_{\text{out}}(m,\vec{P})=\sum_{n=2}^{\infty}\left(\frac{V}{h^3}\right)^{n-1}\frac{1}{n!}\prod_{i=1}^{n}\int dm_i\rho_{\text{in}}(m_i)\int d^3p_i\,\delta\Big(m-\sum_{i=1}^{n}E_i\Big)\,\delta^3\Big(\sum_{i=1}^{n}\vec{p}_i-\vec{P}\Big). \tag{3}$$

One recovers Frautschi's $\rho_{\text{out}}(m)$ for $\vec{P}\equiv 0$. Now, for $\alpha>0$,

$$\begin{aligned}\int_0^{\infty}e^{-m\alpha}\int e^{i\vec{P}\cdot\vec{r}}\rho_{\text{out}}(m,\vec{P})d^3P\,dm&=\sum_{n=2}^{\infty}\left(\frac{V}{h^3}\right)^{n-1}\frac{1}{n!}[R(\alpha,r)]^n\\&=\frac{h^3}{V}\left(e^{VR/h^3}-1-\frac{VR}{h^3}\right),\end{aligned} \tag{4}$$

where the real function $R(\alpha,r)$ is defined by

$$R(\alpha,r)\equiv\int_0^{\infty}dm\rho_{\text{in}}(m)\int d^3p\,e^{-E(p)\alpha}e^{i\vec{p}\cdot\vec{r}}=\frac{4\pi\alpha}{\alpha^2+r^2}\int_0^{\infty}dm\rho_{\text{in}}(m)m^2K_2(m(\alpha^2+r^2)^{1/2}). \tag{5}$$

Suppose $\rho_{\text{in}}(m)$ is a polynomial in m, that is,

$$\rho_{\text{in}}(m)=\sum_l a_l m^l. \tag{6}$$

Let us calculate $\rho_{\text{out}}(m)$. Now,

$$\begin{aligned}R(\alpha,r)&=\sum_l\frac{4\pi\alpha}{\alpha^2+r^2}a_l\int_0^{\infty}dm\;m^{2+l}K_2(m(\alpha^2+r^2)^{1/2})\\&=\frac{4\pi\alpha}{\alpha^2+r^2}\sum_l\frac{2^{l+1}a_l}{(\alpha^2+r^2)^{(l+3)/2}}\Gamma\left(\frac{l+5}{2}\right)\Gamma\left(\frac{l+1}{2}\right).\end{aligned} \tag{7}$$

On integrating (4) with respect to $\vec{r}$ and using (7),

$$(2\pi)^3\int_0^{\infty}e^{-m\alpha}\rho_{\text{out}}(m)dm=\sum_{n=2}^{\infty}\left(\frac{V}{h^3}\right)^{n-1}\frac{1}{n!}4\pi\int_0^{\infty}r^2dr\left[\frac{4\pi\alpha}{\alpha^2+r^2}\sum_l\frac{2^{l+1}a_l}{(\alpha^2+r^2)^{(l+3)/2}}\Gamma\left(\frac{l+5}{2}\right)\Gamma\left(\frac{l+1}{2}\right)\right]^n. \tag{8}$$

Suppose we have a solution which satisfies the strong bootstrap condition,[4] that is, $\rho_{\text{in}}(m)=\rho_{\text{out}}(m)$. It is clear that the existence of such a solution would imply the existence of a solution satisfying the asymptotic bootstrap condition (2). Then, $\rho_{\text{out}}(m)=\sum a_l m^l$ so that (8) becomes

$$(2\pi)^3\sum_l a_l\,\frac{V\Gamma(l+1)}{\alpha^{l+4}}=\sum_{n=2}^{\infty}\frac{1}{h^{3(n-1)}}\frac{4\pi}{n!}\int_0^{\infty}x^2\,dx\left[\sum_l\frac{4\pi 2^{l+1}}{(1+x^2)^{(l+5)/2}}\frac{Va_l}{\alpha^{l+4}}\Gamma\left(\frac{l+5}{2}\right)\Gamma\left(\frac{l+1}{2}\right)\right]^n. \tag{9}$$

Note the appearance of the factor Va_l/α^{l+4} on both sides of (9). This equation should hold for *all* values of α; otherwise, the bootstrap condition $\rho_{\text{out}}(m)=\rho_{\text{in}}(m)$ cannot be satisfied.

Equation (9) implies, for a given input spectrum, that is, a given set of (positive) numbers $\{a_l\}$, a relationship between V and α, $V=V(\alpha)$. Therefore, for (6) to be a bootstrap solution the volume *must* be a function of the temperature. If, however, the volume *does not* depend on the temperature then (9) *cannot* hold for all values of α and (6) cannot be a bootstrap solution.

The above analysis can be elucidated for the simple case when $\rho_{\text{in}}(m)=C$, a constant. Then (8) becomes

$$(2\pi)^3\int_0^{\infty}e^{-m\alpha}\rho_{\text{out}}(m)\,dm=\sum_{n=2}^{\infty}\left(\frac{V}{h^3}\right)^{n-1}\frac{2\pi}{n!}\frac{(6\pi^2C)^n}{\alpha^{4n-3}}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2}n-\frac{3}{2})}{\Gamma(\frac{5}{2}n)}. \tag{10}$$

If the volume V is *independent* of the temperature α^{-1}, then (10) can be inverted and gives

$$(2\pi)^3\rho_{\text{out}}(m)=\sum_{n=2}^{\infty}\left(\frac{V}{h^3}\right)^{n-1}\frac{2\pi}{n!}\frac{(6\pi^2C)^n}{\Gamma(4n-3)}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2}n-\frac{3}{2})}{\Gamma(\frac{5}{2}n)}m^{4n-4}. \tag{11}$$

Therefore, we *do not* have a bootstrap solution; that is, $\rho_{\text{out}}(m)\neq\rho_{\text{in}}(m)$. However, assume a bootstrap solution $\rho_{\text{out}}(m)=\rho_{\text{in}}(m)=C$; then (10) becomes

$$(2\pi)^3C=\sum_{n=2}^{\infty}\left(\frac{V}{\alpha^4h^3}\right)^{n-1}\frac{2\pi}{n!}(6\pi^2C)^n\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2}n-\frac{3}{2})}{\Gamma(\frac{5}{2}n)}, \tag{12}$$

which has two possible solutions for C; a trivial solution, $C\equiv 0$, and a nontrivial solution $VC/\alpha^4h^3=$ pure real number. Therefore, if the volume is related to the temperature as $V\propto T^{-4}$, then a bootstrap solution exists with $\rho_{\text{in}}(m)=\rho_{\text{out}}(m)=\text{const}$. This can be seen also directly from (1) provided one introduces properly the dependence of the volume on the temperature. This is accomplished by replacing the volume in (1) by the differential operator $(h^3/C)A(d^4/dm^4)$, where $A\equiv VC/\alpha^4h^3$ is the pure real number determined by (12).

It has been reported recently[1] that Nahm has established analytically that $\rho(m)\sim m^{-3}e^{m/T_0}$, as $m\to\infty$, by studying the singularity of the partition function, $Z(T)$, for $T\to T_0$. The solutions found above *do not* have a singularity at $T=T_0$, or better, $T_0=\infty$. The works of Refs. 1 and 2 only considered solutions with a singularity in $Z(T)$, that is, the exponential hadron mass spectrum, and omitted those which give rise to a $Z(T)$ which is regular in T. Note, however, that in Ref. 3 the solutions for $Z(T)$ must always be singular and, hence, nonexponential hadron mass spectra are ruled out (see Sec. IV). In Ref. 3 [see Eq. (12) therein] it was assumed that the volume must be finite which implied both a maximum temperature and an exponential spectrum.

III. THERMODYNAMICS OF STRONG INTERACTIONS

In his formulation of statistical thermodynamics of strong interactions, Hagedorn posed the following mathematical problem[5]:

$$\int_0^{\infty}\sigma(m,V_0)e^{-m/T}dm=\exp\left[\frac{V_0T}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\infty}\rho(m;n)m^2K_2\left(\frac{nm}{T}\right)dm\right], \tag{13}$$

with

$$\frac{\ln\rho(m)}{\ln\sigma(m,V_0)}\to 1\quad\text{as}\quad m\to\infty, \tag{14}$$

where

$$\rho(m;n)\equiv\rho_B(m)-(-1)^n\rho_F(m)\equiv\begin{cases}\rho(m) & \text{for } n \text{ odd}\\ \Delta\rho(m) & \text{for } n \text{ even}\end{cases} \tag{15}$$

and

$$V_0\equiv\tfrac{4}{3}\pi m_\pi^{-3}.$$

Hagedorn argues that for $\rho(m)=3\delta(m-m_\pi)$, $\sigma(E,V_0)$ grows as $\exp(\text{const}E^{3/4})$ and the self-consistency condition (14) can never be established. Frautschi[2] also follows this line of reasoning and concludes that $\rho_{\text{out}}(m)$ must grow at least this fast. Both proceed to give proofs that their respective statistical bootstrap models do not possess solutions which grow less than exponentially.

In this section we shall construct polynomial solutions for the level density which allow for a dependence of the volume on the temperature. Suppose

$$\Delta\rho(m)=0 \quad\text{and}\quad \rho(m)=\sum_l a_l m^l\,. \tag{16}$$

We shall impose the strong bootstrap condition $\sigma(E,V_0)=\rho(E)$, which is more stringent than (14). Then

$$\sum_l a_l\Gamma(l+1)T^{l+1}=\exp\left[\frac{V_0}{2\pi^2}\sum_l a_l\,\Gamma\left(\frac{l+5}{2}\right)\Gamma\left(\frac{l+1}{2}\right)2^{l+1}T^{l+4}\sum_{n\text{ odd}}^{\infty}\frac{1}{n^{l+5}}\right] \tag{17}$$

for *all* values of T. Given a bootstrap solution, a given set of (positive) numbers $\{a_l\}$, we obtain a solution $V_0=V_0(T)$ determined by (17). Therefore, if V_0 is allowed to vary with the temperature T as $V_0=V_0(T)$ then one has a nonexponential bootstrap solution to Hagedorn's mathematical problem.

Consider the simple case when $\rho(E)=\sigma(E,V_0)=C$, a constant. Then (17) becomes

$$CT=\exp[(3/4\pi)V_0CT^4U_5]\,, \tag{18}$$

where $U_5\equiv\sum_{n\text{ odd}}^{\infty}(1/n^5)$ and (18) is to hold for *all* values of T. Equation (18) possesses a unique solution given by

$$\frac{3}{4}\frac{V_0C}{\pi}T^4U_5=1, \tag{19}$$

so that for a given bootstrap solution $\rho(E)=\sigma(E,V_0)=C$, V_0 *must* be related to the temperature T by (19). The functional relationship between V_0 and T is as in the corresponding case in the previous section.

Note that, as in the analysis of the previous section, if V_0 is not allowed to depend on T, then bootstrap solutions of the type (16) could not exist. However, as the model of Ref. 3 and the numerical work of Ref. 4 indicate, bootstrap solutions *do* relate V_0 and T. In fact, the result of Ref. 3 relates T and V_0 as follows:

$$1=\frac{4\pi V_0}{(hc)^3}\frac{(m_\pi c^2)^2}{\beta}K_2(m_\pi c^2\beta)\,, \tag{20}$$

where m_π is the pion mass and $\beta=(kT)^{-1}$.

IV. QUASIPARTICLE METHOD

The quasiparticle method of describing a system of strongly interacting particles was introduced in Ref. 3. There one wrote formulas in terms of a mass spectrum of the form

$$\rho(m)=\sum_i Z_i\delta(m-m_i)+\rho_c(m)\,, \tag{21}$$

where Z_i denotes the multiplicity of the low-lying mass states and $\rho_c(m)$ denotes the continuous part of the spectrum.

In reality, particles do not have a well-defined mass; hence one should write formulas in terms of a continuous spectrum only. Also, this will allow for a more direct comparison of the work of Ref. 3 to that of Frautschi and of Hagedorn [(1) and (13) of the present paper], which are already appropriate for theories where particles are not treated as infinitely narrow resonances.

The fundamental equation (6) of Ref. 3 is

$$\int_0^\infty e^{-m\alpha}\omega(m)dm=\exp\left[-\int_{m_0}^\infty\rho(m)dm\ln\left(1-\frac{V}{h^3}\int d^3p\times\exp[-\alpha(p^2+m^2)^{1/2}]\right)\right]. \tag{22}$$

This is the analog of (1) and (13) in Frautschi's and Hagedorn's work, respectively.

In the previous sections it was shown that (1) and (13), together with the asymptotic bootstrap condition, allow solutions which do not behave exponentially for large values of the mass, if the volume is allowed to depend on the temperature. We shall now show that (22) gives rise to an exponential hadron mass spectrum *without* using the asymptotic bootstrap condition (14). If, in addition, the asymptotic bootstrap condition is required, then the solution to (22) is unique for large values of the mass.

The right-hand side of (22) requires

$$1-\frac{4\pi V}{h^3}m_0^{\,2}\frac{K_2(m_0\alpha)}{\alpha}\geq 0\,. \tag{23}$$

Therefore, if the temperature α^{-1} is allowed to go to infinity, then the volume of a hadron will vanish as T^{-3} or faster as $T\to\infty$. Hence one must require that

$$\frac{h^3\alpha}{4\pi m_0^{\,2}K_2(m_0\alpha)}\geq V\geq V_0\neq 0\,. \tag{24}$$

That is, there exists a nonzero β such that

$$1 - \frac{4\pi}{h^3} V_0 \frac{m_0^2 K_2(m_0\beta)}{\beta} = 0, \tag{25}$$

which implies that $\alpha \geqslant \beta > 0$. (The maximum temperature β^{-1} is to be determined experimentally from the hadron level density.) One must insist that the volume of a hadron *cannot* vanish. This, in turn, implies the existence of a highest temperature. This requirement gives the relationship (25) between the maximum temperature β^{-1}, the threshold mass m_0 (or "lowest" mass), and the volume V_0 of a hadron.

The left-hand side must also be valid only for $\alpha \geqslant \beta \neq 0$. The density of states $\omega(m)$ *cannot* be a polynomial because a polynomial would give for the left-hand side a function valid for $\alpha > 0$. The consistency of both sides of (22) *requires* $\omega(m) \sim g(m)e^{m\beta}$ as $m \to \infty$ with $g(m) = o(e^{m\epsilon})$ as $m \to \infty$ with $\epsilon > 0$. However, such behavior for $\omega(m)$ will make the left-hand side diverge for $\alpha \to \beta$. [For $g(m) \to m^{-1/4}e^{2\sqrt{m}}$ as $m \to \infty$ one has an essential singularity for $\alpha \to \beta$.] That is, one must require that $\rho(m) \sim m^b e^{m\beta}$ as $m \to \infty$ with $b \geqslant -\frac{5}{2}$. The right-hand side would then behave as $\alpha \to \beta$:

$$\lim_{\alpha \to \beta} \exp\left\{ -\int_{m_0}^{\infty} \rho(m)\, dm \ln\left[1 - \frac{4\pi V_0}{h^3} \frac{m^2}{\alpha} K_2(m\alpha)\right]\right\}$$

$$= \begin{cases} B \exp A(\alpha - \beta)^{-5/2-b}, & b > -\frac{5}{2} \quad \text{(26a)} \\ D(\alpha - \beta)^{-C}, & b = -\frac{5}{2} \quad \text{(26b)} \end{cases}$$

where A, B, C, and D are positive constants. Hence,

$$mg(m) \underset{m \to \infty}{\sim} \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{BA^n m^{(5/2+b)n}}{n!\,\Gamma((\frac{5}{2}+b)n)}, & b > -\frac{5}{2} \quad \text{(27a)} \\ [D/\Gamma(C)]\, m^C, & b = -\frac{5}{2}. \quad \text{(27b)} \end{cases}$$

It is quite important that for $b > -\frac{5}{2}$, $g(m) = o(e^{m\epsilon})$ as $m \to \infty$ with $\epsilon > 0$; otherwise the integral on the left-hand side of (22) will not exist. We want to show that the integral function $g(m)$ given by (27a) is of finite order as $m \to \infty$; that is, $g(m) = O(e^{m^\gamma})$ as $m \to \infty$ with $\gamma > 0$. One has that[6] $g(m) = O(e^{m^\gamma})$ as $m \to \infty$ with $\gamma = (\frac{5}{2}+b)/(\frac{7}{2}+b)$. Since $\frac{5}{2}+b > 0$ then $g(m)e^{-m\epsilon} \to 0$ as $m \to \infty$ for $\epsilon > 0$.

This result is rather interesting since one has obtained that $\rho(m)$ *must* behave exponentially for large values of the mass *without* requiring *any* bootstrap condition whatsoever. Recall that in previous works[1-3] the appearance of an exponential hadron mass spectrum cannot follow without the assumption of the bootstrap condition of Hagedorn. We see that in the quasiparticle formalism with *finite-width* particles the bootstrap condition was not invoked; nevertheless the hadron mass spectrum behaves exponentially for large values of the mass.

If, in addition, one requires the *asymptotic* bootstrap condition (14), then the *only* solution which satisfies this added restriction is $\rho(m) \to m^{-5/2}e^{m\beta}$. Therefore, in the quasiparticle formalism with an asymptotic bootstrap condition one obtains a unique behavior for the hadron mass spectrum at large values of the mass. In the works of Nahm, Hamer, and Frautschi, a unique solution is obtained if one adds a strong asymptotic bootstrap condition and does not include the zero- and one-particle terms.

V. CONCLUSION

The existence of polynomial hadron mass spectra in the statistical bootstrap models studied by Hagedorn, Frautschi, and Alexanian demands a temperature-dependent volume. This dependence is such that as the temperature approaches infinity the volume vanishes. The necessity of requiring a maximum temperature in order to exclude point particles from the theory was already derived in Ref. 3 in relation to the quasiparticle formalism. In this note it is shown that the statistical bootstrap models of Hagedorn and Frautschi, when extended to include temperature-dependent volume, also have the feature of relating the existence of a maximum temperature with the absence of point particles.

Temperature is related to average energy; thus polynomial hadron spectra require a dependence of the geometrical size of a hadron on average energy. However, all statistical bootstrap models so far investigated assume a *constant* volume for a hadron. In the work of Ref. 3 the volume is related to the hadron mass spectrum through the parameter β in (25). It is clear that a connection between the hadron mass spectrum and the geometrical size of a particle has been established. If the hadron spectrum behaves exponentially for large values of the mass then one has no point particles and the geometrical size of the particle is determined by (25). However, the possibility still remains that at high energies particles may behave like point particles, in which case the hadron mass spectrum will behave as a polynomial for large values of the mass. Recent experimental data may indicate such a behavior.[7] However, it is not clear how to interpret theoretically the experimental data. Nevertheless, a volume depending on the energy (or temperature) may be required by future experiments.

The question of the width of particles is an important point which emerges out of the present works. In Refs. 1 to 3 one dealt primarily with zero-width resonances in the following sense.

Hagedorn and Frautschi suppose a spectrum increasing faster than $\exp(\mathrm{const}\, m^{3/4})$, a result based on the existence of at least one zero-width particle. However, their asymptotic bootstrap condition, of course, makes no mention of the low-mass states. This is what allowed us to obtain a nonexponential hadron mass spectrum. Therefore, in order to derive the geometrical size of a particle from the works of Hagedorn and Frautschi one *must* introduce a further postulate, for example, a stronger bootstrap condition, as done by Hamer and Frautschi,[4] in order to introduce information about the low-mass states. Note that this may also serve as a means of obtaining a unique solution, as Hamer and Frautschi claim. Of course, one may abandon giving information about low-mass states and instead insist that the geometrical size of a particle should not vanish, in which case exponential hadron spectra are obtained with perhaps a relationship between the parameters of the spectra and the geometrical size of a hadron. On purely physical arguments it is clear that there should be a connection between the existence of low-mass states and the fact that particles have a finite geometrical size.

The different cases (i), (ii), and (iii) discussed in Ref. 3 were also based on the existence of low-lying zero-width resonances. There one needed the asymptotic bootstrap condition to obtain an exponential hadron mass spectrum. We have seen in this note that if the idea of zero-width resonances is totally abandoned in favor of the more realistic finite-width resonances, then there exists no singularity in the partition function except that due to the massive states, case (iii) in Ref. 3, and such singularity emerges *without* using a bootstrap condition.

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Two-Particle Discontinuities of One-Loop Graphs in the Dual-Resonance Model*

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The two-particle discontinuities of the planar and nonplanar single-loop graphs are calculated for $s < 4(10-\alpha_0)$ BeV^2 for several values of the Regge intercept α_0, of the dimensionality D of the oscillators, and of the squared momentum transfer t. The results are not at all what was expected.

I. INTRODUCTION

Several years ago Kikkawa, Sakita, and Virasoro (KSV)[1] proposed that unitarity be incorporated in dual-resonance models (DRM) by regarding the N-point beta function as the Born approximation to a complete theory for the strong-interaction scattering amplitude. The full amplitude would be a sum over Feynman-like multiloop diagrams defined to have their discontinuities cor-