# Critical Remarks on the Asymptotic Symmetry Scheme

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(Received 2 June 1971)

The asymptotic symmetry scheme proposed by Oneda and Matsuda is analyzed rather critically on two counts. First of all, we discuss the underlying reasons given for the scheme's basic assumption, that single-particle matrix elements of the SU(3) raising operator $V^K$ are effectively not renormalized, even when the symmetry is broken, and we find that some of the arguments are not completely convincing. Second, it is shown that most of the results of the scheme can be obtained without using this basic assumption, and that it is only most of the results derived from equal-time commutators of the form $[\dot{V}^K, A^i]=0$ which are necessary consequences of this assumption. These latter results include all of their intermultiplet mass sum rules together with a few other predictions such as the degeneracy of the $\Sigma^0$ and $\Lambda^0$, and, in general, these are not as well satisfied experimentally as the other results.

## I. INTRODUCTION

In a recent series of papers, Oneda and Matsuda[1] have defined a particular scheme of asymptotic symmetry and used it to derive various sum rules for masses and coupling constants in broken SU(3) and SU(2) symmetry. Agreement with experimental data is reasonably good for the coupling constants and for several of the mass sum rules, although a few mass formulas involving baryons are not quite as successful; in particular, $\Sigma^0$ and $\Lambda^0$ are mass-degenerate, even in broken SU(2) symmetry. With this type of situation, it is rather difficult to assess the merits of the scheme since it yields both "good" and "not-so-good" predictions. Accordingly, it is interesting to examine carefully the various assumptions used and to find out if some results can be obtained without using all of the assumptions.

The scheme is distinguished by one particular assumption [hereafter denoted by (A)] which basically states that in the infinite-three-momentum frame, in broken symmetry, the values of single-particle matrix elements of the SU(3) raising and lowering operators $V^K$ are effectively unchanged from the corresponding values in the SU(3) limit. A more exact statement of this assumption will be given in Sec. II, but we point out here that normally, matrix elements involving two states which would belong to different irreducible representations (IR) of SU(3) in the symmetric limit, i.e., off-diagonal matrix elements, are of order $O(\epsilon)$ while diagonal elements differ from their symmetric values by a term at most of order $O(\epsilon^2)$ (Ademollo-Gatto theorem),[2] where $\epsilon$ is the small symmetry-breaking parameter. In addition, mass differences within an SU(3) multiplet are taken to be of order $O(\epsilon)$ so that if one uses a formal counting procedure in orders of $\epsilon$, then terms of order $O(\epsilon)$ can normally arise either from mass differences or from off-diagonal matrix elements.

Other assumptions made by Oneda and Matsuda are standard ones. The first two current-algebra[3] relations are used, which means that the charges $A^i(t)$ of the axial-vector currents $A^i_\mu(\vec{x},t)$ are octet operators of the SU(3) group generated by the vector charges $V^i(t)$. Also, in general, the symmetry-breaking part of the Hamiltonian density transforms as the eighth component of an octet operator,[4] although occasionally the operator is slightly restricted so that the equal-time commutator (ETC) $[\dot{V}^{K^+}, A^{\pi^+}]$ vanishes (with $\dot{V}^{K^+}$ being the time derivative of $V^{K^+}$).

To obtain relations involving coupling constants or masses, ETC's such as those mentioned above are sandwiched between one-particle states whose three-momentum is infinite, and the sum over intermediate states is truncated by the use of (A), i.e., all terms involving off-diagonal $V^K$ matrix elements are immediately neglected.

However, simply by using formal counting in orders of $\epsilon$, we can show that many of the results of Oneda and Matsuda do not require (A) at all; in fact, coupling-constant sum rules derived from the current-algebra ETC's $[V^i, A^j]=if^{ijk}A^k$, and mass formulas arising from ETC's of the form $[\dot{V}^K, V^i]=0$, can be obtained immediately by a formal counting argument. It is only some of the results (including all of the intermultiplet mass formulas) derived from ETC's of the form $[\dot{V}^K, A^i]=0$ which are necessary consequences of (A), and these results are generally not as well satisfied experimentally as those which are independent of (A).

In certain cases, we know that some results must follow from more general arguments, although the use of the ETC formalism sometimes makes it appear that (A) is also necessary. An example of this is the derivation of the Gell-Mann–Okubo (GMO) mass formula[5] from the equation $\langle p|[\dot{V}^{K^+}, A^{K^+}]|\Xi^-\rangle = 0$; in this case, all of the leading off-diagonal terms, in addition to the leading diagonal ones, are of order $O(\epsilon)$, but a more detailed examination shows that the leading off-diagonal contributions cancel, multiplet by multiplet. However, a similar treatment of the equation $\langle p|[\dot{V}^{K^+}, A^{\pi^+}]|\Sigma^-\rangle = 0$ shows that such a cancellation does not take place there, with the result that the $\Sigma^0$ and $\Lambda^0$ cannot be proved degenerate simply on the basis of a formal counting argument, but the application of (A) gives the degeneracy at once.

In Sec. II, we begin by explaining assumption (A) in more detail and summarize the arguments which Oneda and Matsuda give in support of (A), and then proceed to give a critical discussion of one or two of these arguments, and also of the pattern of results. We demonstrate in Sec. III how many of the results follow independently of (A), and then give examples to illustrate how a formal counting argument yields the GMO formula, but how it fails to produce the $\Sigma^0$-$\Lambda^0$ degeneracy.

## II. GENERAL CRITICISMS

In this section, we first of all give a fuller statement of assumption (A), together with supporting arguments, and then we take a rather critical look at (A) and these arguments.

Oneda and Matsuda impose their strong condition on the $V^K$ matrix elements only in the infinite-three-momentum frame where $q^2$, the squared momentum transfer, vanishes and where the renormalization of the form factors of $V^K$ appears to be a minimum. Formally, *all* off-diagonal matrix elements of $V^K$ are of order $O(\epsilon)$ when the states involved have the same isospin and hypercharge. However, in the asymptotic symmetry approximation, only those states are retained which have the same space-time quantum numbers,[6] and which are so close to each other in mass as to make mixing an important problem (e.g., $\eta^0$-$X^0$ and $\omega^0$-$\varphi^0$ mixing). The argument for neglecting states with different space-time quantum numbers is that $V^K$ is essentially a scalar operator, thus causing a "momentum barrier" to be set up; also, more distant states with the same $J^{PC}$ are neglected since their contributions are damped by the large mass differences involved.

As an example of how mixing is treated, consider the $\eta^0$-$X^0$ case.[1]

$$\eta^0(\vec{p}) = \pi_8 \cos\theta + \pi_9 \sin\theta\,, \qquad (1a)$$

$$X^0(\vec{p}) = -\pi_8 \sin\theta + \pi_9 \cos\theta\,, \qquad (1b)$$

where $\eta^0(\vec{p})$, $X^0(\vec{p})$ represent the creation operators of the physical states, $\pi_8$ and $\pi_9$ represent the corresponding operators for the symmetric states, $\theta$ is the mixing angle, and the equations are assumed valid when $|\vec{p}|\to\infty$. Then

$$\langle\eta^0(\vec{p})|V^{K^-}|K^+(\vec{p}\,')\rangle = 2E(2\pi)^3\delta^3(\vec{p}-\vec{p}\,')G_+(0)\cos\theta\,, \qquad (2a)$$

$$\langle X^0(\vec{p})|V^{K^-}|K^+(\vec{p}\,')\rangle = 2E(2\pi)^3\delta^3(\vec{p}-\vec{p}\,')G_+(0)(-\sin\theta)\,, \qquad (2b)$$

where $G_+(0)$ is the appropriate form factor [$G_-(0)$ has a vanishing coefficient]. Use of the ETC $[V^{K^+}, V^{K^-}] = (V^3 + \sqrt{3}V^8)$ between $K^+$ states shows that if off-diagonal elements are of order $O(\epsilon)$, then $G_+(0)$ differs by a term of order $O(\epsilon^2)$ from the symmetric value. Similar results are easily obtained for the other form factors where no explicit mixing is taken into account.

Now, with such "close" mixing taken care of, since the off-diagonal terms, formally of order $O(\epsilon)$, are effectively of a higher order than this because of the extra damping effect, the renormalization of the diagonal form factors is effectively of a higher order than $O(\epsilon^2)$. Assumption (A) states that the renormalization of the form factors of $V^K$ at $q^2=0$ is small, and negligible compared with other SU(3)-breaking effects, e.g., mass differences. "We neglect all the nondiagonal elements $\langle b|V^K|A\rangle$ of the vector charge $V^K$ (except for cases when there is a mixing problem for the states under consideration, . . . ) only in the infinite-momentum limit. . . ."[7]

This, as far as the present author is concerned, is the basic assumption of the scheme, together with the main supporting arguments. Let us now consider these more closely.

The first comment to be made is that the decisions as to which mixing to neglect and which to include seem somewhat arbitrary, particularly when the lack of mixing between low-lying and higher-lying multiplets is subsequently given[1] as the main reason for the poorer predictions of the intermultiplet mass formulas, at least for the baryons. In addition, the argument used for neglecting mixing with states of different space-time quantum numbers does not seem to be valid. To begin with, it is not $V^K$ which does the mixing, but $\mathcal{H}'$, the symmetry-breaking part of the Hamiltonian. This is essentially a scalar operator, and it is interesting to consider one or two examples of matrix elements, involving states with different spin and parity, in the infinite-momentum limit.

First, compare the case of two $\frac{1}{2}^+$ baryons with that of one $\frac{1}{2}^+$ and one $\frac{1}{2}^-$ baryon. In the infinite-momentum limit, the respective matrix elements are

$$\bar{u}_1(\vec{p}, r)u_2(\vec{p}, s) = (M_1 + M_2)\delta_{rs} \tag{3}$$

and

$$\bar{u}_1(\vec{p}, r)\gamma_5 u_2(\vec{p}, s) = (M_1 - M_2)h\delta_{rs}, \tag{4}$$

where $h$ is the helicity of the states, and $r$, $s$ denote the spins. Clearly, except when $M_1 \approx M_2$, these elements are of comparable magnitude, i.e., the second is not highly damped when $|\vec{p}| \to \infty$. In fact, in the rest frame, the corresponding values are $2(M_1M_2)^{1/2}\delta_{rs}$ and zero.

For the mesons, certain couplings are forbidden in any frame, e.g., $\langle 0^-|\mathcal{H}'|0^+\rangle \equiv 0$ and $\langle 0^-|\mathcal{H}'|1^-\rangle \equiv 0$. However, for those matrix elements which are allowed, the infinite-momentum limit does not appear to reduce the magnitude of the off-diagonal elements. Consider

$$\begin{aligned}\langle 0^-(\vec{p})|\mathcal{H}'|1^+(\vec{k}, r)\rangle &= g(q^2)\eta^\mu(\vec{k}, r)p_\mu \\ &= g(q^2)\eta^0(\vec{k}, r)(p^0 - k^0)\delta_{r3}, \end{aligned} \tag{5}$$

where $\eta^\mu(\vec{k}, r)$ is the polarization vector for the $1^+$ state, and we have used the conditions $k^\mu\eta_\mu(\vec{k}) = 0$, $\vec{p} = \vec{k}$, and the fact that only the third polarization vector has a nonzero time component. When $|\vec{p}| \to \infty$, the right-hand side of Eq. (5) tends to the value $g(0)\delta_{r3}(m_0^2 - m_1^2)/2m_1$, where $m_0$ and $m_1$ are the masses of the $0^-$ and $1^+$ states, respectively. Moreover, for $\vec{p} = 0$, the right-hand side of Eq. (5) vanishes, since $\eta^0(\vec{k}, 3) = |\vec{k}|/m_1$.

Thus, the argument that, in the infinite-momentum frame, the off-diagonal matrix elements of $V^K$ are effectively smaller than $O(\epsilon)$ does not appear to be well justified.

Another criticism is that it is rather unsatisfactory to have widely differing degrees of success between predictions for baryons and those for mesons, when the same ETC is used in the derivations. It seems even more questionable to reject the results for the baryons while accepting those for the mesons, for no better reason than that the former results are very poor while the latter may perhaps be better; such an *a posteriori* distinction between baryons and mesons is not a welcome feature in a model which ought to treat both types of particle on the same footing. When such a situation appears to arise, as it does for the intermultiplet formulas in broken SU(2),[8] it seems advisable to check whether such differences actually do exist.

For the SU(3) case,[7] the following equation is predicted from $[\dot{V}^{K^0}, A^{\pi^-}] = 0$:

$$(\Omega^-)^2 - (\Xi^{*-})^2 = (\Xi^0)^2 - (\Sigma^0)^2 = (\Xi^0)^2 - (\Lambda^0)^2. \tag{6}$$

Experimentally,[9] the factors (in $\mathrm{GeV}^2$) are 0.46, 0.31, 0.49. The second factor certainly does not fit, since the $\Sigma^0$ and $\Lambda^0$ are not degenerate; hence, the baryon predictions are not particularly good.

For the corresponding meson case, the formula is

$$K_\alpha^2 - \pi_\alpha^2 = \text{const}, \tag{7}$$

where $\alpha$ denotes the particular octet involved. Although this is claimed to be reasonably successful, the following data indicate that this success is only partial:

$$\begin{aligned} &0^-: \; K^2 - \pi^2 = 0.227 \pm 0.003, \\ &1^-: \; K^2 - \rho^2 = 0.21 \pm 0.14, \\ &1^+: \; K_A^2 - A_1^2 = 0.40 \pm 0.18, \\ &2^+: \; K^{**2} - A_2^2 \begin{cases} = 0.35 \pm 0.16 \\ = 0.25 \pm 0.16, \end{cases} \\ &0^+: \; \begin{cases} \kappa^2 - \delta^2 = 0.23 \\ \kappa^2 - \pi_N^2 = 0.13. \end{cases} \end{aligned} \tag{8}$$

Since the members of the $0^+$ octet are still not well established, the results from this multiplet should not really be used as evidence.

Thus, for the SU(3) intermultiplet formulas, neither set of predictions is very good, so that the mesons and baryons are back on the same footing.

For SU(2), the results from the ETC $[\dot{V}^{\pi^+}, A^{K^+}] = 0$ are very poor for the baryons, where we have the equations[8]

$$(\Sigma^-)^2 - (\Sigma^0)^2 = (\Delta^-)^2 - (\Delta^0)^2 = (\Sigma^-)^2 - (\Lambda^0)^2. \tag{9}$$

Experimentally, the outside factors are 0.012 $\mathrm{GeV}^2$ and 0.189 $\mathrm{GeV}^2$. As for the mesons, we obtain

$$(K_\alpha^0)^2 - (K_\alpha^+)^2 \equiv \delta_\alpha = \text{const}. \tag{10}$$

This time, the values of $\delta_\alpha$ are known for only two octets, viz.,

$$\begin{aligned} &0^{-+}: \; \delta_\alpha \approx 0.004 \text{ GeV}^2, \\ &1^{--}: \; \delta_\alpha \approx 0.01 \pm 0.007 \text{ GeV}^2. \end{aligned} \tag{11}$$

At present, all that can be said is that these results are not inconsistent. Thus, once again, the clash between baryon and meson predictions may not be quite as bad as it first appears.

However, since these paradoxes may be avoided by accepting less success for the mesons rather than increased success for the baryons, the basic problem of rather poor predictions arises. But, as mentioned earlier, there does not appear to be strong backing for the main assumption of the scheme, so that these poorer results may possibly disappear. In Sec. III, we show in detail how to obtain the "better" results without this assumption, leaving the above "poorer" predictions as the only necessary consequences of the asymptotic symmetry scheme.

### III. ALTERNATIVE DERIVATION OF RESULTS

In this section we show in some detail how assumption (A) is unnecessary for the derivation of many of the results obtained by Oneda and Matsuda, and we also illustrate the difference between the derivation of the GMO formula and the $\Sigma^0$-$\Lambda^0$ degeneracy using the ETC formalism.

The results may be separated into two classes, depending on whether $V^K$ itself is involved or its time derivative $\dot{V}^K$. In the first category, since the only terms involved are matrix elements of $V^K$ and those of some other charge, the only formally order $O(\epsilon)$ factors are off-diagonal elements of $V^K$. Hence, working to leading order in $\epsilon$ [here, $O(1)$], we can truncate the sum over intermediate states at exactly the same point as Oneda and Matsuda do when they ignore all off-diagonal $V^K$ matrix elements, i.e., there is no difference between the results of an ordinary broken-symmetry theory (in the infinite-momentum frame) and the asymptotic symmetry scheme; in fact, there is no difference from the results of the symmetric theory (as pointed out by Oneda and Matsuda[1]) since we work to order $O(1)$.

When $\dot{V}^K$ appears, the situation is complicated by mass-difference terms, with those corresponding to diagonal matrix elements being of order $O(\epsilon)$ whereas all other mass differences are independent of $\epsilon$ (apart from small corrections). But since we also introduce assumptions about the nature of the symmetry-breaking Hamiltonian density $\mathcal{H}'$, we are able to make several general statements about mass sum rules. To begin with, the fact that $\mathcal{H}'$ transforms like the eighth component of an octet (leading to the vanishing of such ETC's as $[\dot{V}^{K^+}, V^{K^+}]$, $[\dot{V}^{K^+}, V^{\pi^+}]$, and $[\dot{V}^{K^+}, A^{K^+}]$) means that the GMO formula and the decuplet equal-spacing rule must be valid anyway, irrespective of whether (A) is employed or not. However, because the largest symmetry group with which we are dealing is SU(3), and also because we use only the first two current-algebra relations, we are unable to derive intermultiplet mass formulas unless an additional dynamical assumption, such as (A), is made. Hence, all intermultiplet sum rules obtained by Oneda and Matsuda necessarily depend on (A); the remaining difficulty is then to discover which intramultiplet formulas depend on (A) and which do not – in particular, what is the status of the $\Sigma^0$-$\Lambda^0$ degeneracy?

First of all, consider the statement that we must be able to derive the GMO formula from equations such as

$$\langle p | [\dot{V}^{K^+}, V^{K^+}] | \Xi^- \rangle = 0 , \tag{12}$$

$$\langle p | [\dot{V}^{K^+}, A^{K^+}] | \Xi^- \rangle = 0 . \tag{13}$$

In the former case, when we sum over intermediate states, the leading diagonal terms are of order $O(\epsilon)$ on account of the mass-difference factors, whereas the leading off-diagonal terms are of order $O(\epsilon^2)$ since they involve the product of two order $O(\epsilon)$ matrix elements; by working to leading order, we obtain the same result as Oneda and Matsuda do by applying (A), viz., the GMO formula. However, the situation is a little more complicated when $V^{K^+}$ is replaced by $A^{K^+}$; we have in detail

$$\{\langle p | V^{K^+} | \Sigma^0\rangle \langle \Sigma^0 | A^{K^+} | \Xi^-\rangle [(\Sigma^0)^2 - p^2] - \langle p | A^{K^+} | \Sigma^0\rangle \langle \Sigma^0 | V^{K^+} | \Xi^-\rangle [(\Xi^-)^2 - (\Sigma^0)^2]\} + \{\Sigma^0 \to \Lambda^0\}$$
$$+ \sum_n [\langle p | V^{K^+} | n\rangle \langle n | A^{K^+} | \Xi^-\rangle (n^2 - p^2) - \langle p | A^{K^+} | n\rangle \langle n | V^{K^+} | \Xi^-\rangle (\Xi^2 - n^2)] = O(\epsilon^2) . \tag{14}$$

Now, the leading diagonal terms are of order $O(\epsilon)$ as before while the leading off-diagonal terms are also of order $O(\epsilon)$, since each includes only one off-diagonal matrix element of $V^K$. By using the Wigner-Eckart theorem, the leading diagonal terms reduce to the expression

$$2F[2(N^2 + \Xi^2) - (\Sigma^2 + 3\Lambda^2)] ,$$

where $F$ is the reduced matrix element $\langle \underline{8} \| A \| \underline{8} \rangle$ for antisymmetric coupling, and is clearly real; also, because of isospin conservation, we have replaced $p^2$ by $N^2$, the nucleon squared mass. In order to obtain the GMO formula, we must now show that the leading off-diagonal terms cancel, and this means being able to relate off-diagonal $V^K$ matrix elements to each other.

To do this, we use the mixing of IR's induced by symmetry breaking to relate off-diagonal elements such as $\langle A^i | V^K | B^j \rangle$ and $\langle C^i | V^K | D^j \rangle$, where, for example, $|B^j\rangle$ and $|D^j\rangle$ refer to different isospin submultiplets in the $j$th SU(3) multiplet. However, we cannot relate terms like $\langle A^i | V^K | B^j \rangle$ and $\langle C^i | V^K | D^l \rangle$ to each other, since $j$ and $l$ refer to different SU(3) multiplets. More explicitly, when the symmetry-breaking part of the Hamiltonian, $\epsilon\mathcal{H}'$, is "switched on," the originally degenerate states within an SU(3) multiplet are separated into distinct isospin submultiplets; each one of these then acquires small $O(\epsilon)$ admixtures of corresponding submultiplets from other SU(3) multiplets. In the usual perturbation-expansion approach, the physical state $|A^i\rangle$ belonging (mainly) to the $i$th SU(3) multiplet may be expressed to $O(\epsilon)$ as

$$|A^i\rangle \cong |A_0^i\rangle + \alpha_A^{ij}|A_0^j\rangle,$$
$$\langle A^i| \cong \langle A_0^i| + (\alpha_A^{ij})^*\langle A_0^j|, \tag{15}$$

with

$$\alpha_A^{ij} \cong \frac{1}{2}\frac{\langle A_0^j|\epsilon\mathcal{H}'|A_0^i\rangle}{(E_0^i)^2-(E_0^j)^2} = \frac{1}{2}\frac{\langle A_0^j|\epsilon\mathcal{H}'|A_0^i\rangle}{(M_0^i)^2-(M_0^j)^2}, \tag{16}$$

where $|A_0^i\rangle$ is an exact-symmetry state, and $E_0^i$ is the original energy eigenvalue of the $i$th multiplet whose mass is $M_0^i$.

The orthogonality condition for the physical states yields the condition $(\alpha_A^{ij})^* = -\alpha_A^{ji}$; also, the fact that matrix elements of $V^\pi$, the constant SU(2) generators, are not renormalized as long as SU(2) is an exact symmetry, allows us to show that $\alpha_A^{ij}$ is constant within each isospin submultiplet. In order to relate $\alpha_A^{ij}$ to $\alpha_B^{ij}$, where $A$, $B$ refer to different isospin submultiplets within the same SU(3) multiplet, we use the assumption that $\mathcal{H}'$ transforms as the eighth component of an octet to derive results such as the following. For an octet and a decuplet,[10]

$$\alpha_\Sigma^{8,10} = \alpha_\Xi^{8,10}. \tag{17}$$

No other submultiplets can mix. For two separate decuplets, $a$ and $b$,

$$\alpha_\Omega^{ab} = 2\alpha_\Xi^{ab} = -2\alpha_\Delta^{ab}; \quad \alpha_{Y_1}^{ab} = 0. \tag{18}$$

Returning to Eq. (14), we shall use relations like these above to show how the leading contributions from intermediate singlet (denoted by $\Lambda'$) and decuplet ($Y_1^0$) states cancel, multiplet by multiplet. Using the Wigner-Eckart theorem, the $\Lambda'$ contribution to Eq. (14) is

$$\langle p|V^{K^+}|\Lambda'\rangle\langle\Lambda'|A^{K^+}|\Xi^-\rangle(\Lambda'^2-p^2) - \langle p|A^{K^+}|\Lambda'\rangle\langle\Lambda'|V^{K^+}|\Xi^-\rangle[(\Xi^-)^2-\Lambda'^2]$$
$$\cong [(-\tfrac{3}{2})^{1/2}\alpha_\Lambda^{1,8}]\left(-\frac{1}{2\sqrt{2}}G^{1,8}\right)(\Lambda'^2-N^2) - \left(-\frac{1}{2\sqrt{2}}G^{1,8}\right)^*[(\tfrac{3}{2})^{1/2}\quad\alpha_\Lambda^{1,8}]^*(\Xi^2-\Lambda'^2)$$
$$= \frac{\sqrt{3}}{4}[\mathrm{Re}(\alpha_\Lambda^{1,8}G^{1,8})(\Xi^2-N^2) + i\,\mathrm{Im}(\alpha_\Lambda^{1,8}G^{1,8})(2\Lambda'^2-N^2-\Xi^2)], \tag{19}$$

where $\alpha_\Lambda^{1,8}$ is the appropriate mixing parameter, $G^{1,8}$ is the reduced matrix element $\langle \underline{1}\|A\|\underline{8}\rangle$, and Re$X$ and Im$X$ denote the real and imaginary parts of $X$. It is immediately obvious that the real part of the $\Lambda'$ contribution is of order $O(\epsilon^2)$, and this is the important part as far as the GMO formula is concerned, since the leading diagonal contribution is purely real, as previously mentioned.

Similarly, the $Y_1^0$ contribution is

$$\langle p|V^{K^+}|Y_1^0|\rangle\langle Y_1^0|A^{K^+}|\Xi^-\rangle[(Y_1^0)^2-p^2] - \langle p|A^{K^+}|Y_1^0\rangle\langle Y_1^0|V^{K^+}|\Xi^-\rangle[(\Xi^-)^2-(Y_1^0)^2]$$
$$\cong \left(-\frac{1}{\sqrt{2}}\alpha_\Sigma^{10,8}\right)\left(-\frac{1}{2\sqrt{3}}G^{10,8}\right)(Y_1{}^2-N^2) - \left(\frac{1}{2\sqrt{3}}G^{10,8}\right)^*\left(\frac{1}{\sqrt{2}}\alpha_\Sigma^{10,8}-\sqrt{2}\alpha_\Xi^{10,8}\right)^*(\Xi^2-Y_1{}^2)$$
$$= \frac{1}{2\sqrt{6}}[\mathrm{Re}(\alpha_\Sigma^{10,8}G^{10,8})(\Xi^2-N^2) + i\,\mathrm{Im}(\alpha_\Sigma^{10,8}G^{10,8})(2Y_1{}^2-N^2-\Xi^2)], \tag{20}$$

where we have used Eq. (17). Once again, the real part of this contribution is of order $O(\epsilon^2)$, and this illustrates what happens in the cases of the other intermediate multiplets, viz., $\underline{8}$, $\underline{10}^*$, and $\underline{27}$. Hence, by using (19) and (20) (and other similar expressions) in Eq. (14) and taking the real part, we obtain the desired result

$$2(N^2+\Xi^2) - (\Sigma^2+3\Lambda^2) = O(\epsilon^2). \tag{21}$$

At the same time, the imaginary part of Eq. (14) must also be of order $O(\epsilon^2)$, and since it is clear from the above that the individual contributions are of order $O(\epsilon)$, then it must be the combined contribution of all of the intermediate states which is of order $O(\epsilon^2)$. To understand why this is not a new (and unwanted) result, consider the general case of an order $O(\epsilon)$ operator (here, the ETC) between the physical states $\langle p|$ and $|\Xi^-\rangle$; these states are primarily from the same IR, with order $O(\epsilon)$ contributions from other IR's, so that the leading term is an order $O(\epsilon)$ diagonal one and must therefore be real, while the order $O(\epsilon^2)$ terms are in general off-diagonal and so complex.

Now let us turn to the question of the $\Sigma^0$-$\Lambda^0$ degeneracy, using the equation

$$\langle p|[\mathring{V}^{K^+}, A^{\pi^+}]|\Sigma^-\rangle = 0\,. \tag{22}$$

First of all, Eq. (22) is valid only when $\mathcal{H}'$ is made somewhat less general; for example, if $\mathcal{H}' \sim u_8$, the usual quark density, then the ETC vanishes; in fact, it appears that this follows if $\mathcal{H}'$ is confined to terms belonging to the $(3,\bar{3}) \oplus (3,3)$ and $(1,8) \oplus (8,1)$ IR's of the chiral $SU(3) \otimes SU(3)$ group generated by the 16 $V^i$ and $A^i$, i.e., the only two IR's [apart from (1.1)] which do not contain operators with exotic SU(3) quantum numbers. However, at the present time, such a restriction of $\mathcal{H}'$ appears fairly popular.[11]

Next, if we sum over intermediate states, we find that all of the leading terms are again of order $O(\epsilon)$, with the diagonal ones reducing to the expression $2D(\Sigma^2 - \Lambda^2)$, where $D$ is the reduced matrix element $\langle \underline{8}\|A\|\underline{8}\rangle$ for symmetric coupling. However, this time, we do not have the same type of cancellation mechanism which worked in the previous case, yielding the GMO formula. We can see this immediately by considering the singlet $\Lambda'$ contribution, which consists of the solitary order $O(\epsilon)$ term

$$\langle p|V^{K^+}|\Lambda'\rangle\langle\Lambda'|A^{\pi^+}|\Sigma^-\rangle(\Lambda'^2 - p^2)\,. \tag{23}$$

The reason for there being only one term here lies with the "asymmetry" between the SU(3) indices of the vector and axial-vector charges, which means that in any given multiplet, different intermediate states contribute to the two sides of the ETC; in the derivation of the GMO formula, exactly the same states appeared on both sides. For further evidence of noncancellation, consider the decuplet contribution

$$\begin{aligned}\langle p|V^{K^+}|Y_1^0\rangle\langle Y_1^0|A^{K^+}|\Sigma^-\rangle&(Y_1{}^2 - p^2) - \langle p|A^{K^+}|\Delta^0\rangle\langle\Delta^0|V^{K^+}|\Sigma^-\rangle[(\Sigma^-)^2 - (\Delta^0)^2]\\ &\cong \left(-\frac{1}{\sqrt{2}}\alpha_\Sigma^{10,8}\right)\left(\frac{1}{2\sqrt{3}}G^{10,8}\right)(Y_1{}^2 - N^2) - \left(\frac{1}{\sqrt{6}}G^{10,8}\right)^*(-\alpha_\Sigma^{10,8})^*(\Sigma^2 - \Delta^2)\\ &= -\frac{1}{2\sqrt{6}}\{\mathrm{Re}(\alpha_\Sigma^{10,8}G^{10,8})[Y_1{}^2 + 2\Delta^2 - (N^2 + 2\Sigma^2)] + i\,\mathrm{Im}(\alpha_\Sigma^{10,8}G^{10,8})(Y_1{}^2 + 2\Sigma^2 - N^2 - 2\Delta^2)\},\end{aligned} \tag{24}$$

and the real part of this is only of order $O(\epsilon)$.

Since the contributions from these two multiplets do not cancel "internally," and since we cannot relate the reduced matrix elements such as $G^{1,8}$ and $G^{10,8}$ except by going to a larger symmetry group or else by imposing some dynamical conditions, this is sufficient to show that the leading off-diagonal contributions do not cancel in this case, leaving us with the result

$$\Sigma^2 - \Lambda^2 = O(\epsilon)\,. \tag{25}$$

When assumption (A) is applied, all of these off-diagonal terms disappear and we are left with the term $2D(\Sigma^2 - \Lambda^2)$ on the left-hand side of Eq. (22); thus, if the ETC vanishes, we have the $\Sigma^0 - \Lambda^0$ degeneracy. Of course, this can be avoided by removing the restrictions on $\mathcal{H}'$, but the use of (A) still leaves us with the intermultiplet formulas derived from $[\mathring{V}^{K^+}, A^{K^+}] = 0$. An example of one of these is the relation

$$\Xi_b{}^2 - \Sigma_b{}^2 = \delta_a = \text{const}, \tag{26}$$

where $a$ and $b$ refer to (appropriate) arbitrary multiplets of SU(3). However, the agreement with present experimental data[9] is not particularly good, with $\delta_a = 0.32 \pm 0.02\ \mathrm{GeV}^2$ for the $\frac{1}{2}^+$ octet, and $0.42 \pm 0.06\ \mathrm{GeV}^2$ for the $\frac{3}{2}^+$ decuplet.

Hence, it appears that those results which really do depend on assumption (A) are somewhat less well satisfied experimentally than many of the other predictions which follow from the more usual assumptions on symmetry breaking.

## IV. CONCLUSIONS

The aims of this paper have been twofold. First, we have tried to discuss the main assumption of the asymptotic symmetry scheme and its supporting arguments in a rather critical fashion, and we have concluded that these arguments are not altogether convincing. Second, we have shown that many of the results obtained in the scheme can also be derived without the use of the basic assumption, and these results which (not unnaturally) coincide with those from other, more general, schemes are reasonably well satisfied experimentally. However, for the predictions which depend on the basic assumption, agreement with experimental data is not as good.

## ACKNOWLEDGMENTS

I should like to thank all the members of the Mathematical Physics Department of Edinburgh University for their help – and patience – during the completion of the original version of this work; in particular, my thanks go to Dr. P. W. Higgs and Dr. P. Osborne for some enlightening discussions and for their reading of the manuscript. Also I wish to thank Professor N. Kemmer, F.R.S., for hospitality at the Tait Institute, and the University of Edinburgh for the award of a Chalmers Research Scholarship. The present version of the paper was written at Harvard, and I should like to acknowledge useful conversations with Professor S. L. Glashow and Professor A. H. Luther, and to thank Professor R. V. Pound for hospitality there.

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[10]We use the following sign convention for octet and decuplet states:

$$|\Sigma^+\rangle = -|\underline{8};\ Y=0,\ I=I_3=1\rangle;$$

$$|\Xi^-\rangle = -|\underline{8};\ Y=-1,\ I=\tfrac{1}{2},\ I_3=-\tfrac{1}{2}\rangle;$$

all other particle states are related to the corresponding octet states with a positive sign. The charges $V^i$ and $A^i$ are related to the tensor operators in exactly the same way. In the decuplet, $\Delta^{++}$, $Y_1^{*-}$, $Y_1^{*0}$, and $\Xi^{*0}$ all have positive signs; the remaining states have negative signs.

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PHYSICAL REVIEW D VOLUME 5, NUMBER 4 15 FEBRUARY 1972

# Comments on Statistical Bootstrap Models of Hadrons

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(Received 30 August 1971)

The statistical bootstrap models of Hagedorn and Frautschi, modified so that the volume of a hadron is allowed to vary with the temperature, are considered. It is shown that a large class of polynomial solutions for the level density of hadrons is possible. A feature common to polynomial spectra is that the volume of a hadron must vanish as the temperature approaches infinity. The requirement that hadrons have a finite size implies both a maximum temperature and an exponential hadron mass spectrum. Also, the recent formulation in terms of quasiparticles demands an exponential hadron mass spectrum without requiring an asymptotic bootstrap condition. A unique solution, $\rho(m)\sim m^{-5/2}e^{m\beta}$ as $m\to\infty$, is obtained if one assumes the *asymptotic* bootstrap condition of Hagedorn.

## I. INTRODUCTION

Recently there have been several attempts at understanding the dynamics of strongly interacting particles from the statistical point of view.[1-3] Although the different approaches lead to similar results, the underlying features of the models are quite different.

The thermodynamical model of strong interactions and a systematic comparison of theoretical predictions with experiments were started by Hagedorn in 1965. The main success of this program has been to introduce a bootstrap condition in statistical theories of hadrons leading to an exponential hadron mass spectrum with a universal highest temperature.

More recently, Frautschi[2] developed a statistical bootstrap model of hadrons closely related to that of Hagedorn. However, Frautschi opts to work in terms of phase space with explicit momentum conservation. Also, zero- and one-particle states are excluded. The results Frautschi obtains are