$$T_{a;bb4\cdots 4} = -T_{a;44\cdots 4},$$

and therefore (B8) reads

$$\tfrac{1}{2}(n-1)(T_{4;a4\cdots 4} - T_{a;44\cdots 4}) = 0. \tag{B9}$$

Together with Eq. (B5), this yields the desired result,

$$T_{a;44\cdots 4} = 0. \tag{B10}$$

It should also be noted that the commutation relation (B1) does not put any constraints on the terms in $M$ belonging to chiral representations $(A, B)$ with $A = B$, because such terms automatically satisfy (B1). A suitable tensor basis for the representations $(n/2, n/2)$ is provided by the completely symmetric traceless tensors of rank $n$. For such a tensor $t_{UVW\cdots}$, there is just one way to form an isovector, so that

$$M_a[(n/2, n/2)] = t_{a44\cdots 4}. \tag{B11}$$

The commutator of the chiral generator with the component of $M$ is then

$$[\Gamma_a(1\to 1), M_b(n/2, n/2)] = -i\delta_{ab}t_{44\cdots 4} + i(n-1)t_{ab4\cdots 4}, \tag{B12}$$

and hence is automatically symmetric in $a$ and $b$. Thus, the whole content of a commutation relation like (B1) can be summed up in the statement that $M_a$ may receive contributions only from the chiral representations $(A, B)$ with $A = B$.

*Work supported in part through funds provided by the U. S. Atomic Energy Commission under Contract No. AT(30-1)-2098.

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PHYSICAL REVIEW D VOLUME 5, NUMBER 4 15 FEBRUARY 1972

# Bounds on the Pion's Charge Radius*†

D. N. Levin, V. S. Mathur, and S. Okubo
*Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627*
(Received 18 October 1971)

The strongest possible lower and upper bounds on the electromagnetic radius of the pion are derived in terms of the *modulus* of the timelike form factor. Numerical evaluation indicates that the radius is bounded above by the vector-dominance value and that the form factor will not behave as a "dipole" until $t=(2E)^2 > 17\ \text{GeV}^2$, if at all. The location and number of possible zeros of the form factor are discussed.

At the present time there is a rapid accumulation of information on the pion's electromagnetic form factor,[1] $F(t)$, for timelike and spacelike momentum transfer. Colliding-beam measurements of $\sigma(e^+e^- \to \pi^+\pi^-)$ provide direct access to $|F(t)|$ in the timelike region; for example, experiments[2] at Novosibirsk, Orsay, and Frascati have determined $|F(t)|$ for $t \leq 4.4\ \text{GeV}^2$. Data at higher $t$ will be furnished by new colliding-beam facilities under construction. Estimates of $F(t)$ in the spacelike region have been indirectly extracted from electroproduction experiments.[3] More precise information on the spacelike form factor will soon be available from the Serpukhov-UCLA group,[4] which is using $\pi e$ scattering to measure the pion's radius[1] ($r_\pi$).

The analytic properties of $F(t)$ imply the correlation of these ostensibly different experiments, which measure the variation of $F(t)$ in the spacelike region (e.g., the value of $r_\pi^2$) and the behavior of $|F(t)|$ in the timelike region. The standard technique for making that correlation is to use an ordinary dispersion relation (dispersive *equality*) to calculate $r_\pi^2$ from $\text{Im}F(t)$ for timelike $t$. The major disadvantage of this method[5] is that it is necessary to employ an intermediate model in order to construct $\text{Im}F(t)$ from the measured values of $|F(t)|$. This paper considers the problem of using analyticity to draw a *direct* correlation between the spacelike behavior of $F(t)$ and the measured variation of $|F(t)|$ for timelike momentum transfer; *no* intermediate models are necessary. The result is a set of dispersive *inequalities* which give the strongest possible bounds on $r_\pi^2$ in terms of integrals of $|F(t)|$ over the timelike region.

In the following paragraphs the basic inequalities are first stated, then proved and discussed. Next, in order to facilitate the numerical evaluation of the results, we propose a description for $|F(t)|$ in the timelike region. The evaluation of the basic inequalities then leads to lower and upper bounds on $r_\pi^2$ which are strong enough to correlate the values of $r_\pi^2$ and the timelike form factor.

The basic theorem, which bounds $r_\pi^2$ in terms of $|F(t)|$, can be stated as follows: Assume $F(\xi)$ is an analytic function of $\xi$ with a cut on the positive real axis at $4m_\pi^2 = t_0 \leq \xi \leq \infty$; suppose $F(\xi)$ satisfies the "reality" condition, $F^*(\xi^*) = F(\xi)$, and asymptotically approaches some power of $\xi$ as $\xi \to \infty$. Then $r_\pi^2$ is bounded below and above by

$$\frac{-\sinh(4\epsilon) - 4\epsilon}{8m_\pi^2} + I \leq \frac{r_\pi^2}{6} \leq \frac{\sinh(4\epsilon) - 4\epsilon}{8m_\pi^2} + I, \tag{1}$$

where

$$\epsilon \equiv \frac{m_\pi}{2\pi}\int_{t_0}^{\infty} dt \frac{\ln|F(t)|}{t(t-t_0)^{1/2}}, \quad I \equiv \frac{2m_\pi}{\pi}\int_{t_0}^{\infty} dt \frac{\ln|F(t)|}{t^2(t-t_0)^{1/2}}.$$

In order to prove this result, it is convenient to use the following mapping to transform from the $\xi$ plane (complex $t$ plane) into the $z$ plane:

$$(\xi - t_0)^{1/2} = (it_0^{1/2})(1+z)/(1-z). \tag{2}$$

This transformation maps the whole $\xi$ plane into the open unit disk in the $z$ plane, the upper and lower cuts in the $\xi$ plane onto the lower and upper unit semicircles in the $z$ plane, and the points $\xi = 0, t_0, \infty$ into $z = 0, -1, +1$. In the $z$ plane the form factor is represented by $f(z) \equiv F(\xi(z))$. The properties of $F(\xi)$ (see the "reality" condition and Ref. 1) imply that $f(z)$ is analytic on the open unit disk and satisfies $f^*(z^*) = f(z)$, $f(0) = 1$, and $f'(0) = -2t_0 r_\pi^2/3$. In the language of the $z$ plane, the problem is to bound $r_\pi^2$ or $f'(0)$ by integrals of $|f(z)|$ around the unit circle. It is now possible to exploit the factorization theorem[6] of complex analysis, according to which $f(z)$ can be represented as $f(z) = B(z)G(z)$ where

$$B(z) = \prod_{n=1}^{\infty}\left[\frac{\alpha_n^*(\alpha_n - z)}{|\alpha_n|(1-\alpha_n^* z)}\right],$$
$$G(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln|f(e^{i\theta})| d\theta\right]. \tag{3}$$

Here, the $\alpha_n$ are the zeros of $f$ on the open unit disk ($0 < |\alpha_n| < 1$). Now, the chain rule of differentiation gives

$$f'(0) = B'(0)G(0) + B(0)G'(0).$$

Substituting $B(0) = f(0)/G(0)$ and using the values of $f(0)$ and $f'(0)$, we find

$$|(2t_0 r_\pi^2/3) + G'(0)/G(0)| = |B'(0)|\,|G(0)|. \tag{4}$$

Notice that $G(0)$ and $G'(0)$ are known in terms of integrals of $|f(z)|$ over the unit circle. Hence, if we can bound $|B'(0)|$ from above in terms of such integrals, then Eq. (4) gives lower and upper bounds on $r_\pi^2$ in terms of $|f(z)|$ on the unit circle. To do this, observe that Eq. (3) implies that $B(z)$ is analytic on the open unit disk, is continuous on the closed unit disk, and satisfies the boundary condition $|B(e^{i\theta})| = 1$. Therefore, the maximum-modulus theorem[7] says that one of the following two statements is true: (a) $B(z) = 1$ on the closed unit disk and $|B'(0)| = 1 - |B(0)|^2$, or (b) $|B(z)| < 1$ on the open unit disk. In case (b) it is possible to define a function $A(z)$ which is analytic on the open unit disk and continuous on the closed unit disk:

$$A(z) \equiv \frac{B(0) - B(z)}{z[1 - B^*(0)B(z)]}. \tag{5}$$

It can be verified that $|B(e^{i\theta})| = 1$ implies $|A(e^{i\theta})| = 1$. Therefore, the maximum-modulus theorem requires $|A(z)| \leq 1$ on the closed unit disk. In particular, for case (b) we have $|A(0)| \leq 1$, which by definition of $A(z)$ leads to[8] $|B'(0)| \leq 1 - |B(0)|^2$. This completes the demonstration in both cases (a) and (b) that $|B'(0)|$ is bounded by the following integrals of $|f(z)|$ over the unit circle:

$$|B'(0)| \leq 1 - |B(0)|^2 = 1 - |G(0)|^{-2}. \tag{6}$$

The combination of Eqs. (4) and (6) implies that we have succeeded in bounding $r_\pi^2$ by integrals of $|f(z)|$ over the unit circle:

$$|(2t_0 r_\pi^2/3) + G'(0)/G(0)| \leq |G(0)|(1 - |G(0)|^{-2})$$

or

$$\frac{1}{4t_0}\left[\frac{1}{|G(0)|}-|G(0)|-\frac{G'(0)}{G(0)}\right] \leq \frac{r_\pi^2}{6} \leq \frac{1}{4t_0}\left[|G(0)|-\frac{1}{|G(0)|}-\frac{G'(0)}{G(0)}\right].$$

The proof is completed by using Eq. (2) to convert the integrals in $G(0)$ and $G'(0)$ into integrals of $|F(t)|$ over timelike $t$.

An examination of the above proof shows that the resulting bounds on $r_\pi^2$ [Eq. (1)] are the strongest ones which can be derived in terms of $|F(t)|$ in the timelike region. Other authors[9] have derived weaker results for the upper bound, which can be obtained from the *exact* upper bound [Eq. (1)] in the approximation that $\epsilon$ is small.

In order to evaluate the bounds on $r_\pi^2$, it is necessary to know the behavior of $|F(t)|$ for $t_0 \leq t \leq \infty$. The Novosibirsk and Orsay experiments[2] have measured $|F(t)|$ on the interval $0.35 \leq t \leq 1.0$ GeV$^2$; the data are fit by a modified, $P$-wave Breit-Wigner shape:

$$|F(t)|^2=\frac{0.399}{[(0.592-t)+(1.41)b(t)]^2+(1.99)k^6t^{-1}}, \tag{7}$$

where

$$b(t)=k^2[h(t)-0.504]-(0.0425)(t-0.592),$$

$$h(t)=\frac{(0.637)k}{\sqrt{t}}\ln\left(\frac{\sqrt{t}+2k}{0.276}\right),$$

$$k(t)=(0.5)(t-t_0)^{1/2}.$$

Although there is no information for $0.08$ GeV$^2$ $=t_0 \leq t \leq 0.35$ GeV$^2$, there are good theoretical reasons[10] for believing that $|F(t)|$ is also described by Eq. (7) at these energies. Therefore, we will use Eq. (7) for the entire range $t_0 \leq t \leq 1.0$ GeV$^2$. The Frascati experiment[2] indicates a rather large form factor for $2.0 \leq t \leq 4.4$ GeV$^2$; those results are approximated by[11] $|F(t)|^2=(2.16 \text{ GeV}^4)/t^2$. We will use this expression for all $t$ on $1.0 \leq t \leq 4.4$ GeV$^2$, since this curve intersects the Breit-Wigner shape [Eq. (7)] at $t=1.0$ GeV$^2$. Since there are no data for $t>4.4$ GeV$^2$, we will propose a model for $|F(t)|$ at high $t$, characterized by a single free parameter $t_M$. It will be assumed that $|F(t)|^2$ can be crudely represented by a "large," gently falling Frascati curve for $4.4 \text{ GeV}^2 < t \leq t_M$ and by a "small," sharply falling curve (say, $\sim 1/t^4$) for $t_M < t < \infty$. In other words, we assume

$$|F(t)|^2=(2.16 \text{ GeV}^4)/t^2 \quad \text{for } 4.4 \text{ GeV}^2 < t \leq t_M,$$

$$|F(t)|^2=(2.16 \text{ GeV}^4)t_M^2/t^4 \quad \text{for } t_M < t < \infty.$$

The parameter $t_M$ represents a transition point, after which the "dipole" behavior of the form factor manifests itself; if $t_M = \infty$, the form factor falls[11] like a "single pole" at very high $t$.

A computer was used to evaluate the lower and upper bounds on $r_\pi^2$ [Eq. (1)] as functions of $t_M$ [i.e., as functions of the behavior of $|F(t)|$]. The results are displayed in Fig. 1($r_{\text{VD}}^2 \equiv 6/m_\rho^2$, the vector-dominance value for $r_\pi^2$). Note that the upper bound is nearly constant and therefore insensitive to the high-$t$ behavior of $|F(t)|$. The consequences of these results include the following: (1) The derivation of bounds on $r_\pi^2/r_{\text{VD}}^2$ has exploited the assumptions that $F(t)$ is analytic in the cut $t$ plane and that $|F(t)|$ is adequately described for $t_0 \leq t \leq \infty$ by one of our models (i.e., by *some* value of $t_M$). If the measured values of $(t_M, r_\pi^2/r_{\text{VD}}^2)$ fall in the shaded region of Fig. 1, then the bounds are violated and at least one of these two assumptions is incorrect. (2) Barring a violation of these assumptions, we can conclude that both inequalities are satisfied and that $(t_M, r_\pi^2/r_{\text{VD}}^2)$ must lie in the unshaded region of Fig. 1. This restriction represents the expected correlation of the "spacelike and timelike" parameters, $r_\pi^2$ and $t_M$. In particular, it follows that $0.3 \leq r_\pi^2/r_{\text{VD}}^2 \leq 1.0$ and $t_M \geq 17$ GeV$^2$. Thus, "dipole" behavior of $|F(t)|$ will appear only at $t > t_M \geq 17$ GeV$^2$, if at all. Furthermore, it can be shown[12] that the *location* of the point $(t_M, r_\pi^2/r_{\text{VD}}^2)$ in the unshaded region gives us information about the zeros of the form factor: (a) If $r_\pi^2/r_{\text{VD}}^2 = 1.0$ (the upper bound) and $t_M > 17$ GeV$^2$, then $F(\xi)$ has one and only one zero[13] at the real spacelike momentum transfer, $t_{\text{NU}} = -4t_0e^{4\epsilon} \times (e^{4\epsilon}-1)^{-2}$. A calculation based on this expression shows that, as $t_M$ varies between 17 GeV$^2$ and $\infty$, $t_{\text{NU}}$ ranges[14] from $-\infty$ to $-42$ GeV$^2$. (b) If $r_\pi^2/r_{\text{VD}}^2 = 1.0$ (the upper bound) and $t_M = 17$ GeV$^2$, then $F(\xi)$ has no zeros. (c) If $(t_M, r_\pi^2/r_{\text{VD}}^2)$ falls on the lower (but not upper) boundary of the un-

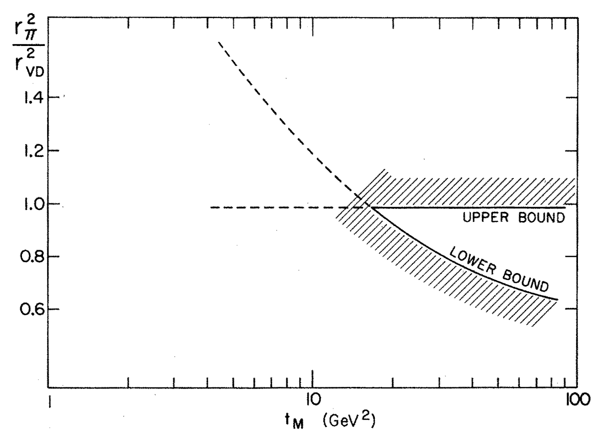


FIG. 1. Lower and upper bounds on $r_\pi^2/r_{\text{VD}}^2$ as functions of $t_M$. As $t_M \to \infty$, the upper bound on $r_\pi^2/r_{\text{VD}}^2$ stays at 1.0, and the lower bound approaches 0.3.

shaded region, then $F(\xi)$ has one and only one zero at a real timelike point, $t_{NL}=4t_0e^{4\epsilon}(e^{4\epsilon}+1)^{-2}$. A computation of $t_{NL}$ shows that it lies just below threshold for all $t_M$. (d) If $(t_M, r_\pi^2/r_{VD}^2)$ falls between but not on the upper and lower boundaries, then $F(\xi)$ has two or more zeros.

To understand the significance of these conclusions, it is necessary to consider the uncertainties in the bounds on $r_\pi^2$ due to experimental error in measurements of $|F(t)|$. A rough estimate shows that the upper and lower bounds on $r_\pi^2$ are uncertain by $\sim\pm 0.25$. The above quantitative results are correspondingly blurred. Calculations of $t_{NU}$ are especially sensitive to experimental errors in $|F(t)|$ and, at present, constitute only estimates of order of magnitude.

*Note added.* After this work was submitted for publication, the authors received related reports from I. Raszillier [I. Raszillier, Institute of Physics (Bucharest) report, 1971 (unpublished); Lett. Nuovo Cimento 2, 349 (1971)]. These papers mention that Eq. (1) has been independently derived in the paper by B. V. Geshkenbeĭn, Yadern. Fiz. 9, 1232 (1969) [Soviet J. Nucl. Phys. 9, 720 (1969)]. However, the phenomenological discussion of the present paper is more elaborate and of immediate interest to experimentalists. Also, Raszillier has demonstrated that Eq. (1) is true even if $|F(t)|$ is replaced everywhere by $|S(t)|$, where $|S(t)|$ is any *upper limit* on the form factor's modulus in the timelike region. This last remark implies that the numerical results of the present paper are valid even if the colliding-beam data are "contaminated" by a significant two-photon process. [See S. J. Brodsky, T. Kinoshita, and H. Terazawa, Phys. Rev. Letters 25, 972 (1970).] To see this, note that charge-conjugation invariance requires that the one-photon and two-photon amplitudes sum incoherently to give the total cross section $\sigma(e^+e^-\to\pi^+\pi^-)$, which is measured in colliding-beam experiments. Thus, the measured total cross section provides an upper bound on the squared modulus of the one-photon amplitude. Therefore, the colliding-beam "measurements" of $|F(t)|$ (obtained by neglecting the two-photon contribution) must always furnish an *upper bound* on the actual value of $|F(t)|$. Combining this fact with Raszillier's result, we conclude that the empirical analysis of this paper is probably valid even if the two-photon process is significantly large. Therefore, any violation of our bounds should be taken seriously.

The authors would like to thank D. R. Palmer for helpful discussions and are indebted to K. Levin and M. Baće for computational assistance.

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*Work supported in part by the U. S. Atomic Energy Commission.

†A preliminary report of some of these results was presented at the 1971 meeting of the Division of Particles and Fields, American Physical Society.

[1]Notation: The pion's form factor is normalized such that $F(0)=1$; the electromagnetic radius $(r_\pi)$ of the pion is defined as $F'(0)=r_\pi^2/6$; timelike momentum transfer corresponds to $t>0$.

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[11]Electroproduction experiments (Ref. 3) suggest that $F(t)$ behaves like a "dipole" or "single pole" at large spacelike momentum transfer. In that case, analyticity requires that the form factor fall as a "dipole" or "single pole" at large timelike $t$.

[12]D. N. Levin, V. S. Mathur, and S. Okubo (unpublished).

[13]In the language used here a zero of order $m>1$ counts as $m$ zeros.

[14]If $r_\pi^2/r_{VD}^2=1.0$ and $t_M=\infty$, then $F(\xi)$ has one and only one zero at $t_{NU}=-42$ GeV$^2$. Heuristically, such a zero means that the deep interior of positively charged (negatively charged) pions contains a concentration of negative (positive) charge.