rents as well.

Therefore, we conclude that on the basis of low-energy theorems such as the above, one does not need to introduce any new terms in either the commutation relations or the divergence conditions provided one consistently enforces gauge invariance and relativity. Whether this conclusion is valid for other current-algebra calculations such as those involving hard-pion processes is an open question.

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$$\langle \vec{k}\lambda | J^\mu(0)|0\rangle = \epsilon_\nu(\vec{k}\lambda)\, T^{\nu\mu}(k),$$

where $k\cdot\epsilon = 0$, and $T^{\nu\mu}(k)$ is a second-rank tensor satisfying $k_\nu T^{\nu\mu}(k)=0$. [See S. Weinberg, Phys. Rev. 135, B1049 (1964), for a discussion of these gauge conditions.] From these two conditions it follows that

$$T^{\nu\mu}(k) = (\delta^{\nu\mu}k^2 - k^\nu k^\mu)\, T(k^2).$$

Hence on the mass shell

$$\epsilon_\nu T^{\nu\mu} = 0 = \langle \vec{k}\lambda | J^\mu | 0\rangle .$$

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Multiparticle Sum Rules*

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Sum rules are derived by applying soft-pion theorems to the dispersion relations for the reactions $\pi + I \to \pi + F$, $I \to 2\pi + F$, $2\pi + I \to F$, with I and F general multiparticle states. In the special case where I contains one particle and F contains two particles, these sum rules have a simple algebraic interpretation: They require that the transition matrix elements for reactions $\pi + I \to F$ or $I \to F + \pi$ can receive contributions only from the representations (A,B) of chiral $SU(2)\otimes SU(2)$ with $A=B$, but not $A=B\pm 1$. Prescriptions are given for dealing with the complications caused by the nonvanishing matrix elements of chiral generators between states of different mass and spin.

I. INTRODUCTION AND SUMMARY

Sum rules of one sort or another have played a large part in the recent development of elementary-particle theory. Given their great importance, it is perhaps surprising that these sum rules have generally been restricted to a limited class, involving transitions between a *single*-particle state and other states, induced by a current or another particle. For instance, the Adler-Weisberger sum rules[1] are derived either by evaluating matrix elements of axial-vector-current-density commutators between *single*-particle states, or by using soft-pion theorems in conjunction with the dispersion relations for scattering of one *single*-particle state into another by a pion; either way, one obtains a sum rule for the matrix elements of an axial-vector current between a one-particle state and other states. When these sum rules are saturated with single-particle states, there emerges a set of commutation relations for matrix elements of the axial-vector current between single-particle states,[2] which imply that these matrix elements generate the algebra of $SU(2)\otimes SU(2)$.

The purpose of this paper is to widen the scope of application of sum rules to general multiparticle processes. In particular, we shall study in detail the sum rules for the processes

$$\begin{aligned}&\pi + I \to \pi + F,\\ &I \to \pi + \pi + F,\\ &\pi + \pi + I \to F,\end{aligned} \tag{1.1}$$

where I and F are general multiparticle states, and π is a massless pion. As long as we take the two pion momenta along the same fixed direction $\vec{n}$, and keep the momenta of all the particles in the states I and F fixed, the amplitude for these processes will satisfy a simple single-variable dispersion relation. Furthermore, the dispersion relation for the part of the amplitude antisymmetric in the pion isovector indices is presumably unsubtracted. We derive our sum rule by imposing on this dispersion relation the condition that the amplitude should vanish when either pion energy vanishes.

Such sum rules are interesting in principle, but they involve matrix elements which would be very difficult to measure. In order to use these sum rules, it is in practice necessary to assume that they can be reasonably well saturated by summing over one-particle intermediate states, plus those semidisconnected contributions from multiparticle states which also yield terms meromorphic in the pion energy. Even with this approximation, the multiparticle sum rules do not reduce to a simple set of commutation relations, except in the simplest multiparticle processes: those in which the initial state I contains one particle and the final state F contains two particles, or vice versa. However, within this limited context the multiparticle sum rules take quite a remarkable form:

$$[\Gamma_a(1\to 1), \Gamma_b(1\to 2)] = [\Gamma_b(1\to 1), \Gamma_a(1\to 2)]. \tag{1.2}$$

Here $\Gamma_a(1\to 1)$ and $\Gamma_a(1\to 2)$ are suitably normalized matrix elements of the pion current $\Box^2\pi_a$ (where $a = 1, 2, 3$) for processes in which one particle goes to one particle or to two particles, with $\Gamma_a(1\to 1)$ interpreted as a single-particle operator which (like the isospin or charge) acts additively on the particle labels appearing on the matrix element $\Gamma_b(1\to 2)$ (see Fig. 1).

It must be emphasized here that the matrix $\Gamma_a(1\to 1)$ will in general induce transitions between single-particle states of different mass – indeed, in evaluating the commutator (1.2), we must include in the intermediate state all the resonances needed to saturate the dispersion relation from which (1.2) was derived. This raises an important problem: When we add contributions to the commutator (1.2), in which the particle labels on the matrix element $\Gamma_a(1\to 2)$ run over particles of varying mass, what should we keep fixed? We cannot fix the energies *and* the momenta of these particles as the mass varies, so should we fix energies, or momenta, or something else? This is an old problem, which has troubled every attempt to introduce symmetry operators which do not preserve the mass of the particle states on which they act.

Fortunately, in the present case there is a well-defined answer to this problem, which is dictated by the same arguments that are used to derive (1.2). The prescription is, that in evaluating the commutators in Eq. (1.2), we must keep fixed the *celerity* of each particle. The "celerity" of a particle of momentum $\vec{p}$ and energy E is in general defined as the three-vector

$$\vec{C} \equiv \vec{n}E - \vec{p}, \tag{1.3}$$

where $\vec{n}$ is a unit vector defining the direction of motion of the pions in reactions (1.1), taken to be fixed throughout our calculations. (Incidentally, the popular "rapidity" variable is just the logarithm of $\vec{n}\cdot\vec{C}$.) Given the mass and celerity of a particle, we can determine its momentum and energy from the formulas

$$\vec{p} = -\vec{C} + \left(\frac{\vec{C}^2 + m^2}{2\vec{n}\cdot\vec{C}}\right)\vec{n}, \tag{1.4}$$

$$E = \frac{\vec{C}^2 + m^2}{2\vec{n}\cdot\vec{C}}. \tag{1.5}$$

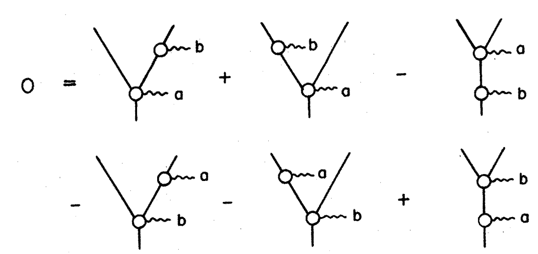

FIG. 1. Graphical representation of the commutation relation (1.2). (In all figures the reaction proceeds upwards, wavy lines denote the pion current, solid lines denote other hadrons.)

Thus, in interpreting commutation relations like Eq. (1.2), we must think of $\Gamma_a(1\to 2)$ as a matrix element for either the scattering process

$$\pi_a + 1 \to 2 + 3 \tag{1.6}$$

or the decay process

$$1 \to \pi_a + 2 + 3 \tag{1.7}$$

in which particles 1, 2, and 3 have fixed celerities $\vec{C}_1$, $\vec{C}_2$, and $\vec{C}_3$, but in which the labels describing the types of particles (including their masses and spins) are allowed to vary, with momenta and energies always given by Eqs. (1.4) and (1.5). The momentum and energy of the pion in these reactions are always taken to have the form

$$\vec{p}_\pi = \vec{n}\omega, \quad p_\pi^0 = \omega, \tag{1.8}$$

so the pion carries zero celerity, and, therefore,

$$\vec{C}_1 = \vec{C}_2 + \vec{C}_3. \tag{1.9}$$

Conservation of energy or momentum in the scattering reaction (1.6) would give a pion energy

$$\omega = \frac{\vec{C}_2{}^2 + m_2{}^2}{2\vec{n}\cdot\vec{C}_2} + \frac{\vec{C}_3{}^2 + m_3{}^2}{2\vec{n}\cdot\vec{C}_3} - \frac{\vec{C}_1{}^2 + m_1{}^2}{2\vec{n}\cdot\vec{C}_1}. \tag{1.10}$$

This quantity can be positive or negative; if (1.10) is positive then $\Gamma_a(1\to 2)$ does describe the scattering reaction (1.6), while if (1.10) is negative then $\Gamma_a(1\to 2)$ describes the decay process (1.7). In summing over particle states of varying mass in Eq. (1.2), we will always encounter some terms in which (1.10) is negative and others in which (1.10) is positive, so commutation relations like (1.2) really provide algebraic constraints on an over-all amplitude which describes *both* scattering and decay.

It follows from the prescription of fixed celerity, that the single-particle operator $\Gamma_a(1\to 1)$ will in general change not only the mass but also the *direction of motion*, and hence the spin, of the particles on which it acts. Thus, $\Gamma_a(1\to 1)$ is not precisely the same as the pion-emission matrix X_a introduced in earlier work,[2] which was defined only for collinear processes. However, it is shown here that $\Gamma_a(1\to 1)$ is related to X_a by a mass- and celerity-dependent Wigner rotation so that $\Gamma_a(1\to 1)$, like X_a, generates an $SU(2)\otimes SU(2)$ algebra, consisting of $\Gamma_a(1\to 1)$ and the isospin. In consequence, any multiparticle transition matrix may be broken up into a series of terms which transform according to the various irreducible representations of this algebra. From this group-theoretical viewpoint, our main result (1.2) just says that $\Gamma_a(1\to 2)$ consists only of the chiral representations

$$(0,0),\ (\tfrac{1}{2},\tfrac{1}{2}),\ (1,1),\ (\tfrac{3}{2},\tfrac{3}{2}),\ \ldots, \tag{1.11}$$

but *not* of the representations

$$(0,1),\ (1,0),\ (\tfrac{1}{2},\tfrac{3}{2}),\ (\tfrac{3}{2},\tfrac{1}{2}),\ (1,2),\ \ldots, \tag{1.12}$$

which could otherwise contribute to an isovector transition matrix.

This sort of elegant algebraic constraint is not so easy to apply to real scattering and decay problems, for the simple practical reason that we do not know many of the matrix elements of the $SU(2)\otimes SU(2)$ generators X_a or $\Gamma_a(1\to 1)$. However, although unknown, these matrix elements are knowable: What is needed is a major program of measurement of decay amplitudes for the "cascade" decays like $A_1\to\rho+\pi$, $\rho\to\pi+\pi$, in which a sequence of resonant states decay into one another by single-pion emission. Given our present ignorance of the "mixing angles" in X_a and $\Gamma_a(1\to 1)$, perhaps the main lesson to be learned from the present work is not so much the particular algebraic results, that chiral representations (1.12) are excluded from the transition matrix $\Gamma_a(1\to 2)$, but instead the more general conclusion, that multiparticle sum rules can set algebraic constraints on the amplitudes for transitions among states containing several particles of fixed celerity.

II. KINEMATICS

We shall consider inelastic reactions of the general form

$$\pi_a(q) + I(P) \to \pi_b(q') + F(P'), \tag{2.1}$$

$$I(P) \to \pi_a(-q) + \pi_b(q') + F(P'), \tag{2.2}$$

$$\pi_a(q) + \pi_b(-q') + I(P) \to F(P'), \tag{2.3}$$

$$\pi_b(-q') + I(P) \to \pi_a(-q) + F(P'), \tag{2.4}$$

where I and F are arbitrary multiparticle initial and final states with total four-momenta P^μ and P'^μ, while π_a and π_b are massless pions with isovector indices a and b and four-momenta $\pm q^\mu$ and $\pm q'^\mu$. It is convenient to regard all four reactions as special cases of a single process, with the specific reaction determined by the signs of q^0 and q'^0. Thus, it will be understood that we are dealing with (2.1) or (2.2) or (2.3) or (2.4) according as $q^0>0$, $q'^0>0$ or $q^0<0$, $q'^0>0$ or $q^0>0$, $q'^0<0$ or $q^0<0$, $q'^0<0$.

The pion four-momenta are related by the energy-momentum conservation condition

$$q^\mu - q'^\mu = P'^\mu - P^\mu. \tag{2.5}$$

All the particles in the states I and F will be taken to have fixed momenta, so P^μ and P'^μ are also fixed, and the matrix element here may be regarded as a function of q^μ alone. However, there is a constraint: In order that both pions should remain on their mass shells, we must have

$$0=q'^2=(q+P-P')^2=2q\cdot(P-P')+(P-P')^2 \tag{2.6}$$

and also

$$0=q^2=(q'-P+P')^2=-2q'\cdot(P-P')+(P-P')^2. \tag{2.7}$$

We plan eventually to apply a soft-pion theorem, so we must require that the physical region for our reaction should include either the point $q^\mu=0$, or the point $q'^\mu=0$, or both. Equation (2.6) or (2.7) can only be satisfied at such points if $(P-P')^2$ vanishes, so the momentum transfer here must be proportional to a fixed lightlike vector:

$$P^\mu-P'^\mu=n^\mu\Delta, \tag{2.8}$$

$$n^0=|\vec{n}|=1, \tag{2.9}$$

with Δ a fixed energy transfer, which may be positive or negative. The constraint (2.6) now reads $q\cdot n=0$, and since q and n are both lightlike, this implies that q^μ is also proportional to n^μ:

$$q^\mu=n^\mu\omega. \tag{2.10}$$

Finally, (2.5), (2.8), and (2.10) give the other pion four-momentum as

$$q'^\mu=n^\mu(\omega+\Delta). \tag{2.11}$$

The matrix element may thus be regarded as a function of a *single* variable ω.

It is convenient to write the S matrix for the reactions (2.1)–(2.4) in terms of a "semicovariant" matrix element M:

$$S_{FI}^{ba}=2\pi\delta^4(P+q-P'-q')(4|\omega|\,|\omega+\Delta|)^{-1/2}M_{FI}^{ba}(\omega). \tag{2.12}$$

In accordance with our previous remarks, this formula should be understood to give the S matrix for reaction (2.1) when $\omega>0$, $\omega+\Delta>0$; for reaction (2.2) when $\omega<0$, $\omega+\Delta>0$; for reaction (2.3) when $\omega>0$, $\omega+\Delta<0$; and for reaction (2.4) when $\omega<0$, $\omega+\Delta<0$. It follows from this convention, and from Bose statistics, that M obeys the crossing relation

$$M_{FI}^{ba}(\omega)=M_{FI}^{ab}(-\omega-\Delta). \tag{2.13}$$

III. SINGLE-VARIABLE DISPERSION RELATIONS

We have specified that all of the hadrons in the initial and final states I and F have fixed momenta, so that only the pion four-momenta are allowed to vary. The point of this assumption is that then the "semicovariant" matrix element M is free of kinematic singularities in the pion four-momentum components q^μ and q'^μ, because we have already extracted the pion "wave functions" $(2|q^0|)^{-1/2}$ and $(2|q'^0|)^{-1/2}$ in defining M, while the corresponding "wave functions" of the hadrons in I and F have fixed arguments. Since q^μ and q'^μ are linear functions of ω, it follows that $M(\omega)$ is free of kinematic singularities in ω.

Indeed, this is one multiparticle problem where a dispersion relation in ω can be proved by simply recapitulating the classic proof[3] of the dispersion relation for elastic scattering of a photon. We note first that $M(\omega)$ may be written as the Fourier transform of a time-ordered product

$$M_{FI}^{ba}(\omega)=\int e^{i\omega n\cdot(x-y)}\,\Box_x^{\,2}\Box_y^{\,2}\langle F|T\{\pi_b(y)\pi_a(x)\}|I\rangle\, d^4x, \tag{3.1}$$

with $\vec{\pi}(x)$ any renormalized pion field. To extend this matrix element off the real axis, we define the closely related function

$$A_{FI}^{ba}(z)\equiv\begin{cases}\displaystyle\int e^{izn\cdot(x-y)}\Box_x^{\,2}\Box_y^{\,2}\langle F|\theta(y-x)[\pi_b(y),\pi_a(x)]|I\rangle\, d^4x & \text{for } \mathrm{Im}z>0\\ \displaystyle -\int e^{izn\cdot(x-y)}\Box_x^{\,2}\Box_y^{\,2}\langle F|\theta(x-y)[\pi_b(y),\pi_a(x)]|I\rangle\, d^4x & \text{for } \mathrm{Im}z<0,\end{cases} \tag{3.2}$$

where $\theta(x)$ is the usual step function, equal to unity for $x^0>0$ and zero for $x^0<0$. The time-ordered and retarded products are related by

$$T\{\pi_b(y)\pi_a(x)\}=\theta(y-x)[\pi_b(y),\pi_a(x)]+\pi_a(x)\pi_b(y),$$

so M is related to F by

$$M_{FI}^{ba}(\omega)=A_{FI}^{ba}(\omega+i\epsilon)+\int e^{i\omega n\cdot(x-y)}\langle F|\Box^2\pi_a(x)\Box^2\pi_b(y)|I\rangle\, d^4x. \tag{3.3}$$

The commutators in Eq. (3.2) are supposed to vanish when $x^\mu-y^\mu$ is outside the light cone, so $x^\mu-y^\mu$ remains within the forward light cone for $\mathrm{Im}z>0$ and within the backward light cone for $\mathrm{Im}z<0$; hence, $A(z)$ is an analytic function of z, with singularities only on the real axis. Further, the discontinuity of $A(z)$ across the real axis is immediately given by Eq. (3.2) as

$$A_{FI}^{ba}(\omega+i\epsilon)-A_{FI}^{ba}(\omega-i\epsilon)=\int e^{i\omega n\cdot(x-y)}\langle F|[\Box^2\pi_b(y),\Box^2\pi_a(x)]|I\rangle\,d^4x$$
$$=\sum_N(2\pi)^4\delta^4(p_I-p_N+\omega n)\langle F|\Box^2\pi_b(0)|N\rangle\langle N|\Box^2\pi_a(0)|I\rangle$$
$$-\sum_N(2\pi)^4\delta^4(p_N+\omega n-p_F)\langle F|\Box^2\pi_a(0)|N\rangle\langle N|\Box^2\pi_b(0)|I\rangle, \tag{3.4}$$

the sums running over a complete set of intermediate states N. Aside from possible subtractions, the analytic amplitude A then satisfies a dispersion relation, which for $\mathrm{Im}z>0$ reads

$$A_{FI}^{ba}(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{A_{FI}^{ba}(\omega+i\epsilon)-A_{FI}^{ba}(\omega-i\epsilon)}{\omega-z}\right)d\omega$$
$$=-i(2\pi)^3\sum_N\delta^3(\vec{p}_I-\vec{p}_N-(p_I^0-p_N^0)\vec{n})\frac{\langle F|\Box^2\pi_b(0)|N\rangle\langle N|\Box^2\pi_a(0)|I\rangle}{p_N^0-p_I^0-z}$$
$$+i(2\pi)^3\sum_N\delta^3(\vec{p}_N-\vec{p}_F-(p_N^0-p_F^0)\vec{n})\frac{\langle F|\Box^2\pi_a(0)|N\rangle\langle N|\Box^2\pi_b(0)|I\rangle}{p_F^0-p_N^0-z}. \tag{3.5}$$

To calculate $M_{FI}^{ba}(\omega)$, we recall that

$$i(p_F^0-p_N^0-\omega-i\epsilon)^{-1}+2\pi\delta(p_F^0-p_N^0-\omega)=i(p_F^0-p_N^0-\omega+i\epsilon)^{-1},$$

so that (3.3) and (3.5) give the dispersion relation for M:

$$M_{FI}^{ba}(\omega)=-i(2\pi)^3\sum_N\delta^3(\vec{p}_I-\vec{p}_N-(p_I^0-p_N^0)\vec{n})\frac{\langle F|\Box^2\pi_b(0)|N\rangle\langle N|\Box^2\pi_a(0)|I\rangle}{p_N^0-p_I^0-\omega-i\epsilon}$$
$$+i(2\pi)^3\sum_N\delta^3(\vec{p}_N-\vec{p}_F-(p_N^0-p_F^0)\vec{n})\frac{\langle F|\Box^2\pi_a(0)|N\rangle\langle N|\Box^2\pi_b(0)|I\rangle}{p_F^0-p_N^0-\omega+i\epsilon}. \tag{3.6}$$

To display the crossing symmetry of M explicitly, we recall that

$$p_F^0=p_I^0-\Delta, \tag{3.7}$$

$$\vec{p}_F=\vec{p}_I-\vec{n}\Delta. \tag{3.8}$$

The dispersion relation (3.6) may thus be written

$$M_{FI}^{ba}(\omega)=-i(2\pi)^3\sum_N\delta^3(\vec{p}_I-\vec{p}_N-(p_I^0-p_N^0)\vec{n})\left(\frac{\langle F|\Box^2\pi_b(0)|N\rangle\langle N|\Box^2\pi_a(0)|I\rangle}{p_N^0-p_I^0-\omega-i\epsilon}+\frac{\langle F|\Box^2\pi_a(0)|N\rangle\langle N|\Box^2\pi_b(0)|I\rangle}{p_N^0-p_I^0+\omega+\Delta-i\epsilon}\right) \tag{3.9}$$

in manifest agreement with the crossing relation (2.13).

In general, the dispersion relation (3.6) or (3.9) will not be valid without subtractions. For this reason, it is useful to introduce the odd amplitude

$$M_{FI}^{(-)ba}(\omega)\equiv\left(\frac{M_{FI}^{ba}(\omega)-M_{FI}^{ab}(\omega)}{2\omega+\Delta}\right). \tag{3.10}$$

The numerator vanishes at the crossing-symmetric point $\omega=-\Delta/2$, so the division by $2\omega+\Delta$ does not introduce a kinematic singularity. Also, the numerator is expected to behave as $|\omega|\to\infty$ like ω^α, where α is the intercept of the ρ trajectory; since $\alpha<1$, the amplitude $M^{(-)}$ would then vanish for $|\omega|\to\infty$. Hence, we expect $M_{FI}^{(-)ba}(\omega)$ to satisfy an unsubtracted dispersion relation. By either repeating the steps that led to (3.9), or (less properly) by using (3.9) itself, we find this dispersion relation to be

$$M_{FI}^{(-)ba}(\omega)=-i(2\pi)^3\sum_N\delta^3(\vec{p}_I-\vec{p}_N-(p_I^0-p_N^0)\vec{n})\left(\frac{\langle F|\Box^2\pi_b(0)|N\rangle\langle N|\Box^2\pi_a(0)|I\rangle-\langle F|\Box^2\pi_a(0)|N\rangle\langle N|\Box^2\pi_b(0)|I\rangle}{(p_N^0-p_I^0+\omega+\Delta-i\epsilon)(p_N^0-p_I^0-\omega-i\epsilon)}\right). \tag{3.11}$$

Up to this point, we have not bothered to specify whether the states labeled F, I, and N are "in" or "out" states. Of course, it is understood that when we label the final and initial states, these are, respectively, "out" and "in" states, while the sum over intermediate states N can be taken over either all "out" states or all "in" states. It will be very convenient to expand this sum into a double sum, and write (3.11) as

$$M_{FI}^{(-)ba}(\omega)=-i(2\pi)^3\sum_{NN'}\delta^3(\vec{p}_I-\vec{p}_N-(p_I^0-p_N^0)\vec{n})\,[\langle F,\mathrm{out}|\Box^2\pi_b(0)|N',\mathrm{in}\rangle(S^{-1})_{N'N}\langle N,\mathrm{out}|\Box^2\pi_a(0)|I,\mathrm{in}\rangle$$

$$-\langle F,\text{out}|\Box^2\pi_a(0)|N',\text{in}\rangle(S^{-1})_{N'N}\langle N,\text{out}|\Box^2\pi_b(0)|I,\text{in}\rangle]$$

$$\times(p_N^0-p_F^0+\omega+\Delta-i\epsilon)^{-1}(p_N^0-p_I^0-\omega-i\epsilon)^{-1}, \tag{3.12}$$

where S^{-1} is the inverse S matrix

$$(S^{-1})_{N'N}\equiv\langle N',\text{in}|N,\text{out}\rangle\,. \tag{3.13}$$

This version of Eq. (3.11) has the advantage that matrix elements of the pion current $\Box^2\vec{\pi}$ are always taken with an "out" state on the left and an "in" state on the right.

IV. SINGLE-PARTICLE DOMINANCE

At this point, we make our one approximation, and assume that $M(\omega)$ is approximately meromorphic in ω, with poles arising from a large number of resonant states with negligible width. This approximation is of course not new: It has been used in conjunction with current algebra from earliest times,[4] it forms the basis for the algebraic realizations of chiral symmetry,[2] and it is a key ingredient in the Veneziano model and its successors.[5]

In our present context, it is important to recognize that poles in $M(\omega)$ can arise not only from single-particle intermediate states N, but also from multiparticle states, provided that the matrix elements $\langle F|\pi|N\rangle$ and $\langle N|\pi|I\rangle$ are suitably disconnected. In the most general pole contribution in Eq. (3.12), the initial and final states are divided into two groups of particles:

$$I=I_1+I_2,\quad F=F_1+F_2\,. \tag{4.1}$$

The intermediate states N and N' are divided into two groups plus a single-particle state:

$$N=N_1+N_2+\alpha(\vec{p}),\quad N'=N_1'+N_2'+\alpha'(\vec{p})\,, \tag{4.2}$$

and the matrix elements break up into disconnected pieces

$$\langle F,\text{out}|\Box^2\pi_b(0)|N',\text{in}\rangle\to\langle F_2,\text{out}|\Box^2\pi_b(0)|\,\alpha'(\vec{p}\,')+N_2',\text{in}\rangle_c\,S_{F_1N_1'}\,,$$

$$(S^{-1})_{N'N}\to(S^{-1})_{N_2'N_2}(S^{-1})_{N_1'N_1}\delta_{\alpha'\alpha}\delta^3(\vec{p}\,'-\vec{p})\,,$$

$$\langle N,\text{out}|\Box^2\pi_a(0)|I,\text{in}\rangle\to\langle\alpha(\vec{p})+N_1,\text{out}|\Box^2\pi_a(0)|I_1,\text{in}\rangle_c\,S_{N_2I_2}\,,$$

where c denotes "connected part" (see Fig. 2). The S matrices all cancel, leaving us with $N_1=F_1$, $N_2'=I_2$, so that (3.12) becomes

$$M_{FI}^{(-)ba}(\omega)=-i(2\pi)^3\sum_{F_1+F_2=F}\;\sum_{I_1+I_2=I}\;\sum_{\alpha}\int d^3p\,\delta^3(\vec{p}_{I_1}-\vec{p}_{F_1}-\vec{p}-(p_{I_1}^0-p_{F_1}^0-p^0)\vec{n})$$

$$\times[\langle F_2,\text{out}|\Box^2\pi_b(0)|\,\alpha(\vec{p})+I_2,\text{in}\rangle_c\langle\alpha(\vec{p})+F_1,\text{out}|\Box^2\pi_a(0)|I_1,\text{in}\rangle_c-(a\leftrightarrow b)]$$

$$\times(p_\alpha^0+p_{I_2}^0-p_{F_2}^0+\omega-i\epsilon)^{-1}(p_\alpha^0+p_{F_1}^0-p_{I_1}^0-\omega-i\epsilon)^{-1}\,, \tag{4.3}$$

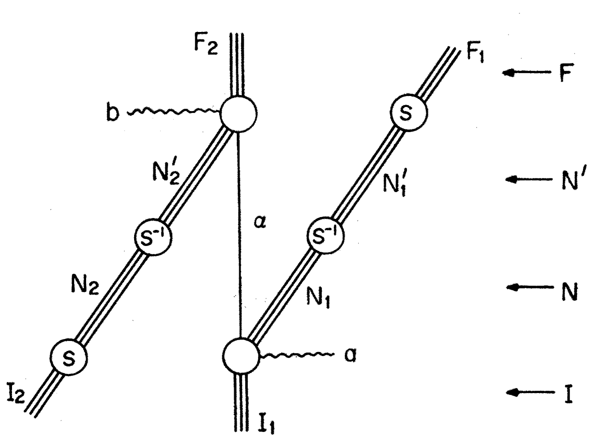

FIG. 2. Graphical representations of the general pole terms in the dispersion relation (3.12).

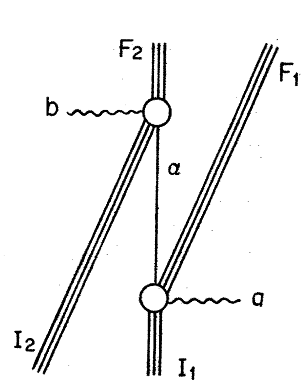

FIG. 3. The same as Fig. 2, after cancellation of the S matrices.

the sums running over all ways of dividing the initial and final states into two groups of particles (see Fig. 3).

It is very convenient at this point to introduce a new variable to characterize physical states, the "celerity" $\vec{C}$. For a state of momentum $\vec{P}$ and energy P^0, the celerity is

$$\vec{C} \equiv \vec{n}P^0 - \vec{P}. \tag{4.4}$$

According to Eqs. (3.7) qnd (3.8), the total celerity is conserved in our over-all reaction

$$\vec{C}_F = \vec{C}_I, \tag{4.5}$$

and the δ function in (4.3) ensures the conservation of celerity in intermediate states

$$\vec{C} \equiv \vec{n}p^0_\alpha - \vec{p}_\alpha = \vec{C}_{I_1} - \vec{C}_{F_1} = \vec{C}_{F_2} - \vec{C}_{I_2}. \tag{4.6}$$

Knowing the celerity and the mass of the particle α, we can easily calculate its energy and momentum,

$$p^0_\alpha = \frac{\vec{C}^2 + m_\alpha{}^2}{2\vec{n}\cdot\vec{C}}, \tag{4.7}$$

$$\vec{p}_\alpha = \vec{n}\left(\frac{\vec{C}^2 + m_\alpha{}^2}{2\vec{n}\cdot\vec{C}}\right) - \vec{C}. \tag{4.8}$$

Hence, we can drop the label $\vec{p}$ in the matrix elements in (4.3), it being understood that the momentum of the particle α is fixed by the conservation of celerity. The integral of the δ function in (4.3) is

$$\int d^3p\, \delta^3(\vec{C}_{F_1} - \vec{C}_{I_1} + p^0\vec{n} - \vec{p}) = \frac{p^0_\alpha}{\vec{n}\cdot(\vec{C}_{I_1} - \vec{C}_{F_1})}, \tag{4.9}$$

so Eq. (4.3) reads

$$M^{(-)ba}_{FI}(\omega) = -i(2\pi)^3 \sum_{F_1+F_2=F}\ \sum_{I_1+I_2=I}\ \sum_\alpha \frac{p^0_\alpha}{\vec{n}\cdot(\vec{C}_{I_1} - \vec{C}_{F_1})}$$
$$\times[\langle F_2, \text{out}|\Box^2\pi_b(0)|\alpha + I_2, \text{in}\rangle_c \langle \alpha + F_1, \text{out}|\Box^2\pi_a(0)|I_1, \text{in}\rangle_c - (a \leftrightarrow b)]$$
$$\times(p^0_\alpha + p^0_{I_2} - p^0_{F_2} + \omega - i\epsilon)^{-1}(p^0_\alpha + p^0_{F_1} - p^0_{I_1} - \omega - i\epsilon)^{-1}, \tag{4.10}$$

it being understood that the energy and momentum of particle α are fixed by the conservation of celerity.

The reader can easily check that the pole structure displayed here is precisely the same as would be found in perturbation theory from Feynman diagrams like Fig. 3. From this point of view, a factor p^0_α enters in (4.10) to cancel the normalization factors $(p^0_\alpha)^{-1/2}$ appearing in the noncovariant matrix elements of the pion current, while the factor $\vec{n}\cdot(\vec{C}_{I_1} - \vec{C}_{F_1})$ arises from the denominator of the virtual-particle propagator in Fig. 3:

$$(p_{I_1} + n\omega - p_{F_1})^2 + m_\alpha{}^2 = (p_{I_1} - p_{F_1})^2 + m_\alpha{}^2 + 2n\cdot(p_{I_1} - p_{F_1})\omega = -2\vec{n}\cdot(\vec{C}_{I_1} - \vec{C}_{F_1})(\omega - \omega_{\text{pole}}). \tag{4.11}$$

V. CHIRAL SUM RULES

It will now be assumed that our massless pions are the Goldstone bosons of an underlying chiral symmetry of the strong interactions. It follows then that the matrix element M must vanish when either q^μ or q'^μ vanishes, so that[6]

$$M^{ba}_{FI}(0) = M^{ba}_{FI}(-\Delta) = 0. \tag{5.1}$$

There are exceptions to this condition, but we will ignore them for the moment, and simply assume that (5.1) is valid; the exceptions will be taken up at the end of this section.

As long as we consider only purely inelastic reactions with $\Delta \neq 0$, it follows from (5.1) that the odd amplitude $M^{(-)}$ also vanishes for zero ω:

$$M^{(-)ba}_{FI}(0) = 0. \tag{5.2}$$

In the elastic case $\Delta = 0$, the value of $M^{(-)}(\omega)$ at $\omega = 0$ depends on the *derivative* of $M(\omega)$ at $\omega = 0$, which might receive contributions from equal-time commutators of the axial-vector currents. To avoid these complications, we will confine our attention to reactions with $\Delta \neq 0$.

Our sum rule follows immediately from Eqs. (5.2) and (4.10):

$$0=\sum_{F_1+F_2=F}\sum_{I_1+I_2=I}\sum_{\alpha}\frac{p_\alpha^0}{\vec{n}\cdot(\vec{C}_{I_1}-\vec{C}_{F_1})}\left[\langle F_2,\text{out}|\Box^2\pi_b(0)|\alpha+I_2,\text{in}\rangle_c\langle\alpha+F_1,\text{out}|\Box^2\pi_a(0)|I_1,\text{in}\rangle_c-(a\leftrightarrow b)\right]$$
$$\times(p_\alpha^0+p_{I_2}^0-p_{F_2}^0)^{-1}(p_\alpha^0+p_{F_1}^0-p_{I_1}^0)^{-1}. \tag{5.3}$$

It is convenient at this point to introduce a new notation for matrix elements of the pion current. Given two states I and F with four-momenta P^μ and P'^μ, we define the reduced matrix element Γ_a by

$$\langle F,\text{out}|\Box^2\pi_a(0)|I,\text{in}\rangle_c\equiv 2F_\pi^{-1}(P'^2-P^2)N_F N_I(\Gamma_a)_{F,I}, \tag{5.4}$$

where $F_\pi\simeq 190$ MeV is a convenient normalization factor, and N is the product of the normalization factors $(2\pi)^{-3/2}$ and $(2p^0)^{-1/2}$, one for each particle in the state. We note that the "mass" difference P'^2-P^2 between states of equal celerity may be written in terms of the energy difference as

$$\begin{aligned}P'^2-P^2&=\vec{P}'^2-(P'^0)^2-\vec{P}^2+(P^0)^2\\&=(\vec{n}\cdot\vec{P}')^2-(P'^0)^2-(\vec{n}\cdot\vec{P})^2+(P^0)^2\\&=(\vec{n}\cdot\vec{P}-P^0)(\vec{n}\cdot\vec{P}'+P'^0-\vec{n}\cdot\vec{P}-P^0)\\&=2(\vec{n}\cdot\vec{C})(P^0-P'^0).\end{aligned}$$

Thus, the energy differences arising from the definition of Γ_a cancel the energy denominators in Eq. (5.3), while the factor p_α^0 in (5.3) is canceled by the two factors of $(p_\alpha^0)^{-1/2}$ appearing in the normalization factors $N_{\alpha+I_2}$ and $N_{\alpha+F_1}$. Our sum rule now reads

$$0=\sum_{F_1+F_2=F}\sum_{I_1+I_2=I}\frac{(\vec{n}\cdot\vec{C}_{F_2})(\vec{n}\cdot\vec{C}_{I_1})}{\vec{n}\cdot(\vec{C}_{I_1}-\vec{C}_{F_1})}$$
$$\times\sum_\alpha\left[(\Gamma_b)_{F_2,\alpha+I_2}(\Gamma_a)_{\alpha+F_1,I_1}-(a\leftrightarrow b)\right]. \tag{5.5}$$

As always, the momentum of particle α is fixed by the conservation of celerity in the transitions $\alpha+I_2\to F_2$ and $I_1\to\alpha+F_1$.

Now let us come back to the exceptions to Eq. (5.1). This condition will be violated only if the amplitude for "scattering" of an axial-vector current from the process $I\to F$ has poles,[7] arising from diagrams in which one of the axial-vector currents interacts with an incoming or outgoing hadron, without changing its mass. Such poles do not arise for reactions in which all participating hadrons are mesons of zero strangeness, because in this case the interaction with the axial-vector current must change the meson's G parity, and there are no strictly degenerate nonstrange meson states of opposite G parity. Poles *can* invalidate Eq. (5.1) for reactions involving baryons or strange mesons, but even so, the sum rule (5.5) will still be valid. Note that for each such pole there is also a missing term in Eq. (4.3), in which the matrix element of $\Box^2\pi_b$ or $\Box^2\pi_a$ is taken between single-particle states of equal mass, and therefore vanishes. Experience shows[2] that the pole terms which violate Eq. (5.1) always simply supply the terms, involving transitions between single-particle states of equal mass, which would otherwise be missing in Eq. (5.5).

VI. THE FIVE-POINT SUM RULE

Our general result, Eq. (5.5), does not take the form of a simple algebraic statement about the Γ matrices. There are, however, reactions $I\to F$ for which (5.5) reduces to a statement about commutators. One such reaction is the well-explored case in which I and F contain one particle each. Apparently the only other reactions for which (5.5) takes a simple algebraic form are those "five-point" reactions in which I contains one particle and F contains two particles, or vice versa.

Let us consider a reaction of this latter type:

$$\beta+\pi_a\to\gamma+\delta+\pi_b, \tag{6.1}$$

where β, γ, and δ are single-particle states with celerities $\vec{C}_1$, $\vec{C}_2$, and $\vec{C}_3$, respectively. The only terms which contribute to the first two sums in Eq. (5.5) are those in which I_1 contains just particle β and I_2 is empty, while either

F_1 empty, $F_2=\gamma+\delta$,

or

$F_1=\gamma$, $F_2=\delta$,

or

$F_1=\delta$, $F_2=\gamma$.

(In general, neither F_2 nor I_1 may be empty.) The factor $\vec{n}\cdot\vec{C}_{I_1}$ in (5.5) is then fixed,

$$\vec{n}\cdot\vec{C}_{I_1}=\vec{n}\cdot\vec{C}_1,$$

while the other celerities cancel,

$$\vec{n}\cdot(\vec{C}_{I_1}-\vec{C}_{F_1})=\vec{n}\cdot(\vec{C}_{F_2}-\vec{C}_{I_2})=\vec{n}\cdot\vec{C}_{F_2}.$$

Thus, Eq. (5.5) here takes the form

$$0=\sum_\alpha\left[(\Gamma_b)_{\gamma+\delta,\alpha}(\Gamma_a)_{\alpha,\beta}+(\Gamma_b)_{\delta,\alpha}(\Gamma_a)_{\alpha+\gamma,\beta}\right.$$
$$\left.+(\Gamma_b)_{\gamma,\alpha}(\Gamma_a)_{\alpha+\delta,\beta}-(a\leftrightarrow b)\right]. \tag{6.2}$$

In order to bring out the algebraic significance of this result, let us now define an additive one-particle transition matrix $\Gamma_b(1\to 1)$, by specifying that the products of this matrix with any transition matrix M that takes one particle into two particles are[8]

$$(\Gamma_b(1\to 1)M)_{\gamma+\delta,\beta}=\sum_\alpha [(\Gamma_b)_{\gamma,\alpha}(M)_{\alpha+\delta,\beta}+(\Gamma_b)_{\delta,\alpha}(M)_{\gamma+\alpha,\beta}], \tag{6.3}$$

$$(M\Gamma_b(1\to 1))_{\gamma+\delta,\beta}=\sum_\alpha (M)_{\gamma+\delta,\alpha}(\Gamma_b)_{\alpha,\beta}. \tag{6.4}$$

Then Eq. (6.2) reads

$$[\Gamma_b(1\to 1),\ \Gamma_a(1\to 2)]=[\Gamma_a(1\to 1),\ \Gamma_b(1\to 2)], \tag{6.5}$$

where $\Gamma_a(1\to 2)$ is the submatrix of Γ_a referring to transitions from one particle to two particles.

It is shown in Appendix A that the general one-particle matrix elements $(\Gamma_a)_{\beta,\alpha}$ are related to the *collinear* pion transition-matrix elements $(X_a)_{\beta,\alpha}$ defined in earlier work, by the formula

$$(\Gamma_a)_{\beta\sigma_\beta,\alpha\sigma_\alpha}=\sum_\lambda D^{(j_\beta)}_{\sigma_\beta\lambda}[R^{-1}(\vec{C},\ m_\beta)] \times(X_\alpha(\lambda))_{\beta,\alpha}D^{(j_\alpha)}_{\lambda\sigma_\alpha}[R(\vec{C},\ m_\alpha)]. \tag{6.6}$$

Here σ_β and σ_α are the components of the spin along the pion direction $\vec{n}$, which we now separate from the particle-type labels α and β; R is a rotation which depends both on the celerity (which is equal for α and β) and the mass; and $(X_a(\lambda))_{\beta,\alpha}$ is a suitably normalized matrix element for the transition $\alpha\to\beta+\pi_a$, in which α and β have helicity λ and momenta parallel or antiparallel to the pion momentum. The collinear matrix elements $X_a(\lambda)$ and the isospin matrices T_a were shown,[2] on the basis of the single-particle saturated inelastic Adler-Weisberger sum rules, to form a representation of the SU(2)⊗SU(2) algebra:

$$[X_a(\lambda),\ X_b(\lambda)]=i\epsilon_{abc}T_c, \tag{6.7}$$

$$[T_a,\ X_b(\lambda)]=i\epsilon_{abc}X_c(\lambda), \tag{6.8}$$

$$[T_a,\ T_b]=i\epsilon_{abc}T_c. \tag{6.9}$$

In consequence, the matrix (6.6) for transitions among single-particle states of arbitrary (but equal) celerity also generates an SU(2)⊗SU(2) algebra similar to (6.7)–(6.9). Finally, this implies that the additive single-particle matrix $\Gamma_a(1\to 1)$ defined by Eqs. (6.3) and (6.4) generates an SU(2) ⊗SU(2) algebra, in the sense that for an arbitrary one-to-two-particle transition matrix M, we have

$$[\Gamma_a(1\to 1),[\Gamma_b(1\to 1),M]]-[\Gamma_b(1\to 1),[\Gamma_a(1\to 1),M]] =i\epsilon_{abc}[T_c,\ M], \tag{6.10}$$

as well as the isospin commutation rules

$$[T_a,[\Gamma_b(1\to 1),M]]-[\Gamma_b(1\to 1),[T_a,M]] =i\epsilon_{abc}[\Gamma_c(1\to 1),\ M], \tag{6.11}$$

$$[T_a,[T_b,M]]-[T_b,[T_a,M]]=i\epsilon_{abc}[T_c,\ M]. \tag{6.12}$$

The commutation relations (6.10)–(6.12) allow us to expand any matrix M in a series of terms $M(A,\ B)$ whose commutation relations with T_a and $\Gamma_a(1\to 1)$ define a representation $(A,\ B)$ of the algebra SU(2)⊗SU(2). In particular, the pion-emission matrix $\Gamma_a(1\to 2)$ is an isovector in the sense that

$$[T_a,\ \Gamma_b(1\to 2)]=i\epsilon_{abc}\Gamma_c(1\to 2). \tag{6.13}$$

Hence, the only chiral representations which can contribute to this matrix are those in which A, B, and 1 satisfy a triangle inequality, so that

$$B=A-1,\quad\text{or}\quad B=A,\quad\text{or}\quad B=A+1.$$

It is shown in Appendix B that the five-point sum rule (6.5) simply rules out the representations with $B=A\pm 1$, so that $\Gamma_a(1\to 2)$ must consist only of the representations $(A,\ B)$ with

$$B=A. \tag{6.14}$$

VII. OTHER SUM RULES

The general sum rule (5.5) applies to processes in which the states I and F may contain arbitrary numbers of particles, not just to the processes discussed in the last section, in which I contains one particle and F contains two. However, the sum rules in this more general context do not seem to lend themselves to any simple algebraic interpretation.

For example, consider the case in which both the initial state I and final state F contain two particles each. The sum rule (5.5) for this case is shown graphically in Fig. 4. The first eight terms add up to the commutators

$$[\Gamma_b(1\to 1),\ \Gamma_a(2\to 2)]-[\Gamma_a(1\to 1),\ \Gamma_b(2\to 2)],$$

just as in Eq. (5.5). However, the next 10 terms cannot be put into so simple a form. In particular, the celerity factors in (5.5) do not cancel or remain constant in the last eight terms of Fig. 4, because neither F_1 nor I_2 are empty here.

There are of course a great many other multiparticle sum rules, for both the limited class of five-point reactions, and the wider class of reactions involving more than five incoming and outgoing particles. For instance:

(a) Instead of the isospin-odd amplitude $M^{(-)}$ which has isospin 1 in the $\pi\pi$ channel, we could instead consider the part of the isospin-even amplitude which has isospin 2 in the $\pi\pi$ channel:

$$M_{FI}^{ba(T=2)}(\omega) \equiv M_{FI}^{ba}(\omega) + M_{FI}^{ab}(\omega) - \tfrac{2}{3}\delta_{ab} M_{FI}^{cc}(\omega).$$

This amplitude presumably satisfies an unsubtracted dispersion relation,[9] so the requirement that it vanish at $\omega = 0$ would lead to additional sum rules, just as in the single-particle case.[2]

(b) Instead of two-pion reactions, we could consider reactions involving other massless particles, such as

$$\pi + I \to \gamma + F,$$

$$\gamma + I \to \gamma + F,$$

etc. Dispersion relations may be easily derived for these processes, so presumably the use of soft-photon theorems would yield algebraic constraints relating photon and pion multiparticle transition amplitudes.

(c) Instead of reactions involving two massless or nearly massless particles, we could try to derive sum rules from the Adler "self-consistency" condition[6] for a reaction involving only *one* pion, or from the soft-photon theorem[10] for a reaction involving only *one* photon. The trouble here is that with only one massless particle involved in the process, it is much more difficult to derive a useful dispersion relation. For instance, consider the reactions

$$\pi + I \to F \quad \text{or} \quad \gamma + I \to F,$$

where I and F are arbitrary multiparticle states. What should we keep fixed as the pion or photon momentum varies? We might try to separate out a massive particle α from the final state

$$F = \alpha + F'$$

and keep fixed the momenta of all the particles in I and F'. Particle α will then have an energy-momentum four-vector

$$P_\alpha^\mu = \Delta^\mu + q^\mu,$$

where q^μ is the pion or photon momentum and Δ^μ is the fixed momentum transfer

$$\Delta^\mu = P_I^\mu - P_{F'}^\mu.$$

Hence, in order to keep particle α on the mass shell as q^μ varies, we must have

$$-m_\alpha^2 = \Delta^2 + 2q \cdot \Delta .$$

But in order to apply a soft-pion or soft-photon theorem, it is necessary that the point $q^\mu = 0$ should be in the physical region, so that we must take the invariant momentum transfer here as

$$\Delta^2 = -m_\alpha^2 .$$

Hence, the pion or photon momentum must satisfy

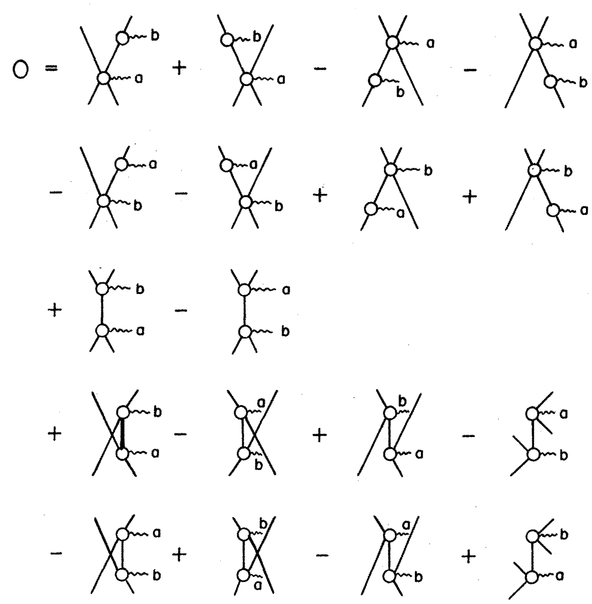

FIG. 4. Graphical representation of the sum rule (5.5), for the case where the states I and F each contain two particles.

the constraint

$$q \cdot \Delta = 0.$$

But q is lightlike and Δ is timelike, so their scalar product cannot vanish. This difficulty could perhaps be circumvented by simply assuming from the beginning that all invariant amplitudes are approximately meromorphic in all invariant squared subenergies, as in the Veneziano model.[5]

(d) There are an unlimited number of superconvergence relations which could be used to derive multiparticle sum rules. The trouble here, as in (c) above, is to know which dispersion relations to use. Again, this problem could be avoided by assuming the invariant amplitudes to be meromorphic. One might hope that the sum rules derived in this way could account for some of the algebraic features of dual-resonance models on a more general basis.

Evidently a good deal of work will have to be done to judge how fruitful these developments are likely to be.

ACKNOWLEDGMENTS

I wish to thank Francis Low for pointing out that it would be easy to derive dispersion relations for the forward scattering of a massless particle, even in multiparticle reactions. I am also grateful for the hospitality of the Collège de France and the University of Turin, where parts of this work were carried out.

APPENDIX A: LORENTZ TRANSFORMATION OF THE TRANSITION MATRIX $\Gamma_a(1\to1)$

We wish to show here that the general one-pion transition amplitude $(\Gamma_a)_{\alpha,\beta}$ is related to the collinear one-pion transition amplitude $(X_a)_{\alpha,\beta}$ by a rotation, as in Eq. (6.6).

According to Eq. (5.4), the matrix element $(\Gamma_a)_{\beta,\alpha}$ is defined by the formula

$$\langle\beta\,\vec{p}_\beta\sigma_\beta|\Box^2\pi_a(0)|\alpha\vec{p}_\alpha\sigma_\alpha\rangle=\frac{2F_\pi^{-1}(m_\alpha^{\,2}-m_\beta^{\,2})}{(2\pi)^3(4p^0_\alpha p^0_\beta)^{1/2}}(\Gamma_a(\vec{C}))_{\beta\sigma_\beta,\alpha\sigma_\alpha}. \tag{A1}$$

We are now explicitly displaying the spin components σ_β, σ_α in the $\vec{n}$ direction as well as the momenta $\vec{p}_\beta$ and $\vec{p}_\alpha$, which for a celerity $\vec{C}$ are given by Eq. (4.7) as

$$\vec{p}_{\alpha,\beta}=-\vec{C}+\left(\frac{m_{\alpha,\beta}^{\;2}+\vec{C}^2}{2\vec{n}\cdot\vec{C}}\right)\vec{n}. \tag{A2}$$

Under an arbitrary Lorentz transformation $\Lambda^\mu{}_\nu$, the single-particle states in (A1) transform into[10]

$$U[\Lambda]|\alpha\vec{p}_\alpha\sigma_\alpha\rangle=\sum_{\sigma'_\alpha}\left(\frac{p_\alpha'^{0}}{p_\alpha^0}\right)^{1/2}D^{(j_\alpha)}_{\sigma'_\alpha\sigma_\alpha}\left[L^{-1}\left(\frac{\vec{p}'_\alpha}{m_\alpha}\right)\Lambda L\left(\frac{\vec{p}_\alpha}{m_\alpha}\right)\right]|\alpha\vec{p}'_\alpha\sigma'_\alpha\rangle\,, \tag{A3}$$

where $L(\vec{p}/m)$ is the "boost" which takes a particle of mass m from rest to momentum $\vec{p}$; $D^{(j)}[R]$ is the usual spin-j unitary representation of the three-dimensional rotation group; and p'^μ is the transformed momentum

$$p_\alpha'^\mu=\Lambda^\mu{}_\nu p^\nu_\alpha. \tag{A4}$$

Since $\Box^2\pi_a$ is a scalar, we can use (A3) in (A1), and find

$$\frac{2F_\pi^{-1}(m_\alpha^{\,2}-m_\beta^{\,2})}{(2\pi)^3(4p_\alpha'^0p_\beta'^0)^{1/2}}(\Gamma_a(\vec{C}))_{\beta\sigma_\beta,\alpha\sigma_\alpha}=\sum_{\sigma'_\beta\sigma'_\alpha}D^{(j_\beta)\,-1}_{\sigma_\beta\sigma'_\beta}\left[L^{-1}\left(\frac{\vec{p}'_\beta}{m_\beta}\right)\Lambda L\left(\frac{\vec{p}_\beta}{m_\beta}\right)\right]\times\langle\beta\,\vec{p}'_\beta\sigma'_\beta|\Box^2\pi_a(0)|\alpha\vec{p}'_\alpha\sigma'_\alpha\rangle D^{(j_\alpha)}_{\sigma'_\alpha\sigma_\alpha}\left[L^{-1}\left(\frac{\vec{p}'_\alpha}{m_\alpha}\right)\Lambda L\left(\frac{\vec{p}_\alpha}{m_\alpha}\right)\right]. \tag{A5}$$

We shall restrict our attention here to Lorentz transformations $\Lambda^\mu{}_\nu$ which leave the pion momenta fixed:

$$\Lambda^\mu{}_\nu n^\nu=n^\mu. \tag{A6}$$

The fact that $\vec{p}_\alpha$ and $\vec{p}_\beta$ here have equal celerity can be simply expressed in the statement that $p^\mu_\alpha-p^\mu_\beta$ is proportional to n^μ, so for Lorentz transformations which satisfy Eq. (A6), $p_\alpha'^\mu-p_\alpha'^\mu$ will also be proportional to n^μ, and hence p'_α and p'_β will again have equal celerity $\vec{C}'$. Thus, Lorentz transformations of this type may be viewed simply as transformations of the celerity, the transformed momenta being given in terms of $\vec{C}'$ and masses by formulas like Eq. (A2). To be more explicit, the transformed celerity is given by

$$\begin{aligned}C'^i&=-p'^i+E'n^i\\&=-\Lambda^i{}_jp^j-\Lambda^i{}_0E+\Lambda^0{}_jp^jn^i+\Lambda^0{}_0En^i\\&=-\Lambda^i{}_j(-C^j+n^jE)-\Lambda^i{}_0E+\Lambda^0{}_j(-C^j+n^jE)n^i+\Lambda^0{}_0En^i\,.\end{aligned}$$

Using (A6), we find that all terms involving E cancel, so that

$$C'^i=\Lambda^i{}_jC^j-n^i\Lambda^0{}_jC^j\,. \tag{A7}$$

For such Lorentz transformations, Eq. (A5) may be written simply as the transformation rule

$$\Gamma_a(\vec{C})_{\beta\sigma_\beta,\alpha\sigma_\alpha}=\sum_{\sigma'_\beta\sigma'_\alpha}D^{(j_\beta)\,-1}_{\sigma_\beta\sigma'_\beta}\left[L^{-1}\left(\frac{\vec{p}'_\beta}{m_\beta}\right)\Lambda L\left(\frac{\vec{p}_\beta}{m_\beta}\right)\right]\Gamma_a(\vec{C}')_{\beta\sigma'_\beta,\alpha\sigma'_\alpha}D^{(j_\alpha)}_{\sigma'_\alpha\sigma_\alpha}\left[L^{-1}\left(\frac{\vec{p}'_\alpha}{m_\alpha}\right)\Lambda L\left(\frac{\vec{p}_\alpha}{m_\alpha}\right)\right]. \tag{A8}$$

Now note that for any celerity $\vec{C}$, there exists a Lorentz transformation $\Lambda(\vec{C})$ for which the transformed celerity (A7) lies in the $\vec{n}$ direction. Without loss of generality, we can take $\vec{n}$ to lie in the 3-direction and $\vec{C}$ to lie in the 1–3 plane; the desired Lorentz transformation $\Lambda(\vec{C})$ is then

$$\Lambda(\vec{C})^\mu{}_\nu=\begin{pmatrix}1&0&-\xi&\xi\\0&1&0&0\\\xi&0&1-\frac{1}{2}\xi^2&\frac{1}{2}\xi^2\\\xi&0&-\frac{1}{2}\xi^2&1+\frac{1}{2}\xi^2\end{pmatrix}, \tag{A9}$$

where

$$\xi \equiv C_1/C_3 .$$

This does satisfy (A6), so the transformed celerity is given by (A7). Applying (A7) and (A9) to the celerity $\vec{C} = (C^1, 0, C^3)$, we find a transformed celerity pointing in the $\vec{n}$ direction

$$\vec{C}' = (0, 0, C^3). \tag{A10}$$

But if $\vec{C}'$ is in the $\vec{n}$ direction, then so are $\vec{p}'_\alpha$ and $\vec{p}'_\beta$, so that the matrix $\Gamma_a(\vec{C}')$ is nothing but the collinear pion transition matrix defined in earlier work[2]:

$$(\Gamma_a(\vec{C}'))_{\beta\sigma_\beta,\alpha\sigma_\alpha} = (X_a(\sigma'_a))_{\beta\alpha}\delta_{\sigma'_\beta\sigma'_\alpha} \text{ for } \vec{C}' \propto \vec{n}. \tag{A11}$$

(Recall that invariance with respect to boosts along the $\vec{n}$ direction makes X_a independent of the celerity, while invariance with respect to rotations around the $\vec{n}$ direction makes X_a conserve helicity.) Thus Eq. (A8) is the desired formula (6.6), with the rotation R given by

$$R(\vec{C}, m) = L^{-1}(\vec{p}'/m)\Lambda(\vec{C})L(\vec{p}/m), \tag{A12}$$

where $\vec{p}$ is defined by Eq. (4.8) as a function of $\vec{C}$ and m, and p' is the transformed four-vector $\Lambda(\vec{C})p$.

APPENDIX B: CHIRAL CONSTITUENTS OF THE TRANSITION MATRIX $\Gamma_a(1\to 2)$

We wish to prove here that any isovector matrix M_a, which has a symmetric commutator with the chiral generator $\Gamma_b(1\to 1)$,

$$[\Gamma_a(1\to 1), M_b] = [\Gamma_b(1\to 1), M_a], \tag{B1}$$

must belong to one of the representations (A, B) of the chiral algebra $SU(2)\times SU(2)$, with $A = B$. Since M_a is an isovector, the only possible chiral representations to which it could belong are of the types (A, B) with $A = B+1$ or $A = B$ or $A = B-1$. Thus, the theorem to be proven here just states that any isovector matrix M_a, which belongs to a representation (A, B) with $A = B\pm 1$, and which satisfies (B1), must vanish.

It proves very convenient here to use a four-dimensional tensor basis. Consider a four-tensor $T_{U;VW\cdots}$ of rank $n+1$, which obeys the following conditions:

(i) $T_{U;VW\cdots}$ is symmetric and traceless in the n indices $V, W, \ldots$.

(ii) The contractions of the index U with any one of the n indices $V, W, \ldots$ also vanish.

$$T_{U;VW\cdots} = 0. \tag{B2}$$

(iii) The completely symmetric part of $T_{U;VW\cdots}$ vanishes.

$$T_{U;VW\cdots} + T_{V;UW\cdots} + T_{W;UV\cdots} + \cdots = 0. \tag{B3}$$

(Here $U, V, W, \ldots$ run over the four values 1, 2, 3, 4.)

Condition (i) alone would restrict $T_{U;VW\cdots}$ to belong to the reducible representation

$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{n}{2},\frac{n}{2}\right) = \left(\frac{n-1}{2},\frac{n-1}{2}\right)\oplus\left(\frac{n-1}{2},\frac{n+1}{2}\right)\oplus\left(\frac{n+1}{2},\frac{n-1}{2}\right)\oplus\left(\frac{n+1}{2},\frac{n+1}{2}\right).$$

Condition (ii) then eliminates the component $((n-1)/2, (n-1)/2)$, while condition (iii) eliminates the component $((n+1)/2, (n+1)/2)$, so that $T_{U;VW\cdots}$ provides a tensor realization of the representation

$$\left(\frac{n-1}{2},\frac{n+1}{2}\right)\oplus\left(\frac{n+1}{2},\frac{n-1}{2}\right). \tag{B4}$$

Setting $V, W, \ldots$ equal to 4 and U equal to a in Eq. (B3), we find

$$T_{a;44\cdots 4} = -nT_{4;a4\cdots 4}. \tag{B5}$$

(Here $a, b, c, \ldots$ run over the three values 1, 2, 3.) Hence, there is essentially only one way of forming an isovector from T, and we can take the part of M_a belonging to the representation (B4) as

$$M_a\left[\left(\frac{n-1}{2},\frac{n+1}{2}\right)\oplus\left(\frac{n+1}{2},\frac{n-1}{2}\right)\right] = T_{a;44\cdots 4}. \tag{B6}$$

The commutator of this matrix with the chiral generator is then given by an infinitesimal $O(4)$ rotation

$$[\Gamma_a(1\to 1), T_{b;4\cdots 4}] = -i\delta_{ab}T_{4;4\cdots 4} + inT_{a;b4\cdots 4},$$

so condition (B1) reads, for the representation (B4),

$$T_{a;b4\cdots 4} = T_{b;a4\cdots 4}. \tag{B7}$$

We wish to prove that (B6) must vanish for any four-tensor $T_{U;VW\cdots}$, which satisfies conditions (i), (ii), (iii), and also Eq. (B7).

To this end, consider the commutator of the chiral generator $\Gamma_c(1\to 1)$ with Eq. (B7):

$$-i\delta_{ac}T_{4;b4\cdots 4} - i\delta_{bc}T_{a;44\cdots 4} + i(n-1)T_{a;bc4\cdots 4}$$
$$= -i\delta_{bc}T_{4;a4\cdots 4} - i\delta_{ac}T_{b;44\cdots 4} + i(n-1)T_{b;ca4\cdots 4}.$$

Contracting b with c, we find

$$T_{4;a4\cdots 4} - T_{a;44\cdots 4} = \tfrac{1}{2}(n-1)(T_{b;ba4\cdots 4} - T_{a;bb4\cdots 4}). \tag{B8}$$

However, T is entirely traceless, so

$$T_{b;ba4\cdots 4} = -T_{4;a4\cdots 4},$$

$$T_{a;bb4\cdots 4} = -T_{a;44\cdots 4},$$

and therefore (B8) reads

$$\tfrac{1}{2}(n-1)(T_{4;a4\cdots 4} - T_{a;44\cdots 4}) = 0. \qquad \text{(B9)}$$

Together with Eq. (B5), this yields the desired result,

$$T_{a;44\cdots 4} = 0. \qquad \text{(B10)}$$

It should also be noted that the commutation relation (B1) does not put any constraints on the terms in M belonging to chiral representations (A, B) with $A = B$, because such terms automatically satisfy (B1). A suitable tensor basis for the representations $(n/2, n/2)$ is provided by the completely symmetric traceless tensors of rank n. For such a tensor $t_{UVW\cdots}$, there is just one way to form an isovector, so that

$$M_a[(n/2, n/2)] = t_{a44\cdots 4}. \qquad \text{(B11)}$$

The commutator of the chiral generator with the component of M is then

$$[\Gamma_a(1\to 1), M_b(n/2, n/2)] = -i\delta_{ab}t_{44\cdots 4} + i(n-1)t_{ab4\cdots 4}, \qquad \text{(B12)}$$

and hence is automatically symmetric in a and b. Thus, the whole content of a commutation relation like (B1) can be summed up in the statement that M_a may receive contributions only from the chiral representations (A, B) with $A = B$.

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Bounds on the Pion's Charge Radius*†

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The strongest possible lower and upper bounds on the electromagnetic radius of the pion are derived in terms of the *modulus* of the timelike form factor. Numerical evaluation indicates that the radius is bounded above by the vector-dominance value and that the form factor will not behave as a "dipole" until $t = (2E)^2 > 17$ GeV^2, if at all. The location and number of possible zeros of the form factor are discussed.

At the present time there is a rapid accumulation of information on the pion's electromagnetic form factor,[1] $F(t)$, for timelike and spacelike momentum transfer. Colliding-beam measurements of $\sigma(e^+e^- \to \pi^+\pi^-)$ provide direct access to $|F(t)|$ in the timelike region; for example, experiments[2] at Novosibirsk, Orsay, and Frascati have determined $|F(t)|$ for $t \leq 4.4$ GeV^2. Data at higher t will be furnished by new colliding-beam facilities under construction. Estimates of $F(t)$ in the spacelike region have been indirectly extracted from electroproduction experiments.[3] More precise information on the spacelike form factor will soon be available from the Serpukhov-UCLA group,[4] which