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Dual-Pion Model Satisfying Current-Algebra Constraints*

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The dual-pion model, previously constructed with $m_\pi{}^2=-\frac{1}{2}$ and $m_\rho{}^2=0$, is generalized to arbitrary pion mass while maintaining the relation $m_\rho{}^2=m_\pi{}^2+\frac{1}{2}$. As in the conventional dual-resonance model, the price for doing this is the introduction of ghosts. In the special case $m_\pi=0$, the N-pion amplitudes are shown to possess Adler zeros, and, as a consequence, to satisfy the constraints that current algebra imposes on on-mass-shell hadron amplitudes. The axial charges are explicitly constructed and shown to close with the isospin to form the algebra of SU(2)× SU(2). This model should provide a useful theoretical laboratory for studying the consequences of combining duality and current-algebra constraints. The construction of currents is a challenging problem.

I. INTRODUCTION

A great challenge to the practitioners of the dual-resonance art is the construction of a model possessing a realistic spectrum. As a first step one would like to have a ghost-free model with massless pions, a ρ-f^0 trajectory with intercept $\frac{1}{2}$ and no tachyon, satisfying current-algebra constraints. In this paper we present a model that accomplishes all but one of these goals, its main defect being that it has ghosts.

To put this work into perspective we review the status of existing models. Last spring an apparently ghost-free dual-pion model was constructed[1] for the unrealistic masses $m_\pi{}^2=-\frac{1}{2}$ and $m_\rho{}^2=0$ (in units with $\alpha'=1$). Subsequently, two different modifications of this model were suggested,[2,3] each of which allows for a massless pion and is probably ghost-free. However, they also have a massless ρ and do not appear to satisfy current-algebra constraints. Brower,[4] on the other hand, formulated a chiral-invariant model, which has ghosts and a tachyon. It has been suggested[5] that any ghost-free dual-resonance model must have a trajectory with intercept unity. To my knowledge, no counterexample to this rule has been found so far. Ghosts are the price to pay (admittedly a high one) for changing the masses from their values in the ghost-free dual-pion model. It was known to Neveu and me last spring that squares of masses could be increased by λ^2 while preserving the cyclic symmetry of the N-pion amplitudes if $k_i\cdot k_j$ were replaced by $k_i\cdot k_j+\frac{1}{2}\lambda^2$ for adjacent lines i and j and unchanged for nonadjacent lines. The reason we did not discuss this in our published papers was that we did not know how to factorize the modified amplitudes. Subsequently, but prior to the present work, others[6] have expressed this scheme in an operator formalism using $(N+4)$-component vectors for the N-point amplitude. The amplitudes constructed in this paper are the same ones, but they are obtained in a formalism that makes their factorization properties and spectrum clearer than in the other approaches. To motivate the construction we begin with a discussion of the variable-intercept version of the conventional model.

Following the notation of Ref. 1, we use harmonic-oscillator operators α_m^μ for which

$$[\alpha_m^\mu,\alpha_n^\nu]=-mg^{\mu\nu}\delta_{m,-n}\,. \tag{1.1}$$

The conventional model with intercept unity for the leading trajectory is built up from the ground-state emission vertex operator

$$V_0(k)=\exp(ik\cdot x)\exp\left(\sqrt{2}k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}\right)\exp\left(-\sqrt{2}k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}\right) \tag{1.2}$$

and the propagator

$$D=(L_0-1)^{-1},$$

where

$$L_0=-\tfrac{1}{2}\alpha_0\cdot\alpha_0-\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n=R-p^2.$$

The most commonly used method for extending this model to the leading intercept $\alpha_0=1-\lambda^2$ is to keep the same vertex operator (but requiring that $k^2=\lambda^2-1$), and to modify the propagator to the form

$$D=\int_0^1(1-x)^{-\lambda^2}x^{L_0+\lambda^2-2}dx.$$

A less familiar formulation of the same model turns out to be better suited for generalization to dual-pion models. In this formulation one defines a vertex

$$V(k)=V_5V_0(k)$$

and propagator

$$D=(L_0+\tfrac{1}{2}\lambda^2-1)^{-1},$$

where

$$V_5=\exp\left(\lambda\sum_{n=1}^{\infty}\frac{\alpha_{-n}^{(5)}}{n}\right)\left|0_5\right\rangle\left\langle 0_5\right|\exp\left(\lambda\sum_{n=1}^{\infty}\frac{\alpha_{n}^{(5)}}{n}\right),$$

$$L_0=-\tfrac{1}{2}\alpha_0\cdot\alpha_0-\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n+\tfrac{1}{2}\alpha_0^{(5)}\alpha_0^{(5)}+\sum_{n=1}^{\infty}\alpha_{-n}^{(5)}\alpha_n^{(5)},\quad(1.3)$$

$$\alpha_0^{(5)}=\sqrt{2}p_5=\lambda,$$

and

$$[\alpha_m^{(5)},\alpha_n^{(5)}]=m\,\delta_{m,-n},$$

$$[\alpha_m^{(5)},\alpha_n^{\mu}]=0.$$

The equivalence of the two formulations is an immediate consequence of the formula

$$\left\langle 0_5\right|\exp\left(\lambda\sum_{n=1}^{\infty}\frac{\alpha_n^{(5)}}{n}\right)x^a\exp\left(\lambda\sum_{n=1}^{\infty}\frac{\alpha_{-n}^{(5)}}{n}\right)\left|0_5\right\rangle=(1-x)^{-\lambda^2},$$

where

$$a=\sum_{n=1}^{\infty}\alpha_{-n}^{(5)}\alpha_n^{(5)}.$$

To specify the spectrum of physical states it is necessary to take note of the gauge invariances. One gauge operator is

$$L_1=-\sum_{n=0}^{\infty}\alpha_{-n}\cdot\alpha_{n+1}+\sum_{n=0}^{\infty}\alpha_{-n}^{(5)}\alpha_{n+1}^{(5)}.$$

The other Virasoro operators[7] fail to provide gauge invariances in this case because of the ground-state projection of the fifth mode in V_5. This is the origin of the ghosts. L_1 is *not* the only gauge invariance, however, because the fifth-mode operators can only occur in those combinations contained in

$$\exp\left(\lambda\sum_{n=1}^{\infty}\frac{\alpha_{-n}^{(5)}}{n}\right)\left|0_5\right\rangle.$$

Therefore there must be a gauge operator G_A to embody this fact. The actual construction of G_A is somewhat complicated. However, there does not seem to be much value in doing it, since the gauge condition is quite manageable in the form we have expressed it. Using the gauge conditions it is not hard to construct physical states. For example, at $M^2=\lambda^2$ one finds a vector $\epsilon\cdot\alpha_{-1}|0\rangle$ and a scalar $[(1/\lambda)k\cdot\alpha_{-1}-\sqrt{2}\,\alpha_{-1}^{(5)}]\,|0\rangle$.

II. DUAL-PION MODEL WITH VARIABLE MASSES

In the dual-pion model of Ref. 1, we introduced operators b_m^{μ} with the algebra

$$\{b_m^{\mu},b_n^{\nu}\}=-g^{\mu\nu}\delta_{m,-n},\quad(2.1)$$

m and n taking half-integer values. The pion emission vertex in the model is

$$V_\pi(k)=-\frac{g}{\sqrt{2}}[G_m,V_0(k)]$$

$$=gk\cdot HV_0(k),\quad(2.2)$$

where

$$G_m=\sum_{n=-\infty}^{\infty}\alpha_n\cdot b_{m-n},\quad m=\pm\tfrac{1}{2},\pm\tfrac{3}{2},\ldots$$

$$H^{\mu}=\sum_{m=-\infty}^{\infty}b_m^{\mu},$$

and $V_0(k)$ is given in Eq. (1.2). This vertex, together with the propagator

$$D=(L_0-\tfrac{1}{2})^{-1},$$

where

$$L_0=-\tfrac{1}{2}\alpha_0\cdot\alpha_0-\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n-\sum_{m=1/2}^{\infty}m\,b_{-m}\cdot b_m,$$

gives the dual-pion model with $m_\pi^2=-\frac{1}{2}$ and $m_\rho^2=0$ in the F_2 formulation,[8] the one in which the pion is represented by the ground state.

The generalization to other masses is now straightforward. We introduce new oscillators $b_m^{(5)}$ with the algebra

$$\{b_m^{(5)},b_n^{(5)}\}=\delta_{m,-n}\quad(2.3)$$

and define

$$G_{1/2}=\sum_{n=-\infty}^{\infty}(\alpha_n\cdot b_{1/2-n}-\alpha_n^{(5)}b_{1/2-n}^{(5)})\quad(2.4)$$

and

$$H_5=\sum_{m=-\infty}^{\infty}b_m^{(5)}.$$

The pion emission vertex is then given by

$$V_\pi(k)=-\frac{g}{\sqrt{2}}[G_{1/2},V_5V_0(k)]$$

$$=g\left(k\cdot H\,V_5+\frac{\lambda}{\sqrt{2}}[H_5,V_5]\right)V_0(k),\quad(2.5)$$

where it is now understood that the factor $|0_5\rangle\langle 0_5|$ inside V_5 in Eq. (1.3) is a projection for both the $\alpha_n^{(5)}$ and $b_m^{(5)}$ modes. The appropriate choice for the propagator is then

$$D=(L_0+\tfrac{1}{2}\lambda^2-\tfrac{1}{2})^{-1}=(R+\lambda^2-\tfrac{1}{2}-p^2)^{-1},\quad(2.6)$$

where

$$L_0=-\tfrac{1}{2}\alpha_0\cdot\alpha_0-\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n-\sum_{m=1/2}^{\infty}m\,b_{-m}\cdot b_m$$

$$+\tfrac{1}{2}\alpha_0^{(5)}\alpha_0^{(5)}+\sum_{n=1}^{\infty}\alpha_{-n}^{(5)}\alpha_n^{(5)}+\sum_{m=1/2}^{\infty}m\,b_{-m}^{(5)}b_m^{(5)}.\quad(2.7)$$

Using the techniques of Ref. 8 it is then possible to prove that $G_{1/2}$ is a gauge operator, while the operators $G_{3/2}, G_{5/2}, \ldots$ no longer are. The operator G_A is still a gauge as is the operator $G_B = N_B{}^2 - N_B$, where

$$N_B = \sum_{m=1/2}^{\infty} b_{-m}^{(5)} b_m^{(5)} . \tag{2.8}$$

The duality of the model we have just formulated is not obvious. It can be proved by recasting it in an F_1 formulation[8] and then using the methods of Ref. 1. An alternative method of demonstrating that the N-pion amplitudes have cyclic symmetry is to show the equivalence with the $(N+4)$-component formalism[6] in which the proof is easier.

To illustrate the use and meaning of the formulas given above, the calculation of the $\pi\pi$ amplitude depicted in Fig. 1 is presented in detail:

$$\begin{aligned} A_4 &= \langle 0; k_1 | V_\pi(k_2)(R + \lambda^2 - \tfrac{1}{2} - s)^{-1} V_\pi(k_3) | 0; k_4 \rangle \\ &= \int_0^1 dx\, x^{\lambda^2 - s - 3/2} \langle 0 | V_\pi(k_2) x^R V_\pi(k_3) | 0 \rangle \\ &= g^2 \int_0^1 dx\, x^{\lambda^2 - s - 3/2} (1-x)^{-2k_2 \cdot k_3 - \lambda^2} \\ &\qquad \times \left\langle 0 \left| \left(k_2 \cdot H - \frac{\lambda}{\sqrt{2}} H_5\right) x^R \left(k_3 \cdot H + \frac{\lambda}{\sqrt{2}} H_5\right) \right| 0 \right\rangle \\ &= -g^2 (k_2 \cdot k_3 + \tfrac{1}{2}\lambda^2) \int_0^1 dx\, x^{\lambda^2 - s - 1} (1-x)^{-2k_2 \cdot k_3 - \lambda^2 - 1} . \end{aligned}$$

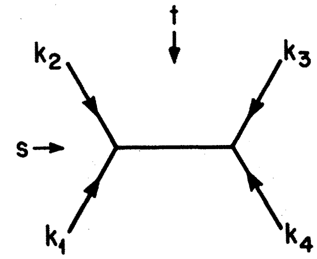

FIG. 1. Kinematics of the four-pion amplitude.

Substituting $\lambda^2 = m_\pi{}^2 + \frac{1}{2}$, we obtain

$$A_4 = \tfrac{1}{2} g^2 \frac{\Gamma(m_\pi{}^2 + \frac{1}{2} - s)\Gamma(m_\pi{}^2 + \frac{1}{2} - t)}{\Gamma(2m_\pi{}^2 - s - t)} = C(s, t). \tag{2.9}$$

This is the formula of Lovelace and Shapiro[9] for the case $m_\rho{}^2 = m_\pi{}^2 + \frac{1}{2}$ and $m_\pi{}^2$ arbitrary, which now has been obtained from a fully factorizable scheme. Taking account of isotopic spin by the Chan-Paton prescription,[10] the full $\pi\pi$ amplitude in the tree approximation becomes

$$\begin{aligned} M_{abcd}(s,t,u) &= \tfrac{1}{2}\operatorname{tr}(\tau_a\tau_b\tau_c\tau_d) C_{st} + \tfrac{1}{2}\operatorname{tr}(\tau_a\tau_b\tau_d\tau_c) C_{su} + \tfrac{1}{2}\operatorname{tr}(\tau_a\tau_c\tau_b\tau_d) C_{tu} \\ &= \delta_{ab}\delta_{cd}(C_{st} + C_{su} - C_{tu}) + \delta_{ac}\delta_{bd}(C_{su} + C_{tu} - C_{st}) + \delta_{ad}\delta_{bc}(C_{st} + C_{tu} - C_{su}). \end{aligned} \tag{2.10}$$

Isolating the amplitudes of definite s-channel isospin,

$$\begin{aligned} M^{(I_s=0)} &= 3C_{st} + 3C_{su} - C_{tu} , \\ M^{(I_s=1)} &= 2(C_{st} - C_{su}), \\ M^{(I_s=2)} &= 2C_{tu} . \end{aligned} \tag{2.11}$$

We specify our normalization by the partial-wave expansion

$$M^{(I)}(s,z) = -\frac{\sqrt{s}}{q_s} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l^I(s)} \sin\delta_l^I(s) P_l(z). \tag{2.12}$$

Identifying the width of the ρ from the residue of its pole (which corresponds to the value one would calculate in a one-loop approximation) gives, to lowest order in m_π,

$$\Gamma_\rho / m_\rho = \tfrac{1}{6} g^2 .$$

Similarly, for the s-wave scattering lengths one finds

$$a_0 = \tfrac{7}{2}\pi g^2 \alpha' m_\pi ,$$

$$a_2 = -\pi g^2 \alpha' m_\pi ,$$

which are in the ratio suggested by Weinberg.[11]

III. THE CASE $m_\pi = 0$

It is well known that the properties of the four-pion amplitude we have been discussing are in striking agreement with current-algebra predictions. We are now in a position to see whether this agreement is fortuitous or a property of the N-pion amplitudes as well. Thus we henceforth consider the special case $m_\pi = 0$, which corresponds to $\lambda = 1/\sqrt{2}$. The spectrum can be investigated by using the gauge operators $G_{1/2}$, G_A, and G_B to construct physical states. All of them with $M^2 = 0, \frac{1}{2}, 1$ are listed in Table I. At $M^2 = \frac{3}{2}$ there are an f^0, ρ', η, several σ's, and possibly other states as well. One of the σ's is a ghost (negative norm), the only such state with $M^2 \leq \frac{3}{2}$. It is interesting that in the $\pi\pi \to \pi\pi$ amplitude there is no spin-zero pole at $M^2 = \frac{3}{2}$. The reason for this turns out to be an exact cancellation between positive-

TABLE I. Low-lying states of the $m_\pi = 0$ dual-pion model.

M^2	I^G	J^P	Name	Operator description
0	1^-	0^-	π	$\|0\rangle$
$\frac{1}{2}$	1^+	1^-	ρ	$\epsilon \cdot b_{-1/2} \|0\rangle$
$\frac{1}{2}$	0^+	0^+	σ	$\sqrt{2}\,(k \cdot b_{-1/2} - b^{(5)}_{-1/2}) \|0\rangle$
1	0^-	1^-	ω	$\epsilon_{\mu\nu\lambda\sigma} \epsilon^{\mu} k^{\nu} b^{\lambda}_{-1/2} b^{\sigma}_{-1/2} \|0\rangle$
1	1^-	1^+	A_1	$[\sqrt{3}\, \epsilon \cdot \alpha_{-1} + (\frac{8}{3})^{1/2} \epsilon \cdot b_{-1/2} k \cdot b_{-1/2} - (\frac{2}{3})^{1/2} \epsilon \cdot b_{-1/2} b^{(5)}_{-1/2}] \|0\rangle$
1	0^-	1^+	H	$(\frac{1}{3})^{1/2} (2\epsilon \cdot b_{-1/2} b^{(5)}_{-1/2} - \epsilon \cdot b_{-1/2} k \cdot b_{-1/2}) \|0\rangle$
1	1^-	0^-	π'	$(2\alpha^{(5)}_{-1} - k \cdot \alpha_{-1} + \sqrt{2}\, b^{(5)}_{-1/2} k \cdot b_{-1/2}) \|0\rangle$

norm and negative-norm σ states. It is *not* a manifestation of another gauge condition that has been omitted. To check this point, the σ couplings to four pions have been explicitly calculated.

We now come to the crucial question: Do the N-pion amplitudes have Adler zeros?[12] In other words, does the N-pion amplitude vanish when any of the pion four-momenta goes to zero? Note that there are no pole terms to take into account because of the nondegeneracy of even- and odd-G-parity states in the model. Consider the N-pion amplitude shown in Fig. 2. This amplitude has cyclic symmetry so we may equally well concentrate attention on any of the pion momenta. k_2 turns out to be a particularly convenient choice.

$$A_N \sim \int_0^1 dx \int_0^1 dy_4 \cdots dy_{N-1} F(y) \prod_{i=4}^{N-1} (1 - xy_4 \cdots y_i)^{-2k_2 \cdot k_i} (1-x)^{-2k_2 \cdot k_3 - 1/2} x^{-2k_1 \cdot k_2 - 1}$$
$$\times \langle 0 | (k_2 \cdot H - \tfrac{1}{2} H_5) x^R (k_3 \cdot H | 0_5 \rangle \langle 0_5 | - \tfrac{1}{2} [H_5, | 0_5 \rangle \langle 0_5 |]) \cdots | 0 \rangle$$
$$\underset{k_2 \to 0}{\sim} \int_0^1 dx\, x^{-1/2} (1-x)^{-3/2} \int_0^1 dy_4 \cdots dy_{N-1} \tilde{F}(y).$$

The remarkable fact about the limit $k_2 \to 0$ is that the x integral becomes a separate factor. The integral is divergent, but it is an easy exercise in analytic continuation to show that it should be identified as $B(\frac{1}{2}, -\frac{1}{2}) = 0$. Thus, as the factor involving y integrations is in general finite, Adler zeros are present.

Mandelstam has shown[13] that other current-algebra implications for on-shell hadron amplitudes are consequences of the existence of the Adler zeros. Because of its simplicity and importance we repeat his argument here. Consider the process $\pi_a + \alpha \to \pi_b + \beta$ depicted in Fig. 3. a and b are isospin labels and α and β are arbitrary on-mass-shell states. The part of the amplitude that is isovector in the t channel must be given by an odd function of q (by Bose statistics). Expanding about $q = 0$,

$$M^{(-)ab}_{\alpha\beta}(p, q, Q) = \epsilon_{abc} [q_\mu M^{\mu c}_{\alpha\beta}(p, Q) + O(q^3)] . \quad (3.1)$$

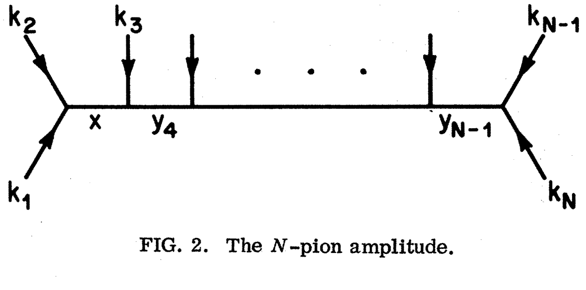

FIG. 2. The N-pion amplitude.

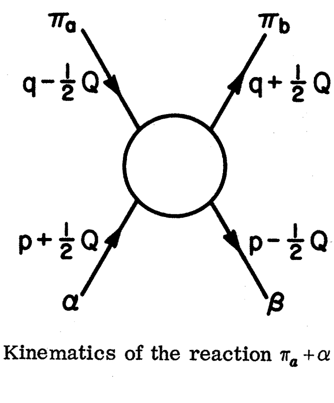

FIG. 3. Kinematics of the reaction $\pi_a + \alpha \to \pi_b + \beta$.

Since this amplitude must vanish for $q=\pm\frac{1}{2}Q$ (the Adler zeros), it follows that for small Q,

$$Q_\mu M^{\mu c}_{\alpha\beta}(p, Q)=O(Q^3). \qquad (3.2)$$

In particular, the amplitude for emitting zero-momentum pion pairs satisfies a divergence condition. By an argument of Weinberg[14] it follows that the corresponding coupling constant must be proportional to a conserved quantity. The only available candidate is the isotopic spin. (This identification is reinforced in our model by explicit consideration of the four-pion amplitude.) Thus we conclude that

$$M^{\mu c}_{\alpha\beta}(p, Q=0)=\gamma p_\mu \langle\alpha|I_c|\beta\rangle, \qquad (3.3)$$

where γ is independent of α and β, and that

$$M^{(-)ab}_{\alpha\beta}(p, q, Q=0)=\gamma p\cdot q\,\epsilon_{abc}\langle\alpha|I_c|\beta\rangle+O(q^3). \qquad (3.4)$$

Taking the value of γ from pion decay and comparing with our four-pion amplitude yields the identification $\gamma=4\pi g^2$ and

$$g^2=(8\pi\alpha' F_\pi^{\,2})^{-1}\approx 1.1, \qquad (3.5)$$

where $F_\pi\approx 190$ MeV is the pion decay constant.

IV. THE AXIAL-VECTOR COUPLING MATRIX

As part of a general discussion of the algebraic consequences of chiral symmetry, Weinberg introduced[15] the axial-vector coupling matrix. The matrix element $(X_a)_{\beta\alpha}$ is proportional to the amplitude for the decay $\alpha\to\beta+\pi_a$ in a collinear frame. The purpose of this section is to obtain an explicit expression for X_a in our model. The methods used are closely related to ones recently invented by Del Giudice, Di Vecchia, and Fubini[16] for studying the collinear emission of massless vector mesons in the conventional model.

It is convenient to choose a particular frame for studying collinear pion emission, even though the results will be covariant. We define two lightlike four-vectors

$$p_0=(\tfrac{1}{2},0,0,-\tfrac{1}{2}) \quad\text{and}\quad \delta=(\tfrac{1}{2},0,0,\tfrac{1}{2}).$$

We then require that a state with mass squared M^2 have the four-momentum $p_0+M^2\delta$. Emission of pions with momentum proportional to δ connects states of this type. To implement this suggestion we define

$$V_{N+1/2}(z)=z^{L_0}V_\pi(-(N+\tfrac{1}{2})\delta)z^{-L_0} \qquad (4.1)$$

and

$$B_{N+1/2}=\frac{1}{2\pi i}\oint\frac{dz}{z}V_{N+1/2}(z). \qquad (4.2)$$

$V_\pi(k)$ and L_0 are the expressions given in Eqs. (2.5) and (2.7), respectively. The operator $B_{N+1/2}$ has the property that $\langle\beta|B_{N+1/2}|\alpha\rangle$ describes pion emission for the collinear on-mass-shell one-particle states α and β, with momenta $p_0+m_\alpha^{\,2}\delta$ and $p_0+m_\beta^{\,2}\delta$, respectively, if $m_\alpha^{\,2}=N+\frac{1}{2}+m_\beta^{\,2}$. The contour integral is well defined if α and β are on the mass shell. If they are on the mass shell, but with $m_\alpha^{\,2}\neq N+\frac{1}{2}+m_\beta^{\,2}$, then the matrix element $\langle\beta|B_{N+1/2}|\alpha\rangle$ vanishes.

To see the connection between the B's and Weinberg's X_a, consider the case of forward $\pi\pi$ scattering with $I_t=1$.

$$C(s,t=0)-C(u,t=0)=\frac{1}{\pi}\int\frac{ds'}{s'-s}\,\mathrm{Im}C(s',t=0)-\frac{1}{\pi}\int\frac{du'}{u'-u}\,\mathrm{Im}C(u',t=0)$$
$$=\sum_{N=0}^{\infty}\left(\frac{1}{N+\frac{1}{2}-s}-\frac{1}{N+\frac{1}{2}-u}\right)\langle 0|B_{N+1/2}B_{-(N+1/2)}|0\rangle \underset{s,u\to 0}{\sim}(s-u)\sum_{N=0}^{\infty}\frac{\langle 0|B_{N+1/2}B_{-(N+1/2)}|0\rangle}{(N+\frac{1}{2})^2}. \qquad (4.3)$$

On the other hand, direct inspection of the functions $C(s,t)$ and $C(u,t)$ in Eq. (2.9) gives for the same limit

$$C(s,t=0)-C(u,t=0)\underset{s,u\to 0}{\sim}\pi g^2(s-u). \qquad (4.4)$$

Therefore the operator

$$X=\frac{1}{g\sqrt{\pi}}\sum_{N=-\infty}^{\infty}\frac{B_{N+1/2}}{|N+\frac{1}{2}|} \qquad (4.5)$$

satisfies the equation $\langle 0|X^2|0\rangle=1$. To make the notation completely clear, we emphasize that by $\langle\alpha|X^2|\beta\rangle$ we mean

$$\langle\alpha|X^2|\beta\rangle=\sum_\gamma\langle\alpha|X|\gamma\rangle\langle\gamma|X|\beta\rangle,$$

where the intermediate states γ are collinear on-mass-shell one-particle states. Only the $B_{N+1/2}$ term in Eq. (4.5) with $m_\alpha^{\,2}-m_\gamma^{\,2}=N+\frac{1}{2}$ contributes to the matrix element $\langle\alpha|X|\gamma\rangle$. Now, the argument of Mandelstam presented in the previous section allows us to conclude that $X^2=1$ is not only true when evaluated between pions, but is an operator identity valid in the space of on-mass-shell physical states. The restriction to physical states is crucial. The Adler zeros, which are the essential ingredient in Mandelstam's argument, were demonstrated to be present in N-pion amplitudes, which contain only physical states. For example, it can be shown by explicit calculation that for the spurious state $|s\rangle = G_{-1/2}|0\rangle$, $\langle s|X^2|s\rangle = 3\langle s|s\rangle$. Thus the operator equation $X^2=1$ is only true in the physical part of the space, which is an indication that it may be difficult to verify by direct manipulation of the operators.

The axial-vector coupling matrix can now be identified as

$$X_a = \tfrac{1}{2}(X + \Omega X \Omega^+)\tau_a, \qquad (4.6)$$

where Ω is the twist operator. By this we mean to indicate an operator whose matrix elements between single-particle states $|\alpha\rangle$ and $|\beta\rangle$, with associated isospin matrices τ_α and τ_β, is

$$\tfrac{1}{4}\langle\alpha|(X+\Omega X\Omega^+)|\beta\rangle \operatorname{tr}(\tau_\alpha \tau_a \tau_\beta). \qquad (4.7)$$

If $|\alpha\rangle$ and $|\beta\rangle$ are states that satisfy the subsidiary conditions and are simultaneously isospin eigenstates, then the expression in (4.7) simplifies to the form

$$\tfrac{1}{4}\langle\alpha|X|\beta\rangle[1-(-1)^{I_\alpha + I_\beta}]\operatorname{tr}(\tau_\alpha\tau_a\tau_\beta) \qquad (4.8)$$

which is nonvanishing only if $I_\alpha=0$ and $I_\beta=1$, or vice versa. By similar reasoning we find for the matrix element of $[X_a, X_b]$

$$\tfrac{1}{4}i\epsilon_{abc}\langle\alpha|\beta\rangle[1+(-1)^{I_\alpha+I_\beta}]\operatorname{tr}(\tau_\alpha\tau_c\tau_\beta),$$

which allows for the identification

$$[X_a, X_b] = i\epsilon_{abc} I_c\,.$$

This, together with

$$[X_a, I_b] = i\epsilon_{abc} X_c\,,$$

$$[I_a, I_b] = i\epsilon_{abc} I_c\,,$$

is the SU(2)×SU(2) algebra of vector and axial-vector charges.

V. CONCLUSION

This work is expected to provide a useful theoretical laboratory for studying the meshing of current-algebra and duality constraints. The physical predictions of the model are of secondary interest in view of its obvious defects – the presence of ghosts and the peculiar way in which G parity arises.

Some interesting questions remaining to be studied are the following:

(1) What is the SU(2)×SU(2) classification of various low-lying states?

(2) How does a small pion mass break the SU(2) ×SU(2) symmetry?

(3) What Lagrangian field theory, if any, is obtained in the zero-slope limit?[17] This limit should yield a chiral-invariant theory of massless pions only, i.e., the nonlinear σ model.

(4) What can be learned from an extension of the model to include fermions?[18]

(5) Most important of all, is it possible to construct the vector and axial-vector currents that this model is likely to possess?

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