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Writing Hamiltonians in Terms of Local Currents*

Arnold A. Dicke
College of Basic Studies, Boston University, Boston, Massachusetts 02215

and

Gerald A. Goldin
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

(Received 23 August 1971)

Several relativistic models are presented which are based on local currents and combine nontrivial internal-symmetry groups with q-number Schwinger terms. Each model is given by specifying a Lie algebra of equal-time current commutators together with a consistent expression for the Hamiltonian as a function of the currents.

I. INTRODUCTION

Several authors have been pursuing the idea of writing nonrelativistic and relativistic models with local currents as the basic dynamical variables.[1] Sugawara proposed such a model, with internal symmetry and finite c-number Schwinger terms.[2] Its simplicity and internal consistency inspired considerable investigation; but Dashen and Frishman showed that it possessed "too much symmetry", leading to consequences not observed in the physical world.[3] Simultaneously, Dashen and Sharp proposed a quark model based on local currents, but for which they were unable to identify the Schwinger terms.[4]

In this paper we present some relativistic models, based on local currents, with nontrivial internal-symmetry groups and q-number Schwinger terms. The models we discuss resemble Sugawara's model in that the equal-time current algebras reduce to his if the Schwinger terms are replaced by c-numbers. However, the Hamiltonians we write in terms of the local currents are not generalizations of Sugawara's Hamiltonian, but resemble more closely the Hamiltonian in a model proposed by Sharp, which we discuss in Sec. II.[5]

For each model we present a Lie algebra of equal-time current commutators, together with an expression for the Hamiltonian as a function of the local currents. Concrete representations in Hilbert space have not yet been obtained for such an algebraic system, except for nonrelativistic models[6]; but some things can be said without making a commitment to a particular representation. For example, the generators of the internal-symmetry group G (the "charges"), obtained formally by integrating the time components of the local currents over all space, cannot be represented as operators in the physical Hilbert space.[7] Nevertheless, the invariance of the Hamiltonian with respect to G can be specified by requiring equations of continuity, such as Eq. (3.20) below, to follow from the current algebra. Relativistic invariance demands some additional constraints on the Hamiltonian density, such as Eq. (3.26) below, which was first pointed out by Schwinger.[8]

In Sec. II we review the model due to Sharp mentioned above, and examine its relationship to Sugawara's model. In Sec. III we incorporate the symmetry group SU(2) by including additional underlying real scalar fields. In Sec. IV we generalize the model of Sec. II to the symmetry group U(n) by including additional underlying complex fields. In each case we propose a form for the Hamiltonian which satisfies the appropriate equation of continuity and the constraints of relativity. Our conclusions are summarized in Sec. V.

II. MODEL FOR CHARGED SCALAR MESONS

Sharp originally wrote down a model for charged scalar mesons in terms of the local currents $j_\mu(\vec{x})$, $S(\vec{x})$, and $\dot{S}(\vec{x})$.[5] These were defined from canonical fields by the equations

$$j_\mu(\vec{x}) = i\{\phi^*(\vec{x})\partial_\mu\phi(\vec{x}) - (\partial_\mu\phi^*(\vec{x}))\phi(\vec{x})\}\ , \qquad (2.1)$$

$$S(\vec{x}) = \phi^*(\vec{x})\phi(\vec{x})\ , \qquad (2.2)$$

$$\dot{S}(\vec{x}) = \phi^*(\vec{x})\pi^*(\vec{x}) + \pi(\vec{x})\phi(\vec{x})\ , \qquad (2.3)$$

with $\pi^*(\vec{x}) = \partial_0\phi(\vec{x})$, and satisfy the equal-time commutation relations

$$[j_0(\vec{x}), j_k(\vec{y})] = 2i\frac{\partial}{\partial x^k}[S(\vec{x})\delta(\vec{x}-\vec{y})]\ , \qquad (2.4)$$

$$[S(\vec{x}), \dot{S}(\vec{y})] = 2iS(\vec{x})\delta(\vec{x}-\vec{y})\ , \qquad (2.5)$$

$$[j_k(\vec{x}), \dot{S}(\vec{y})] = 2ij_k(\vec{x})\delta(\vec{x}-\vec{y})\ , \qquad (2.6)$$

$k = 1, 2, 3$, where all of the other commutators vanish.

Once Eqs. (2.4)–(2.6) have been obtained, the algebra of local observables is taken as the starting point of the theory, and Eqs. (2.1)–(2.3) are dropped. It is important to realize that in a particular representation of Eqs. (2.4)–(2.6) underlying canonical fields may not even exist, and Eqs. (2.1)–(2.3) may no longer hold.[9]

The Hamiltonian density in this model, without the interaction term, is[10]

$$H(\vec{x}) = K_0^*(\vec{x})\frac{1}{S(\vec{x})}K_0(\vec{x}) + \sum_{k=1}^{3}K_k^*(\vec{x})\frac{1}{S(\vec{x})}K_k(\vec{x}) + m^2S(\vec{x})\ , \qquad (2.7)$$

where

$$K_\mu(\vec{x}) = \tfrac{1}{2}[\partial_\mu S(\vec{x}) - ij_\mu(\vec{x})]\ . \qquad (2.8)$$

A rigorous interpretation has already been offered for the "inverse distributions" which appear in expressions such as Eq. (2.7), in the context of a specific representation of the local current algebra.[1] The important requirement is that the inverse operator-valued distribution always appear *sandwiched between* a pair of ordinary operator-valued distributions. These distributions in the numerator must be *proportional* to the distribution in the denominator in the representation in question, in a sense which can be defined.

It is interesting to compare the models discussed in this paper with the Sugawara model, which can be written down for a general compact internal-symmetry Lie group G whose Lie algebra has structure constants f_{abd}.[11] The Sugawara current algebra is

$$[\mathcal{J}_0^a(\vec{x}), \mathcal{J}_0^b(\vec{y})] = if_{abd}\mathcal{J}_0^d(\vec{x})\delta(\vec{x}-\vec{y})\ , \qquad (2.9)$$

$$[\mathcal{J}_0^a(\vec{x}), \mathcal{J}_k^b(\vec{y})] = if_{abd}\mathcal{J}_k^d(\vec{x})\delta(\vec{x}-\vec{y}) + ic\delta_{ab}\frac{\partial}{\partial x^k}\delta(\vec{x}-\vec{y})\ , \qquad (2.10)$$

$$[\mathcal{J}_j^a(\vec{x}), \mathcal{J}_k^b(\vec{y})] = 0\ , \qquad (2.11)$$

where a, b, and d range from 1 to N, N being the number of infinitesimal generators of G; $j, k = 1, 2, 3$; and $c \neq 0$ is a finite constant. Sugawara showed that a satisfactory choice for the Hamiltonian density is

$$H(\vec{x}) = \frac{1}{2c}\sum_{a=1}^{N}\sum_{\mu=0}^{3}\mathcal{J}_\mu^a(\vec{x})\mathcal{J}_\mu^a(\vec{x})\ , \qquad (2.12)$$

which leads to the current-conservation equations

$$\partial_0\mathcal{J}_0^a(\vec{x}) = \sum_{k=1}^{3}\partial_k\mathcal{J}_k^a(\vec{x})\ . \qquad (2.13)$$

Let us look at the restriction imposed on Eqs. (2.4)–(2.6) by requiring $S(\vec{x})$ to be represented by a constant multiple of the identity $\frac{1}{2}cI$. $\dot{S}(\vec{x})$ then equals zero, and the commutation relations (2.5)–(2.6) must be abandoned. If $\dot{S} = 0$, Eq. (2.7) implies that $[H, S(\vec{x})] = 0$, so that we have a consistent model. This is in fact the Sugawara model for the case of a trivial internal-symmetry group; the Hamiltonian density becomes

$$H(\vec{x}) = \frac{1}{2c}\sum_{\mu=0}^{3}j_\mu(\vec{x})j_\mu(\vec{x}) \qquad (2.14)$$

when m^2 approaches zero, which is the same as in the Sugawara model.

The choice $S(\vec{x}) = \frac{1}{2}cI$ might at first be regarded as a way to make unambiguous sense out of $1/S(\vec{x})$ in Eq. (2.7), since the inverse of a nonzero constant can scarcely be ill-defined. But we know that products of distributions at a point need not make mathematical sense. In fact, the expression (2.7) may be *less* singular than a bilinear expression, if the factor $1/S(\vec{x})$ cancels something singular in the numerator.

In Secs. III and IV we seek to generalize the charged-scalar model to admit internal symmetries. We wish to obtain models which have their commutation relations patterned after those of the Sugawara model, except that the constant (c-number) Schwinger terms will be replaced by operator (q-number) Schwinger terms. In order to do this, it is necessary to introduce additional underlying fields – either real (Sec. III) or complex (Sec. IV). Finally, we seek to write a satisfactory Hamiltonian in terms of the local currents for each model thus obtained.

III. GENERALIZATION TO SU(2)

A model with the internal symmetry group SU(2) may be written using three underlying real fields $\Phi=(\varphi_1, \varphi_2, \varphi_3)$, with

$$[\varphi_r(\vec{x}), \pi_s(\vec{y})]=i\delta_{rs}\delta(\vec{x}-\vec{y}) . \tag{3.1}$$

Letting $\varphi=(1/\sqrt{2})(\varphi_1-i\varphi_2)$ and $\varphi^*=(1/\sqrt{2})(\varphi_1+i\varphi_2)$, the relationship between this model and the model of Sec. II becomes apparent when φ_3 is set equal to zero.

We define the local currents

$$J_\mu^a(\vec{x})=\frac{1}{i}\,\partial_\mu\Phi(\vec{x})^\dagger\sigma^a\Phi(\vec{x}) \tag{3.2}$$

and

$$S(\vec{x})=\Phi(\vec{x})\Phi(\vec{x})^\dagger , \tag{3.3}$$

where, for example, Eq. (3.3) is an abbreviation for $S_{ab}(\vec{x})=\varphi_a(\vec{x})\varphi_b(\vec{x})$. In this section the σ^a are the generators of the *regular* representation (or adjoint representation) of SU(2):

$$\sigma^a{}_{bc}=-i\epsilon_{abc} . \tag{3.4}$$

Then

$$\sigma^{aT}=-\sigma^a , \tag{3.5}$$

where the superscript T denotes the transpose.

The following commutation relations obtain among the local currents $J_\mu^a(\vec{x})$, $S(\vec{x})$, and $\dot{S}(\vec{x})$:

$$[J_0^a(\vec{x}), J_0^b(\vec{y})]=i\epsilon_{abd}\delta(\vec{x}-\vec{y})J_0^d(\vec{x}) , \tag{3.6}$$

$$[J_0^a(\vec{x}), S_{rs}(\vec{y})]=\delta(\vec{x}-\vec{y})[S(\vec{x}), \sigma^a]_{rs} , \tag{3.7}$$

$$[J_0^a(\vec{x}), J_k^b(\vec{y})]=i\epsilon_{abd}J_k^d(\vec{x})\delta(\vec{x}-\vec{y}) + i\frac{\partial}{\partial x^k}\{\delta(\vec{x}-\vec{y})\,\mathrm{tr}[\sigma^a S(\vec{x})\sigma^b]\} , \tag{3.8}$$

$$[J_0^a(\vec{x}), \dot{S}_{rs}(\vec{y})]=\delta(\vec{x}-\vec{y})[\dot{S}(\vec{x}), \sigma^a]_{rs} , \tag{3.9}$$

$$[S_{rs}(\vec{x}), \dot{S}_{tu}(\vec{y})]=i\delta(\vec{x}-\vec{y})[\delta_{ts}S_{ru}(\vec{x})+\delta_{tr}S_{su}(\vec{x}) + \delta_{su}S_{rt}(\vec{x})+\delta_{ru}S_{ts}(\vec{x})] , \tag{3.10}$$

$$[\dot{S}_{rs}(\vec{x}), J_k^a(\vec{y})]=\frac{\partial}{\partial x^k}\{\delta(\vec{x}-\vec{y})[\sigma^a, S(\vec{x})]_{rs}\} - i\delta(\vec{x}-\vec{y})\sum_{b=1}^{3}\{\sigma^a, \sigma^b\}J_k^b(\vec{x}) , \tag{3.11}$$

where all of the other commutators vanish.

With φ and φ^* defined as above we can identify $j_\mu(\vec{x})$ of Sec. II with $J_\mu^3(\vec{x})$, and $S(\vec{x})$ of Sec. II with $\frac{1}{2}[S_{11}(\vec{x})+S_{22}(\vec{x})]$.

The free Hamiltonian density

$$H(\vec{x})=\tfrac{1}{2}\sum_{a=1}^{3}[\pi_a(\vec{x})\pi_a(\vec{x})+\nabla\varphi_a(\vec{x})\cdot\nabla\varphi_a(\vec{x})+m^2\varphi_a(\vec{x})^2] \tag{3.12}$$

can be written in more than one way as an explicit function of the local currents. For example, let

$$T^a=\tfrac{1}{2}(S_{bb}+S_{cc}) , \tag{3.13}$$

where a, b, c are a cyclic permutation of the indices 1, 2, 3; and let

$$K_\mu^a=\tfrac{1}{2}(\partial_\mu T^a-iJ_\mu^a) . \tag{3.14}$$

Define

$$H^a(\vec{x})=\sum_\mu K_\mu^{a*}(\vec{x})\frac{1}{T^a(\vec{x})}K_\mu^a(\vec{x})+m^2T^a(\vec{x}) . \tag{3.15}$$

$H^a(\vec{x})$ corresponds to a part of the Hamiltonian density contributed by the underlying fields φ_b and φ_c only, and may be rewritten in terms of these fields as

$$H^a(\vec{x})=\sum_\mu\partial_\mu\frac{1}{\sqrt{2}}(\varphi_b+i\varphi_c)\partial_\mu\frac{1}{\sqrt{2}}(\varphi_b-i\varphi_c) + m^2\frac{1}{\sqrt{2}}(\varphi_b+i\varphi_c)\frac{1}{\sqrt{2}}(\varphi_b-i\varphi_c) . \tag{3.16}$$

The total Hamiltonian density is now easily seen to be

$$H(\vec{x})=\tfrac{1}{2}\sum_{a=1}^{3}H^a(\vec{x}) . \tag{3.17}$$

A second way to write the Hamiltonian resembles more closely the method used in Sec. IV.[12] Define $K_\mu^{jk}=\varphi_j\partial_\mu\varphi_k=\partial_\mu\varphi_k\varphi_j$.[13] Then

$$K_\mu^{jk}=\tfrac{1}{2}(\partial_\mu S_{jk}+\epsilon^{jkl}J_\mu^l) . \tag{3.18}$$

We may now write the Hamiltonian density as

$$H(\vec{x})=\frac{1}{6}\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{\mu=0}^{3}K_\mu^{jk}(\vec{x})\frac{1}{S_{jj}(\vec{x})}K_\mu^{jk}(\vec{x})+\tfrac{1}{2}m^2\,\mathrm{tr}S(\vec{x}) . \tag{3.19}$$

Inverse distributions appear in both forms of the Hamiltonian density only when sandwiched between other distributions, as is required from the discussion in Sec. II.

For a Hamiltonian to be acceptable, it must lead to the equation of continuity

$$i[H, J_0^a(\vec{y})] = \sum_k \frac{\partial}{\partial y^k} J_k^a(\vec{y}) \ , \tag{3.20}$$

where $H = \int d^3x\, H(\vec{x})$.

For $H(\vec{x})$ defined by Eq. (3.19), for example, this result follows from the current commutation relations

$$[K_0^{rs}(\vec{x}), J_0^b(\vec{y})] = \delta(\vec{x}-\vec{y})[\sigma^b, K_0(\vec{x})]_{rs}, \tag{3.21}$$

$$[K_k^{rs}(\vec{x}), J_0^b(\vec{y})] = \delta(\vec{x}-\vec{y})[\sigma^b, K_k(\vec{x})]_{rs} + \frac{\partial}{\partial y^k}[\delta(\vec{x}-\vec{y})(S(\vec{y})\sigma^b)_{rs}] \ , \tag{3.22}$$

and the identity

$$\left[\frac{1}{A}, B\right] = -\frac{1}{A}[A, B]\frac{1}{A}. \tag{3.23}$$

In addition, one needs the identities

$$S_{lj}\frac{1}{S_{jj}}K_\mu^{jk} = K_\mu^{lk} \tag{3.24}$$

and

$$K_\mu^{jk}\frac{1}{S_{jj}}S_{jl} = K_\mu^{lk} \tag{3.25}$$

(where no summation is implied by repeated indices) to complete the calculation. Hopefully, there exist representations of the current algebra such that (3.24) and (3.25) are satisfied. We therefore regard Eqs. (3.24) and (3.25) as constraints on the choice of representation of the current algebra, just as the requirement that the inverse distributions in the Hamiltonian make sense imposes a constraint on the choice of representation.

When one makes the substitution $S(\vec{x}) \to \frac{1}{2}cI$, Eqs. (3.6)–(3.8) reduce to the Sugawara current algebra given by Eqs. (2.9)–(2.11) for the internal-symmetry group SU(2). However, Eqs. (3.17) and (3.19) both reduce to

$$\frac{1}{4c}\sum_a\sum_\mu J_\mu^a(\vec{x})J_\mu^a(\vec{x}),$$

which differs from Sugawara's Hamiltonian by a factor of $\frac{1}{2}$ and is incompatible with the equation of continuity for the Sugawara currents.

The same identities (3.24) and (3.25) that were used to establish the equation of continuity are needed to demonstrate the validity of the Schwinger condition

$$[H(\vec{x}), H(\vec{y})] = i\sum_k\left(\frac{\partial}{\partial x^k} - \frac{\partial}{\partial y^k}\right)[\delta(\vec{x}-\vec{y})\Theta_{0k}(\vec{x})] \ , \tag{3.26}$$

where

$$\Theta_{\mu\nu} = \frac{1}{6}\sum_{k,j}\left(K_\mu^{jk}\frac{1}{S_{jj}}K_\nu^{jk} + K_\nu^{jk}\frac{1}{S_{jj}}K_\mu^{jk} - g_{\mu\nu}K_\alpha^{jk}\frac{1}{S_{jj}}K^{\alpha jk}\right) + \tfrac{1}{2}g_{\mu\nu}m^2\,\mathrm{tr}S. \tag{3.27}$$

IV. GENERALIZATION TO U(n)

Instead of adding more *real* components to the two real fields which underlie the model defined in Eqs. (2.1)–(2.3), we can achieve a different generalization by adding more *complex* fields to the single complex field of this model. Let $\varphi_1, \ldots, \varphi_n$ be complex scalar fields, $\pi_i^*(\vec{x}) = \partial_0\varphi_i(\vec{x})$, $\pi_i(\vec{x}) = \partial_0\varphi_i^*(\vec{x})$, and

$$[\varphi_i(\vec{x}), \pi_j(\vec{y})] = [\varphi_i^*(\vec{x}), \pi_j^*(\vec{y})] = i\delta_{ij}\delta(\vec{x}-\vec{y}), \tag{4.1}$$

with all other commutators vanishing. We can then define the currents by

$$J_\mu^a = (2i)^{-1}\sum_{j,k}(\partial_\mu\varphi_j^*\sigma^a{}_{jk}\varphi_k - \varphi_j^*\sigma^a{}_{jk}\partial_\mu\varphi_k), \tag{4.2}$$

or more succinctly

$$J_\mu^a = (2i)^{-1}(\partial_\mu\varphi^*\sigma^a\varphi - \varphi^*\sigma^a\partial_\mu\varphi), \tag{4.3}$$

and

$$S_{jk} = \varphi_j^*\varphi_k \ . \tag{4.4}$$

In order to be able to write a Hamiltonian in terms of the local currents in a model of this kind, we must let $\frac{1}{2}\sigma^a$ range over the infinitesimal generators of U(n) instead of merely the generators of SU(n) in Eqs. (4.2) and (4.3). That is, in addition to the currents J_μ^a, $a = 1, \ldots, n^2-1$, defined by the traceless Hermitian $n\times n$ matrices which generate SU(n), we must include

$$J_\mu^0 = (2i)^{-1}\sum_{j,k}(\partial_\mu\varphi_j^*\delta_{jk}\varphi_k - \varphi_j^*\delta_{jk}\partial_\mu\varphi_k) \tag{4.5}$$

$$= (2i)^{-1}\sum_j(\partial_\mu\varphi_j^*\varphi_j - \varphi_j^*\partial_\mu\varphi_j) \ . \tag{4.6}$$

The currents now satisfy the Lie algebra

$$[J_0^a(\vec{x}), J_0^b(\vec{y})] = if_{abc}J_0^c(\vec{x})\delta(\vec{x}-\vec{y}), \tag{4.7}$$

$$[J_0^a(\vec{x}), J_k^b(\vec{y})] = if_{abc}J_k^c(\vec{x})\delta(\vec{x}-\vec{y}) + i\frac{\partial}{\partial x^k}(\delta(\vec{x}-\vec{y})\,\mathrm{tr}[S(\vec{x})\{\tfrac{1}{2}\sigma^{aT}, \tfrac{1}{2}\sigma^{bT}\}]), \tag{4.8}$$

$$[S_{rs}(\vec{x}), J_0^a(\vec{y})] = \delta(\vec{x}-\vec{y})[S(\vec{x}), \tfrac{1}{2}\sigma^{aT}]_{rs}, \tag{4.9}$$

$$[\dot{S}_{rs}(\vec{x}), J_0^a(\vec{y})] = \delta(\vec{x}-\vec{y})[\dot{S}(\vec{x}), \tfrac{1}{2}\sigma^{aT}]_{rs}, \tag{4.10}$$

$$[S_{rs}(\vec{x}), \dot{S}_{tu}(\vec{y})] = i\delta(\vec{x}-\vec{y})[\delta_{st}S_{ru}(\vec{x}) + \delta_{ru}S_{ts}(\vec{x})]. \tag{4.11}$$

We shall choose the generators $\frac{1}{2}\sigma^a$ in such a way that

$$\mathrm{tr}\,(\tfrac{1}{2}\sigma^a \tfrac{1}{2}\sigma^b) = d^{-1}\delta_{ab}, \tag{4.12}$$

where d is a constant depending on n. This allows us to write the final equation of the Lie algebra

$$[\dot{S}_{rs}(\vec{x}), J_k^a(\vec{y})] = \frac{\partial}{\partial x^k}\{\delta(\vec{x}-\vec{y})[S(\vec{x}), \tfrac{1}{2}\sigma^{aT}]_{rs}\} - i\delta(\vec{x}-\vec{y})\sum_{b=0}^{n^2-1}\{\tfrac{1}{2}\sigma^{bT}, \tfrac{1}{2}\sigma^{aT}\}J_k^b(\vec{x}). \tag{4.13}$$

In the above the f_{abc} are the structure constants of U(n). All of the other commutators vanish.

For the case $n=1$, the only generator is $\frac{1}{2}\sigma^0 = \frac{1}{2}$; the constant d equals 4, and the anticommutator in Eq. (4.8) has the value $\frac{1}{2}$. With $j_\mu(x) = 2J_\mu^0(x)$, the algebra is identical to that of Eqs. (2.4)–(2.6).

For the case $n=2$, the generators are $\frac{1}{2}$ times the Pauli matrices $\sigma^1, \sigma^2, \sigma^3$ which satisfy

$$\{\tfrac{1}{2}\sigma^{aT}, \tfrac{1}{2}\sigma^{bT}\} = \tfrac{1}{2}\delta_{ab}, \tag{4.14}$$

together with $\frac{1}{2}\sigma^0 = \frac{1}{2}$. The sub-algebra of Eqs. (4.7)–(4.11) and (4.13) corresponding to the group SU(2), i.e., omitting J_μ^0, is the same as the Lie algebra of Eqs. (3.6)–(3.11) based on three underlying real fields transforming according to SU(2); however, one writes a different Hamiltonian.

In order to define the Hamiltonian density in terms of the local currents in each model, we observe that the objects

$$K_\mu^{*jk} = \partial_\mu \varphi_j^* \varphi_k \tag{4.15}$$

and

$$K_\mu^{jk} = \varphi_j^* \partial_\mu \varphi_k \tag{4.16}$$

can be written as linear combinations of the currents:

$$K_\mu^{*jk} = \tfrac{1}{2}\left(\partial_\mu S_{jk} + id\sum_{a=0}^{n^2-1}\tfrac{1}{2}\sigma^a{}_{kj}J_\mu^a\right), \tag{4.17}$$

$$K_\mu^{jk} = \tfrac{1}{2}\left(\partial_\mu S_{jk} - id\sum_{a=0}^{n^2-1}\tfrac{1}{2}\sigma^a{}_{kj}J_\mu^a\right). \tag{4.18}$$

For example,

$$K_\mu^{*jk} - K_\mu^{jk} = \partial_\mu\varphi_j^*\varphi_k - \varphi_j^*\partial_\mu\varphi_k = \partial_\mu\varphi_l^* P^{(jk)}{}_{lm}\varphi_m - \varphi_l^* P^{(jk)}{}_{lm}\partial_\mu\varphi_m, \tag{4.19}$$

where $P^{(jk)}$ is the projection matrix given by

$$P^{(jk)}{}_{lm} = \delta_{jl}\delta_{km}. \tag{4.20}$$

$P^{(jk)}$ can be written as a linear combination of the generators $\frac{1}{2}\sigma^a$:

$$P^{(jk)} = \sum_{a=0}^{n^2-1}\lambda^a{}_{(jk)}\tfrac{1}{2}\sigma^a.$$

Using Eq. (4.12), it is easy to see that $\lambda^a{}_{(jk)} = d\sigma^a{}_{kj}/2$. Thus Eq. (4.19) becomes

$$K_\mu^{*jk} - K_\mu^{jk} = id\sum_{a=0}^{n^2-1}\tfrac{1}{2}\sigma^a{}_{kj}J_\mu^a. \tag{4.21}$$

In the case $n=1$, $K_\mu = \frac{1}{2}(\partial_\mu S - 2iJ_\mu) = \frac{1}{2}(\partial_\mu S - ij_\mu)$, and we recover the model of Sec. II. In the case $n=2$,

$$K_\mu^{11} = \tfrac{1}{2}[\partial_\mu S_{11} - i(J_\mu^0 + J_\mu^3)], \tag{4.22}$$

$$K_\mu^{12} = \tfrac{1}{2}[\partial_\mu S_{12} - i(J_\mu^1 + iJ_\mu^2)], \tag{4.23}$$

$$K_\mu^{22} = \tfrac{1}{2}[\partial_\mu S_{22} - i(J_\mu^0 - J_\mu^3)], \tag{4.24}$$

$$K_\mu^{21} = \tfrac{1}{2}[\partial_\mu S_{21} - i(J_\mu^1 - iJ_\mu^2)]. \tag{4.25}$$

Now we can write the Hamiltonian density in terms of local currents, for the case of the group U(n), as

$$H(\vec{x}) = \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{\mu=0}^{3}K_\mu^{*kj}(\vec{x})\frac{1}{S_{jj}(\vec{x})}K_\mu^{jk}(\vec{x}) + m^2\,\mathrm{tr}S(\vec{x}), \tag{4.26}$$

which follows directly from

$$H(\vec{x}) = \sum_k\sum_\mu \partial_\mu\varphi_k^*(\vec{x})\partial_\mu\varphi_k(\vec{x}) + m^2\sum_k\varphi_k^*(\vec{x})\varphi_k(\vec{x}).$$

Again we need to check explicitly that the Hamiltonian $H = \int d^3x\, H(\vec{x})$ satisfies the continuity equation

$$i[H, J_0^a(\vec{y})] = \sum_k \frac{\partial}{\partial y^k}J_k^a(\vec{y}), \tag{4.27}$$

and thus displays the internal symmetry of the group U(n). We must also verify the Schwinger condition

$$[H(\vec{x}), H(\vec{y})] = i\sum_k\left(\frac{\partial}{\partial x^k} - \frac{\partial}{\partial y^k}\right)[\delta(\vec{x}-\vec{y})\Theta_{0k}(\vec{x})], \tag{4.28}$$

where

$$\Theta_{\mu\nu} = \frac{1}{n}\sum_{i,j}\left(K_\mu^{*ij}\frac{1}{S_{jj}}K_\nu^{ji} + K_\nu^{*ij}\frac{1}{S_{jj}}K_\mu^{ji} - g_{\mu\nu}K_\alpha^{*ij}\frac{1}{S_{jj}}K^{\alpha ji}\right) + g_{\mu\nu}m^2\,\mathrm{tr}S. \tag{4.29}$$

In order to do this, we make use of the commutation relations

$$[K_0^{*jk}(\vec{x}), J_0^a(\vec{y})] = \delta(\vec{x}-\vec{y})[K_0^*(\vec{x}), \tfrac{1}{2}\sigma^{aT}]_{jk}, \tag{4.30}$$

$$[K_0^{jk}(\vec{x}), J_0^a(\vec{y})] = \delta(\vec{x}-\vec{y})[K_0(\vec{x}), \tfrac{1}{2}\sigma^{aT}]_{jk}, \tag{4.31}$$

$$[K_l^{*jk}(\vec{x}), J_0^a(\vec{y})] = \delta(\vec{x}-\vec{y})[K_l^*(\vec{x}), \tfrac{1}{2}\sigma^{aT}]_{jk} + \frac{\partial}{\partial y^l}[(\tfrac{1}{2}\sigma^{aT}S(\vec{y}))_{jk}\delta(\vec{x}-\vec{y})], \tag{4.32}$$

$$[K_l^{jk}(\vec{x}), J_0^a(\vec{y})] = \delta(\vec{x}-\vec{y})[K_l(\vec{x}), \tfrac{1}{2}\sigma^{aT}]_{jk} - \frac{\partial}{\partial y^l}[(S(\vec{y})\tfrac{1}{2}\sigma^{aT})_{jk}\delta(\vec{x}-\vec{y})], \tag{4.33}$$

and Eq. (3.23) for computing commutators with $1/S_{jj}$. In addition, the identities

$$S_{lj}(\vec{x})\frac{1}{S_{jj}(\vec{x})}K_\mu^{jk}(\vec{x}) = K_\mu^{lk}(\vec{x}), \tag{4.34}$$

$$K_\mu^{*kj}(\vec{x})\frac{1}{S_{jj}(\vec{x})}S_{jl}(\vec{x}) = K_\mu^{*kl}(\vec{x}) \tag{4.35}$$

(where no summation over j is implied) enable one to complete the computation.

In order to retrieve the Sugawara-model Lie algebra from Eqs. (4.7)–(4.9), we must have

$$S_{jk}(\vec{x}) \to \tfrac{1}{2}dc\delta_{jk}, \tag{4.36}$$

since

$$\operatorname{tr}\{\tfrac{1}{2}\sigma^{aT}, \tfrac{1}{2}\sigma^{bT}\} = \frac{2}{d}\delta_{ab}. \tag{4.37}$$

However, this substitution causes the Hamiltonian density (4.26) with $m=0$ to become

$$\frac{1}{2cn}\sum_a\sum_\mu J_\mu^a(\vec{x})J_\mu^a(\vec{x}). \tag{4.38}$$

We thus recover the Sugawara Hamiltonian *only* in the case $n=1$; i.e., only in the model of Sec. II.[14] For n greater than 1, the Hamiltonian density (4.38) combined with the Sugawara Lie algebra does not satisfy the equation of continuity.

V. CONCLUSIONS

We have developed some new relativistic models based on local currents, with internal symmetry, in which q-number Schwinger terms appear in the equal-time Lie algebra. While we do not regard these models as candidates for nontrivial physical theories, they provide a richer algebraic structure than was provided by earlier models.

The current algebras themselves resemble the algebra in the Sugawara model except for their respective Schwinger terms; but the Hamiltonians in the present models have a different form from Sugawara's Hamiltonian. Except for the case of a trivial internal-symmetry group, the Hamiltonians in these models do not reduce to the Sugawara Hamiltonian when the Schwinger term is replaced by a c-number.[15]

We have defined the currents formally in terms of underlying fields, and computed their commutation relations on this basis; likewise, we have obtained identities which must hold in order for the Hamiltonian to be satisfactory. Nevertheless, we believe that it will not in general be possible to represent the algebra of currents and the algebra of underlying fields simultaneously. We regard the current algebra as the algebraic starting point of the model, and identities such as (4.34)–(4.35) and the existence of the Hamiltonian as constraints upon the allowable representations of the algebra. In this we hold a different point of view from that expressed by Freundlich and Lurié.[16]

Clearly the next step in pursuing relativistic models based on local currents must be to obtain *bona fide* representations of the current algebras, or of the groups obtained by exponentiating the current commutators.

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[1]Cf. G. Goldin and D. H. Sharp, in *1969 Battelle Rencontres: Group Representations,* edited by V. Bargmann (Springer, Berlin, 1970), p. 300.

[2]H. Sugawara, Phys. Rev. 170, 1659 (1968).

[3]R. Dashen and Y. Frishman, Phys. Rev. Letters 22, 572 (1969).

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[5]D. H. Sharp, Phys. Rev. 165, 1867 (1968).

[6]G. Goldin, J. Math. Phys. 12, 462 (1971).

[7]C. A. Orzalesi, Rev. Mod. Phys. 42, 405 (1970).

[8]J. Schwinger, in *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963), pp. 89–134; Phys. Rev. 130, 406 (1963); 130, 800 (1963).

[9]For example, C. Newman [Ph.D. thesis, Princeton University, 1971 (unpublished)] discusses representations of a "superlocal" theory based on local currents, in which the underlying fields do not exist.

[10]In the future, equations such as (2.7) will be written

$$H(\vec{x}) = \sum_\mu K_\mu^*(\vec{x})\frac{1}{S(\vec{x})}K_\mu(\vec{x}) + m^2\operatorname{tr}S(\vec{x}),$$

where it is understood that the summation over μ is an ordinary summation and not a Minkowskian summation.

[11]Sugawara originally wrote his model down for chiral SU(3) ×SU(3), which is not explicitly discussed in the present paper. S. Coleman, D. Gross, and R. Jackiw, Phys. Rev. 180, 1359 (1969), among others, wrote down

the generalization to an arbitrary symmetry group.

[12]J. Grodnik (private communication).

[13]For the purpose of arriving at an expression for the Hamiltonian as a function of the local currents, one may ignore such terms as the $\delta(0)$, which technically ought to appear in the last equality for $\mu=0$; this was already assumed in the defining equation (3.2). The justification for the Hamiltonian expression obtained in this manner lies in the fact that it can be shown to satisfy the desired continuity equation and Schwinger condition using the current commutators.

[14]The Sugawara constant c in Eqs. (4.36) and (4.38) for the case $n=1$ is one-fourth the constant c in Eq. (2.14), since we have defined $J^0_\mu = \frac{1}{2} j_\mu$.

[15]This result differs from an analogous statement in a slightly different context by Y. Freundlich and D. Lurié, Phys. Rev. D 1, 1660 (1970).

[16]See Freundlich and Lurié (Ref. 15).

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Dilatation and Dimensional Transformations*

Antonio Aurilia and Yasushi Takahashi
Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada

and

Hiroomi Umezawa
Department of Physics, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201
(Received 12 October 1971)

The effect of a field redefinition on the dilatation transformation is considered. It is proposed that the generator of the dilatation transformation is modified if the divergence of the dilatation current can be expressed in the form of the divergence of a local current as a consequence of field equations. The notion of the dimensional transformation is introduced to generalize the case when the dilatation current does not satisfy the above condition. The dimensional transformation is worked out in detail in the case of a massive neutral scalar field.

I. INTRODUCTION

In this paper an attempt is made to clarify some fundamental points with regard to the definition of the dilatation transformation, and also to show explicitly a close relation between the dilatation transformation and dimensional analysis.

The first problem arises in the following situation. We consider some dynamical system described by field variables $\phi^{(\kappa)}(x)$ $(\kappa=1, 2, \ldots)$ and some constants such as the mass and the coupling constant with some dimension. In the dilatation transformation, the field variables $\phi^{(\kappa)}$ are transformed according to their dimension whereas the dimensional constants are held fixed.[1] Hence, the appearance of the dimensional constants results in the violation of invariance under the dilatation transformation in the dimensionally consistent theory. Let us suppose, however, that the dimensional constants can be eliminated by a field redefinition. In terms of new field variables thus introduced, we now have a dilatation-invariant theory. Which of the two dilatation transformations, in terms of the old or new variables, is the legitimate one? Or, can we define the dilatation transformation irrespective of the choice of field variables? We shall discuss this point in the following two sections. The argument presented is further generalized in Sec. IV to clarify the relation between the dilatation transformation and the dimensional analysis. For this purpose, we introduce the notion of the *dimensional transformation*, associated with *dimensional invariance*. Dimensional invariance can never be violated in the sense that any dimensionally consistent theory must be invariant under the dimensional transformation.

It may be admitted that the above generalization looks trivial at a glance. Indeed, the dimensional transformation does not provide us with anything new except to show that our theory is dimensionally consistent. We point out, however, that the dimensional transformation plays a vital role in the discussion of the spontaneous breakdown of the dilatation transformation. This will be the subject of a subsequent paper, in which we shall show that the dilatation transformation of Heisenberg operators turns into the dimensional transformation at the level of physical (or asymptotic) fields.

Our argument makes use of the following five relations.[2]

Relation (1). For a spatially closed function F, it holds that