# Nondynamical Structure of Collinear Processes*

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The dynamics-independent structure of collinear reactions involving particles of arbitrary spins is investigated. The constraints on the $M$ matrix are discussed and the consequences on the experimental observables are derived. These consequences simplify the tests of many conservation laws and make it also easier to determine experimentally the invariant amplitudes in the $M$ matrix. A generous supply of examples is worked out in considerable detail.

## I. INTRODUCTION

In this paper we will derive the nondynamical structure of collinear reactions of particles with arbitrary spins, that is, of reactions in the forward and backward directions. In a general sense, the study is an application of a number of papers[1] in which the nondynamical structure of completely general particle reactions was described. The special study of forward and backward reactions seems especially topical nowadays. Most recent models of high-energy reactions (e.g., the peripheral or Regge models) concentrate (perhaps by default) on the near-forward and near-backward directions, and correspondingly experimental information is also less scarce in that angular region than in other regions. It is plausible that a listing of nondynamical features of particle reactions in the forward and backward directions will also help in deciding on the merits of the various dynamical schemes proposed to explain particle reactions in that region.

Another motivation for this study is the fact that since the number of invariant amplitudes for a given reaction decreases sharply in the forward and backward directions, the tests of conservation laws and the methods of determination of intrinsic quantum numbers of participating particles, which are direct consequences of the relationship between invariant amplitudes and the experimental observables, can be expected to simplify considerably. To be sure, this theoretical simplification does not necessarily mean greater experimental accessibility, since the requirement of collinearity might, in practice, be a difficult one to satisfy. Nevertheless, it is possible that new methods of testing conservation laws and of determining intrinsic quantum numbers (for instance, of the type of the Adair test) might emerge from such a study.

In Sec. II we will discuss the constraints on the $M$ matrix for collinear processes, and will count the number of invariant amplitudes that survive these constraints. This counting is important since it is the difference between the number of bilinear products of invariant amplitudes on the one hand, and the number of linearly independent observables on the other, that is directly responsible for creating tests of conservation laws and methods for determining quantum numbers.

Section III lists the changes in the structure of observables in terms of the bilinear combinations of invariant amplitudes that are brought about by the results of Sec. II for the reaction $0+s\to 0+s'$. Section IV further extends these results for composite reactions, i.e., reactions of the type $s+s'\to s''+s'''$. Finally, Sec. V gives a number of concrete reactions as examples of the results developed in the previous sections. Section VI is the summary and conclusions. An Appendix clarifies some of the notation used here.

## II. STRUCTURE OF THE $M$ MATRIX AND THE NUMBER OF AMPLITUDES

Consider a collinear reaction of four particles of arbitrary spins $s_a$, $s_b$, $s_c$, and $s_d$,

$$a+b\to c+d. \tag{2.1}$$

Let us choose the angular momentum quantization axis, $z$, in this one preferred direction. Since the orbital angular momentum has no component in this direction, the conservation of the $z$ component of the total angular momentum can be written as

$$s_{az}+s_{bz}=s_{cz}+s_{dz}. \tag{2.2}$$

Equation (2.2) places a constraint on the invariant amplitudes and decreases their number from $\prod_{i=a,\ldots,d}(2s_i+1)$ which is the number in the general case in the absence of any conservation laws except rotation invariance. The number of remaining amplitudes can be counted by remarking that a given value of $|s_{az}+s_{bz}|=s_a+s_b-Q$ can be realized $Q+1$ ways if $Q\leq 2s_b-1$, and in $2s_b+1$ ways if $Q\geq 2s_b$. A similar statement can be made about the final state, and matching the two to satisfy Eq. (2.2), after some calculations, gives the following

results for $N_0^{(c)}$, the number of amplitudes for collinear processes in the presence of rotation invariance only:

$$N_0^{(c)} = (2s_d+1)[1-(s_a^2+s_b^2+s_c^2+\tfrac{1}{3}s_d^2) + (s_a+s_b+s_c-\tfrac{1}{3}s_d)+2(s_as_b+s_bs_c+s_as_c)] \tag{2.3}$$

if $s_c+s_d > s_c-s_d > s_a-s_b$;

$$N_0^{(c)} = 2(2s_b-\alpha)(2s_b-\alpha+1)(\tfrac{2}{3}s_b+\tfrac{1}{6}\alpha+\tfrac{1}{6}) + 2(2s_b+1)(2s_d-2s_b+\alpha)(s_d+s_b-\tfrac{1}{2}\alpha+\tfrac{1}{2}) + (2s_b+1)(2s_d+1)(2s_c-2s_d+1) \tag{2.4}$$

if $s_c+s_d > s_a-s_b > s_c-s_d$; and

$$N_0^{(c)} = (2s_b+1)(2s_d+1)(2s_c+1) \tag{2.5}$$

if $s_a-s_b > s_c+s_d > s_c-s_d$.

Here

$$\alpha \equiv s_a+s_b-s_c-s_d \tag{2.6}$$

and we assumed

$$s_a+s_b > s_c+s_d, \quad \text{and} \quad s_a > s_b, \quad s_c > s_d. \tag{2.7}$$

Since the number of amplitudes is invariant under the interchange of the two sides of a reaction, Eq. (2.7) does not represent a restriction of the most general case.

As one would expect, having imposed one linear constraint on the amplitudes, the resulting expression is trilinear in the $s_i$'s instead of the quadrilinear expression

$$N_0 = (2s_a+1)(2s_b+1)(2s_c+1)(2_d+1) \tag{2.8}$$

for the general nonlinear case. Both cases are tabulated for the first few values of the spins in Table I.

Now let us see how we can decide which of the invariant amplitudes survive in the collinear case. For the notation used here, see also the Appendix.

The basic condition we must satisfy in this case is that the $M$ matrix be axially symmetric around the collinear direction (which we will call the $z$ axis). Thus, it can contain only $z$ and $x^2+y^2$. We now have to ascertain which of the unit vectors[2] $\hat{l}$, $\hat{m}$, and $\hat{n}$ correspond to $z$. Using the definitions of these three unit vectors for an almost collinear situation and then going to the limit of collinearity, we see that for the collinear case

$$|z| = \hat{l}, \quad |x|, |y| = \hat{m}, \hat{n}. \tag{2.9}$$

The requirement of collinearity, therefore, consists at least of having only *eee* or *oee* term sets.[3] Consequently, the only product sets surviving will be *eee* and *oee*. Since these contribute only to the *eee*, *oee*, *eoo*, and *ooo* subclasses, the other four subclasses will vanish. This fact has immediate and interesting consequences. Let us remember the relationship between the product sets and the various conservation laws, as given in Table IV of Ref. 4. It will be recalled that tests of conservation laws arise from the fact that the assumption of a conservation law knocks out certain product sets. We see, however, that the two remaining product sets are not among those which are knocked out by invariance under $B$, $TP$, $TPB$, $CB$, $CTP$, or $CPTB$ and hence these invariances cannot be tested in collinear reactions.

In terms of the characteristic quantities,[4] we can express the above results by saying that for collinear reactions the characteristic quantity consists of the pair $\mu_M$, $\nu_M$. Thus any transformation whose characteristic quantity is $\mu_M$, $\nu_M$, or $\mu_M+\nu_M$, cannot be tested.

## III. CONSTRAINTS ON THE $0+s \to 0+s'$ REACTION

We will now further study the constraints of collinearity on the $M$ matrix and the observables, in the case of the reaction

$$0+s \to 0+s', \tag{3.1}$$

which will then help us to discuss the most general four-particle reaction in Sec. IV.

We will talk explicitly about forward reactions. Backward constraints are very similar.

The form of the $M$ matrix for (3.1) in the forward direction will be[5]

$$M = \sum_{k=s'-s}^{s'+s} a_k S_{[k]}(s', s) : T_{[k]}(l_1, l_2, \ldots, l_k). \tag{3.2}$$

Here we assumed $s' \geq s$, which does not restrict the generality of our results. The quantity $T_{[k]}(l_1, \ldots, l_k)$ is the irreducible tensor of rank $k$ containing exclusively $l$'s.

The form given by Eq. (3.2) follows easily from the constraints discussed in Sec. II. In particular, it is a consequence of the requirements discussed in Sec. II in connection with Eq. (2.9), which allow only terms with $l$'s or with $mm+nn$'s. The latter, however, by the relationship

$$ll+mm+nn = 0 \tag{3.3}$$

can be transformed into terms containing only $ll$'s.

Thus we have $2s+1$ terms. Every second one of these is in the *eee* term set and the others in the *eeo* term set. For bosons, if $s'-s$ is even, we have $s+1$ terms in *eee* and $s$ in *oee*; if $s'-s$ is odd, it is the other way around. For fermions, we have $s+\frac{1}{2}$ terms in each of the two term sets.

If we then look at the corresponding product sets, we have for bosons in either case $(s+1)(s+1)+s^2 = 2s^2+2s+1$ products in product set *eee* and

TABLE I. Number of invariant amplitudes in collinear processes $s_a+s_b\to s_c+s_d$. In each entry the first number is the number of amplitudes for the general, noncollinear case, and the second number (in parenthesis) is the number of collinear amplitudes.

| $s_a,s_b$ \ $s_c,s_d$ | 0,0 | 1,0 | $\frac{1}{2},\frac{1}{2}$ | 2,0 | $\frac{3}{2},\frac{1}{2}$ | 1,1 | $\frac{5}{2},\frac{1}{2}$ | 2,1 | $\frac{3}{2},\frac{3}{2}$ | 2,2 | $\frac{5}{2},\frac{3}{2}$ | $\frac{5}{2},\frac{5}{2}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0,0 | 1 (1) | | | | | | | | | | | |
| 1,0 | 3 (1) | 9 (3) | | | | | | | | | | |
| $\frac{1}{2},\frac{1}{2}$ | 4 (2) | 12 (4) | 16 (6) | | | | | | | | | |
| 2,0 | 5 (1) | 15 (3) | 20 (4) | 25 (5) | | | | | | | | |
| $\frac{3}{2},\frac{1}{2}$ | 8 (2) | 24 (6) | 32 (8) | 40 (8) | 64 (14) | | | | | | | |
| 1,1 | 9 (3) | 27 (7) | 36 (10) | 45 (9) | 72 (16) | 81 (19) | | | | | | |
| $\frac{5}{2},\frac{1}{2}$ | 12 (2) | 36 (6) | 48 (8) | 60 (10) | 96 (16) | 108 (18) | 144 (22) | | | | | |
| 2,1 | 15 (3) | 45 (9) | 60 (12) | 75 (13) | 120 (22) | 135 (25) | 180 (28) | 225 (37) | | | | |
| $\frac{3}{2},\frac{3}{2}$ | 16 (4) | 48 (10) | 64 (14) | 80 (14) | 128 (24) | 144 (19) | 192 (20) | 240 (40) | 256 (48) | | | |
| 2,2 | 25 (5) | 75 (13) | 100 (18) | 125 (7) | 128 (32) | 225 (37) | 300 (42) | 375 (55) | 400 (56) | 625 (85) | | |
| $\frac{5}{2},\frac{3}{2}$ | 24 (4) | 72 (12) | 96 (16) | 120 (18) | 192 (20) | 216 (34) | 288 (40) | 360 (52) | 384 (48) | 600 (80) | 576 (80) | |
| $\frac{5}{2},\frac{5}{2}$ | 36 (6) | 108 (16) | 144 (22) | 180 (24) | 288 (40) | 324 (46) | 442 (54) | 540 (70) | 576 (80) | 900 (110) | 864 (104) | 1296 (146) |

| $s_a,s_b$ \ $s_c,s_d$ | $\frac{1}{2},0$ | $\frac{3}{2},0$ | $1,\frac{1}{2}$ | $\frac{5}{2},0$ | $\frac{3}{2},1$ | $2,\frac{1}{2}$ | $\frac{5}{2},1$ | $2,\frac{3}{2}$ | $\frac{5}{2},2$ |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{2},0$ | 4 (2) | | | | | | | | |
| $\frac{3}{2},0$ | 8 (2) | 16 (4) | | | | | | | |
| $1,\frac{1}{2}$ | 12 (4) | 24 (6) | 36 (10) | | | | | | |
| $\frac{5}{2},0$ | 12 (2) | 24 (4) | 36 (6) | 36 (6) | | | | | |
| $\frac{3}{2},1$ | 24 (6) | 48 (10) | 72 (16) | 72 (12) | 144 (19) | | | | |
| $2,\frac{1}{2}$ | 20 (4) | 40 (8) | 60 (12) | 60 (10) | 120 (22) | 100 (18) | | | |
| $\frac{5}{2},1$ | 36 (6) | 72 (12) | 108 (18) | 108 (16) | 216 (34) | 180 (28) | 324 (46) | | |
| $2,\frac{3}{2}$ | 40 (8) | 80 (14) | 120 (22) | 120 (18) | 240 (40) | 128 (32) | 360 (52) | 400 (56) | |
| $\frac{5}{2},2$ | 60 (10) | 120 (18) | 180 (28) | 180 (24) | 360 (52) | 300 (42) | 540 (70) | 600 (80) | 900 (110) |

$(s+1)s+s(s+1)=2s^2+2s$ in product set *oee*. For fermions, we have $2(s+\frac{1}{2})^2=2s^2+2s+\frac{1}{2}$ in each product set.

The total population in product sets has been given previously[3] for the general (noncollinear) case. Subtracting from it the collinear population of product sets, we get the amount of reduction brought about by collinearity. This reduction is for bosons

$$\tfrac{1}{8}(2s+1)^2[(2s'+1)^2-1]+\tfrac{1}{8}(2s'+1)^2-\tfrac{1}{8} \quad \text{for } eee, \tag{3.4}$$

$$\tfrac{1}{8}(2s+1)^2[(2s'+1)^2-3]-\tfrac{1}{8}(2s'+1)^2+\tfrac{3}{8} \quad \text{for } oee, \tag{3.5}$$

and for fermions

$$\tfrac{1}{8}(2s+1)^2(2s'+1)^2 \quad \text{for } eee, \tag{3.6}$$

$$\tfrac{1}{8}[(2s'+1)^2-4](2s+1)^2 \quad \text{for } oee. \tag{3.7}$$

We see that except for $s=s'=0$, there is always a reduction (i.e., simplification) through collinearity. We also see that the larger $s'$ and $s$ are, the greater the simplification, but the reduction is not a function of $s'-s$ as one might possibly believe intuitively.

As we have seen in Sec. II, the restrictions due to collinearity correspond to having characteristic quantities $\mu$ *and* $\nu$. The restrictions are therefore the same as those which would follow from *two* symmetries together with characteristic quantities $\mu$ and $\nu$ (or $\mu$ and $\mu+\nu$, or $\mu+\nu$ and $\nu$), respectively. In this sense, therefore, collinearity in the forward direction is equivalent to $B$ and $PT$, or $B$ and $PBT$, or $PBT$ and $PT$. The analogy is only formal, of course, since collinearity can be applied to any reaction, while the restrictions for $PT$ and $B$ make sense only for reactions which are self-transforming under these symmetries. The formal analogy, however, is useful, since one can immediately list the restrictions due to collinearity by giving the combined list of restrictions, e.g., for $PT$ and $B$.

Such a list, however, would not be quite complete, since the constraints for forward collinearity are not given fully by the characteristic quantities $\mu$ and $\nu$. The reason is that these quantities would still allow separate terms of $mm$ and $nn$, while we demand that they appear only as $mm+nn$. Imposing this extra constraint in addition to those obtained from characteristic quantities $\mu$ and $\nu$ does give the full restrictions. Thus we see that collinearity simplifies the structure of the reaction even beyond what one would expect from the simultaneous imposition of two symmetries.

Examples for these results, given in Sec. V, will illustrate the somewhat abstract considerations of this section.

## IV. COMPOSITE REACTIONS

The realization that almost all constraints of collinear processes are given by the simultaneous assignment of characteristic quantities $\mu$ and $\nu$ greatly facilitates writing down the collinear $M$ matrix for composite reactions. Assume that we want the $M$ matrix for

$$A+B\to C+D, \tag{4.1}$$

which we will call $M_1$. Similarly we will call $M_2$ and $M_3$ the $M$ matrices for

$$A+0\to C+0 \tag{4.2}$$

and

$$0+B\to 0+D, \tag{4.3}$$

respectively. We will denote by the two superscripts the behavior with respect to characteristic quantities $\nu$ and $\mu$, respectively.

Then $M_1^{++}$ will have contributions from two different types of terms. One is the type in which the component superscripts multiply out to $++$. These are $M_2^{++}\otimes M_3^{++}$, $M_2^{+-}\otimes M_3^{+-}$, $M_2^{-+}\otimes M_3^{-+}$, and $M_2^{--}\otimes M_3^{--}$. In the other type the superscripts multiply out to $--$, that is, $M_2^{++}\otimes M_3^{--}$, $M_2^{+-}\otimes M^{-+}$, $M_2^{-+}\otimes M_3^{+-}$, and $M_2^{--}\otimes M_3^{++}$. The reason why both types make contributions is that in the tensor-composition tables[5] in the expansion of one particular product, terms with a certain signature and terms with the conjugate signature alternate.

In addition, we must also impose the constraints due to the requirement that only $mm+nn$ be present. This can be done on the constituents and then be carried over to the composite reactions also.

An example for composite reaction will be given in Sec. V.

## V. EXAMPLES

### A. 0+1→0+0

The $M$ matrix in general,[6] assuming only rotation invariance, is

$$M=a_1\vec{S}\cdot\hat{l}+a_2\vec{S}\cdot\hat{m}+a_3\vec{S}\cdot\hat{n}. \tag{5.1}$$

For collinear processes, the number of terms is one. In particular, we have

$$M_c=a_1\vec{S}\cdot\hat{l}. \tag{5.2}$$

The remaining term is of the *oee* type, hence the remaining term set is *oee* and the remaining subclass is *oee*. In fact we have

$$L(0)=|a_1|^2, \quad L(nn)=\tfrac{1}{3}|a_1|^2, \quad L(ll)=\tfrac{2}{3}|a_1|^2. \tag{5.3}$$

Furthermore,

$$L(mm)+L(nn)=2L(nn)=-L(ll) \tag{5.4}$$

so that we really have only two independent observables and one relation between them,

$$L(ll)=-\tfrac{3}{2}L(0)\,. \tag{5.5}$$

Now let us turn to the conservation laws. If we want to restrict ourselves to test in *one* reaction, we must use self-transforming reactions. $0+1\to 0+0$ can possibly be such only for $I$, $P$, $TP$, $PTB$, $C$, $PC$, $TCB$, and $PTCB$.

### 1. Parity Conservation

In general we have

$$M^{+}=a_2\vec{S}\cdot\hat{m}\,, \tag{5.6}$$

$$M^{-}=a_1\vec{S}\cdot\hat{l}+a_3\vec{S}\cdot\hat{n}\,. \tag{5.7}$$

Thus we have for the collinear case

$$M_c^{+}=0\,, \tag{5.8}$$

$$M_c^{-}=a_1\vec{S}\cdot\hat{l}\,. \tag{5.9}$$

Thus we get the result that for the positive parity product there are no collinear processes at all, while for the negative parity product there is no difference between the parity-conserving and parity-nonconserving cases. Thus, in this example, measurement in the collinear directions yields the following information (nondynamically):

(i) If the observables are all zero, parity is conserved and the parity product is positive.

(ii) If the observables are not all zero, there is no way to conclude anything about whether parity is conserved, i.e., we can then have either parity nonconservation, or parity conservation with a negative parity product.

### 2. TB Invariance

The characteristic quantity is $\lambda_M+\mu_M$. Thus in general we have

$$M^{+}=a_3\vec{S}\cdot\hat{n}\,,\quad M^{-}=a_1\vec{S}\cdot\hat{l}+a_2\vec{S}\cdot\hat{m}\,. \tag{5.10}$$

For the collinear case we have

$$M_c^{+}=0, \tag{5.11}$$

$$M_c^{-}=a_1\vec{S}\cdot\hat{l}\,. \tag{5.12}$$

Thus if $TB$ invariance holds, all observables should vanish.

### 3. PTB Invariance

The characteristic quantity is $\mu_M+\nu_M$. Thus in general we have

$$M^{+}=a_1\vec{S}\cdot\hat{l}\,,\quad M^{-}=a_2\vec{S}\cdot\hat{m}+a_3\vec{S}\cdot\hat{n}\,. \tag{5.13}$$

For the collinear case then we have

$$M_c^{+}=a_1\vec{S}\cdot\hat{l}\,, \tag{5.14}$$

$$M_c^{-}=0\,. \tag{5.15}$$

Thus, for collinear reactions, $PTB$ cannot be tested, which is what we already know.

### 4. Other Tests

The tests $XC$ (with $X=P$, $TB$, or $PTB$) are the same as of $X$, except on a different type of self-transforming reaction.

#### B. $0+\frac{1}{2}\to 0+\frac{1}{2}$

Here the $M$ matrix in general[7] consists of four terms,

$$M=b_0+b_1\vec{\sigma}\cdot\hat{l}+b_2\vec{\sigma}\cdot\hat{m}+b_3\vec{\sigma}\cdot\hat{n}\,. \tag{5.16}$$

In the collinear case this will be reduced to two. We have

$$M_c=b_0+b_1\vec{\sigma}\cdot\hat{l}\,. \tag{5.17}$$

The remaining terms are of the *eee* and *oee* type. Thus the product sets will be of the same type and therefore the surviving subclasses will be *eee*, *oee*, and *eoo*.

The complete structure of these remaining subclasses has been given in Ref. 7, Table II. In view of Eq. (5.17), however, there will be relations between the observables even in these subclasses. In particular, we will have four relations among the eight remaining observables, since they will depend only on four bilinear products of $b_i$'s.

We have

$$\begin{aligned} &L(0,0)=L(l,l)\,,\quad L(l,0)=L(0,l)\,,\\ &L(m,m)=L(n,n)\,,\quad L(n,m)=-L(m,n)\,. \end{aligned} \tag{5.18}$$

The second and the fourth relations are the more trivial type, which can be obtained from the transformations of the type $n\leftrightarrow m$, $l\leftrightarrow l$. The first and third, however, are not of this type.

Now we turn to the conservation laws. The reaction in question can be a self-transforming reaction for $I$, $P$, $TP$, $PTB$, $B$, $PB$, $T$, $PT$, $CB$, $PCB$, $TC$, and $PTC$, but not for $C$, $PC$, $TCB$, and $PTCB$, since a spin-$\frac{1}{2}$ particle cannot be its own antiparticle. Of the eleven possible transformations (not counting $I$), the ones that can be tested in collinear reactions are $P$, $PTB$, $B$, $T$, $CB$, and $TC$.

### 1. Parity

We have in general

$$M^{+}=b_0+b_2\vec{\sigma}\cdot\hat{m}\,, \tag{5.19}$$

$$M^{-}=b_1\vec{\sigma}\cdot\hat{l}+b_3\vec{\sigma}\cdot\hat{n}\,. \tag{5.20}$$

In the collinear case

$$M_c^{+}=b_0\,, \tag{5.21}$$

$$M_c^{-}=b_1\vec{\sigma}\cdot\hat{l}\,. \tag{5.22}$$

Thus tests of parity conservation in collinear reactions are

$$L(l,0)=L(n,m)=0\,. \tag{5.23}$$

These are all the independent tests.

To determine the product of intrinsic parities, we have

$$L(0,0)=L(l,l)=\pm L(m,m)=\pm L(n,n)\,. \tag{5.24}$$

This is a simplification over the noncollinear case, since it permits us to compare either of $L(0,0)$ and $L(l,l)$ with either of $L(m,m)$ and $L(n,n)$ (the last two being indistinguishable in collinear reactions).

### 2. *Time Reversal*

In general we have

$$M^{+}=b_0+b_2\vec{\sigma}\cdot\hat{m}+b_3\vec{\sigma}\cdot\hat{n} \tag{5.25}$$

and in the collinear case

$$M_c^{+}=b_0\,. \tag{5.26}$$

Thus

$$L(l,0)=L(0,l)=0\,, \tag{5.27}$$

$$L(n,m)=L(m,n)=0\,, \tag{5.28}$$

$$L(0,0)=L(m,m)=L(l,l)=L(n,n)\,. \tag{5.29}$$

Again, these are simpler than the general tests. The Eq. (5.27) says that two mirror relations vanish and the third simplifies the general nonmirror relation of

$$L(0,0)=L(m,m)-L(l,l)+L(n,n)\,. \tag{5.30}$$

### 3. *PTC Theorem*

The self-transforming reaction is of the type

$$a+b\to\bar{a}+\bar{b} \tag{5.31}$$

and it is hard to think of a plausible physical example. Formally, however, we can proceed to write

$$M^{+}=b_0+b_1\vec{\sigma}\cdot\hat{l}+b_2\vec{\sigma}\cdot\hat{m} \tag{5.32}$$

so that

$$M_c^{+}=b_0+b_1\vec{\sigma}\cdot\hat{l}\,. \tag{5.33}$$

Thus, as expected, *PTC* cannot be tested in collinear reactions.

A similar discussion can be carried out for the other transformations. The $M$ matrices are given in Table II.

We see from that table that in this simple case in collinear reactions all testable symmetries have the same tests as time-reversal invariance. The type of self-transforming reaction on which such tests can be carried out may of course be different in different cases.

## C. 0+1→0+1

The $M$ matrix in general,[8] assuming only rotation invariance, is

$$\begin{aligned}M=&A_0+A_1S_{[1]}\cdot\hat{m}+A_2S_{[1]}\cdot\hat{l}+A_3S_{[1]}\cdot\hat{n}\\&+A_{21}S_{[2]}:\hat{l}\hat{l}+A_{22}S_{[2]}:\hat{n}\hat{n}+A_{23}S_{[2]}:\hat{l}\hat{n}\\&+A_{24}S_{[2]}:\hat{l}\hat{m}+A_{25}S_{[2]}:\hat{n}\hat{m}\,.\end{aligned} \tag{5.34}$$

TABLE II. $M$ matrices for the $0+\frac{1}{2}\to 0+\frac{1}{2}$ reactions under various transformation laws, in the collinear direction.

| Transformation | Nonzero coefficient in $M^{+}$ | Nonzero coefficient in $M_c^{+}$ |
|---|---|---|
| $I$ | $b_0, b_1, b_2, b_3$ | $b_0, b_1$ |
| $P$ | $b_0, b_2$ | $b_0$ |
| $T$ | $b_0, b_2, b_3$ | $b_0$ |
| $B$ | $b_0, b_3$ | $b_0$ |
| $PT$ | $b_0, b_1, b_2$ | $b_0, b_1$ |
| $PB$ | $b_0, b_1$ | $b_0, b_1$ |
| $TC$ | $b_0, b_2, b_3$ | $b_0$ |
| $TB$ | $b_0, b_1, b_3$ | $b_0, b_1$ |
| $CB$ | $b_0, b_3$ | $b_0$ |
| $PTC$ | $b_0, b_1, b_2$ | $b_0, b_1$ |
| $PTB$ | $b_0$ | $b_0$ |
| $PCB$ | $b_0, b_1$ | $b_0, b_1$ |

For collinear processes this reduces to

$$M_c = A_0 + A_2 S_{[1]} \cdot \hat{l} + A_{21} S_{[2]} : \hat{l}\hat{l} . \tag{5.35}$$

We see then that $(2s+1)=3$ terms are left, as expected. Since the example was worked out in detail in Ref. 8 in terms of the form of this reaction viewed as a composite reaction, we have to convert the constraints into this notation. This will give us an opportunity to check some of our results on composite reactions.

We have the constituent reactions

$$0+1 \to 0+0 \tag{5.36}$$

and

$$0+0 \to 0+1, \tag{5.37}$$

and their $M$ matrices

$$M_1 = a_1 T(l) : S' + a_2 T(m) : S' + a_3 T(n) : S' \tag{5.38}$$

and

$$M_2 = b_1 T(l) : S + b_2 T(m) : S + b_3 T(n) : S . \tag{5.39}$$

Then we have for the composite reaction

$$\begin{aligned} M &= M_1 \otimes M_2 \\ &= C_{11} T(l) : ST(l) : S' + C_{22} T(m) : ST(m) : S' \\ &\quad + C_{33} T(n) : ST(n) : S' + C_{13} T(n) : ST(l) : S' \\ &\quad + C_{31} T(l) : ST(n) : S' + C_{21} T(l) : ST(m) : S' \\ &\quad + C_{12} T(m) : ST(l) : S' + C_{23} T(n) : ST(m) : S' \\ &\quad + C_{32} T(m) : ST(n) : S' . \end{aligned} \tag{5.40}$$

The relationship between coefficients $A_{ij}$ and $C_{ij}$ are

$$\begin{aligned} &C_{22} = -A_0 - \tfrac{1}{3}A_{21} - \tfrac{1}{3}A_{22}, \quad C_{13} = -iA_1 + \tfrac{1}{2}A_{23}, \\ &C_{23} = iA_2 + \tfrac{1}{2}A_{25}, \quad C_{11} = -A_0 + \tfrac{2}{3}A_{21} - \tfrac{1}{3}A_{22}, \\ &C_{31} = iA_1 + \tfrac{1}{2}A_{23}, \quad C_{32} = -iA_2 + \tfrac{1}{2}A_{25}, \\ &C_{33} = -A_0 - \tfrac{1}{3}A_{21} + \tfrac{2}{3}A_{22}, \quad C_{12} = iA_3 + \tfrac{1}{2}A_{24}, \\ &C_{21} = -iA_3 + \tfrac{1}{2}A_{24}, \end{aligned} \tag{5.41}$$

or

$$\begin{aligned} &A_0 = -\tfrac{1}{3}(C_{11} + C_{22} + C_{33}), \quad A_1 = \tfrac{1}{2}i(C_{13} - C_{31}), \\ &A_2 = \tfrac{1}{2}i(C_{32} - C_{23}), \quad A_{21} = C_{11} - C_{22}, \\ &A_{23} = C_{13} + C_{31}, \quad A_{25} = C_{23} + C_{32}, \\ &A_{22} = C_{33} - C_{22}, \quad A_3 = \tfrac{1}{2}i(C_{21} - C_{12}), \\ &A_{24} = C_{21} + C_{12} . \end{aligned} \tag{5.42}$$

We have, using Eq. (4.14),

$$M_1^{++} = a_1 T(l) : S', \quad M_2^{++} = b_1 T(l) : S, \tag{5.43}$$

$$M_1^{+-} = a_2 T(m) : S', \quad M_2^{+-} = b_2 T(m) : S, \tag{5.44}$$

$$M_1^{-+} = a_3 T(n) : S', \quad M_2^{-+} = b_3 T(n) : S, \tag{5.45}$$

$$M_1^{--} = 0, \quad M_2^{--} = 0 . \tag{5.46}$$

So that

$$\begin{aligned} M_0^{++} &= C_{33} T(n) : S'T(n) : S + C_{22} T(m) : S'T(m) : S \\ &\quad + C_{11} T(l) : S'T(l) : S , \end{aligned} \tag{5.47}$$

$$M_0^{--} = C_{23} T(m) : S'T(n) : S + C_{32} T(n) : S'T(m) : S . \tag{5.48}$$

We actually expect only three terms, and indeed we get $C_{11} = C_{22}$, $C_{32} = -C_{23}$. So we finally have

$$\begin{aligned} M_0^c &= \mathfrak{C}_1 [T(n) : S'T(n) : S + T(m) : S'T(m) : S] \\ &\quad + \mathfrak{C}_2 T(l) : S'T(l) : S \\ &\quad + \mathfrak{C}_3 [T(m) : S'T(n) : S - T(n) : S'T(m) : S] . \end{aligned} \tag{5.49}$$

In general, the reaction $0+1 \to 0+1$ has the full observable structure of a composite reaction, consisting of 31 subclasses. The great simplification due to Eq. (3.2) in the collinear direction reduces the number of subclasses to seven. They are given in Table III. Many of the general subclasses vanished entirely, others fused together.

The remaining subclasses contain 39 observables, but clearly only $3^2$, or 9, can be linearly independent. So we must have 30 relationships among them.

If we had imposed only the constraints due to characteristic quantities $\nu$ and $\mu$, we would have had not three but four amplitudes (because $C_{11} = C_{22}$ would not be valid). In that case we would have had $4^2$ or 16 linearly independent observables, and thus 39 – 16, or 23, relationships. Thus the remaining seven relationships in the forward direction are specific consequences of the additional constraints due to the requirement that only $\nu\nu + mm$ can appear and not $\nu\nu$ or $mm$ separately.

The 30 relationships which hold for the forward collinear case can be constructed immediately from Table III. They are the following:

$$\begin{aligned} &L(0, nn) = L(nn, 0), \quad L(0, ll) = L(ll, 0), \\ &L(0, nn) = -\tfrac{1}{2}L(0, ll), \quad L(nn, ll) = -2L(ll, nn), \\ &L(0, 0) + 3L(0, nn) = -9L(nn, ll), \\ &L(l, l) = 2L(0, 0) - 6L(0, nn), \\ &2L(nm, nm) = -\tfrac{1}{9}L(0, 0) + 2L(nn, nn) - \tfrac{1}{3}L(0, nn), \\ &L(nl, nl) = \tfrac{1}{4}L(m, m), \quad L(m, nl) = -L(nl, m), \\ &L(n, n) = L(m, m), \quad L(m, nl) = L(lm, n), \\ &L(lm, lm) = \tfrac{1}{4}L(m, m), \quad L(lm, n) = -L(n, lm), \\ &L(0, nm) = 0, \quad 6L(nn, l) = -L(0, l), \end{aligned} \tag{5.50}$$

TABLE III. Observables for collinear reactions of the type $0+1\to 0+1$.

| Subclass $\alpha$ | $\lvert\mathcal{C}_1\rvert^2$ | $\lvert\mathcal{C}_2\rvert^2$ | $\lvert\mathcal{C}_3\rvert^2$ |
|---|---|---|---|
| $L(0,0)$ | 2 | 1 | 2 |
| $3L(0,nn)$ | −1 | 1 | −1 |
| $3L(0,ll)$ | 2 | −2 | 2 |
| $3L(nn,0)$ | −1 | 1 | −1 |
| $3L(ll,0)$ | 2 | −2 | 2 |
| $9L(nn,nn)$ | 5 | 1 | −4 |
| $9L(nn,ll)$ | −1 | −2 | −1 |
| $9L(ll,nn)$ | −1 | −2 | −1 |
| $9L(ll,ll)$ | 2 | 4 | 2 |
| $L(l,l)$ | 2 | 0 | 2 |
| $2L(lm,lm)$ | 1 | 0 | −1 |
| Subclass $\beta$ | $\mathrm{Re}\mathcal{C}_1\mathcal{C}_2^*$ | | |
| $2L(ln,ln)$ | 1 | | |
| $L(m,m)$ | 2 | | |
| $L(l,l)$ | 2 | | |
| $2L(nm,nm)$ | 1 | | |
| Subclass $\gamma$ | $\mathrm{Im}\mathcal{C}_1\mathcal{C}_2^*$ | | |
| $L(m,ln)$ | −1 | | |
| $L(ln,m)$ | 1 | | |
| $L(lm,n)$ | −1 | | |
| $L(n,lm)$ | 1 | | |
| Subclass $\delta$ | $\mathrm{Im}\mathcal{C}_1\mathcal{C}_3^*$ | | |
| $L(0,nm)$ | 0 | | |
| $3L(nn,nm)$ | −3 | | |
| $3L(ll,nm)$ | 0 | | |
| $L(nm,0)$ | 0 | | |
| $3L(nm,nn)$ | 3 | | |
| $3L(nm,ll)$ | 0 | | |
| Subclass $\epsilon$ | $\mathrm{Im}\mathcal{C}_1\mathcal{C}_3^*$ | | |
| $L(0,l)$ | −4 | | |
| $3L(nn,l)$ | 2 | | |
| $3L(ll,l)$ | −4 | | |
| $L(l,0)$ | −4 | | |
| $3L(l,nn)$ | 2 | | |
| $3L(l,ll)$ | −4 | | |
| Subclass $\zeta$ | $\mathrm{Re}\mathcal{C}_2\mathcal{C}_3^*$ | | |
| $L(m,n)$ | 2 | | |
| $2L(ln,lm)$ | −1 | | |
| $L(n,m)$ | −2 | | |
| $2L(lm,ln)$ | 1 | | |
| Subclass $\eta$ | $\mathrm{Im}\mathcal{C}_2\mathcal{C}_3^*$ | | |
| $L(m,lm)$ | −1 | | |
| $L(ln,n)$ | −1 | | |
| $L(lm,m)$ | −1 | | |
| $L(n,ln)$ | −1 | | |

$$L(ll,nm)=0,\quad 3L(ll,l)=L(0,l),$$
$$L(nm,0)=0,\quad L(l,0)=L(0,l),$$
$$L(nm,ll)=0,\quad 6L(l,nn)=-L(0,l),$$
$$L(nn,nm)=-L(nm,nn),\quad 3L(l,ll)=L(0,l),$$
$$4L(nl,lm)=-L(m,n),\quad L(m,lm)=L(nl,n),$$
$$L(n,m)=-L(m,n),\quad L(lm,m)=L(n,nl),$$
$$4L(lm,nl)=L(m,n),\quad L(n,nl)=L(nl,n).$$

For the sake of variety, instead of looking at the tests of conservation laws, let us explore this time the consequences of collinearity for the determination[9] of the $M$ matrix. We see that in the collinear reaction the set $L(0,0)$, $L(0,l)$, $L(n,n)$, $L(l,l)$, and $L(m,n)$ is sufficient for a complete determination (apart from possible *discrete* ambiguities). Thus in this case measurements of tensor polarization is not necessary. If in addition parity is also conserved, only three measurements will suffice and $L(0,0)$, $L(n,n)$, and $L(l,l)$ will do the trick.

## VI. CONCLUSION

We have seen that the structure of reactions in the special case of collinearity simplifies substantially. As a result, certain symmetries are automatically satisfied, provided the kinematics of spin are properly handled and hence their true validity becomes untestable. Therefore, dynamical models, with applicability mainly in the collinear directions, need not take special pains to incorporate explicitly those symmetries, since collinearity will take care of it automatically. The simplified structure also permits an easier determination of invariant amplitudes in the collinear directions. The extra relationships among observables, brought about by collinearity might also be useful as checks on complicated types of polarization experiments.

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## APPENDIX

In this Appendix, I summarize some of the notation in this paper that has been taken over from previous papers.

The unit vectors used to span the momentum space are defined as

$$\hat{l}=\frac{\vec{q}'-\vec{q}}{|\vec{q}'-\vec{q}|},\quad \hat{m}=\frac{\vec{q}'\times\vec{q}}{|\vec{q}'\times\vec{q}|},\quad \hat{n}=\hat{l}\times\hat{m},\tag{A1}$$

where $\vec{q}$ is the momentum of $A$ and $\vec{q}'$ the momentum of $C$ in the center-of-mass system of the reac-

tion

$$A+B\to C+D. \tag{A2}$$

The $M$ matrix for the reaction

$$A+0\to C+D \tag{A3}$$

(where 0 denotes a particle with spin 0) is written in the form

$$M=\sum_{J,r} a_J^r S_{[J]}(s,s'):T^r_{[J]}, \tag{A4}$$

where $S_{[J]}(s,s')$ is a $J$th-rank spin tensor, depending on the spins $s$ of $A$ and $s'$ of $C$; $T^r_{[J]}$ is a $J$th-rank momentum tensor, containing $l$'s, $m$'s, and $n$'s, and in it the superscript $r$ serves to distinguish the various tensors of the same rank. The colon denotes complete contraction of tensorial indices. The $a^r_J$'s are zero-rank tensor amplitudes, functions of the invariants that can be constructed out of the momenta characterizing the kinematics. The indices $J$ and $r$ on the $a$'s indicate that each $S:T$ combination has its own amplitude.

The various $a$'s in the sum (A4) can be classified into term sets as follows: (a) Count the number of $l$'s, the number of $m$'s, and the number of $n$'s in the $T^r_{[J]}$ that belongs to a certain $a^r_J$. (b) Define the signature of the $a^r_J$ by determining whether the three numbers found in (a) are even or odd (there will thus be eight different signatures: *eee*, *eeo*, *eoe*, ..., *ooo*, where *e* denotes "even" and *o* denotes "odd"). (c) All $a$'s having the same signature belong to the same term set.

The experimental observables (denoted by $L$) are characterized by the preparation of the initial state, and the measurements carried out in the final state. Each of these two specifications can be given in terms of a spin-momentum tensor similar to those appearing in Eq. (A4) for $M$ [except that they will involve $S(s,s)$ and $S(s',s')$, respectively]. Thus, using a shorthand notation, the experimental observables can be characterized as $L(t,u)$, where $t$ is the collection of $l$'s, $m$'s, and $n$'s that appears in the $T^r_{[J]}$ describing the initial state (the order of the $l$'s, $m$'s, and $n$'s does not matter), and $u$ pertains to the final state in a similar way. The actual experimental quantities measured in the laboratory are either these $L(t,u)$'s, or linear combinations thereof, depending on the details of the experimental setup.

The $L(t,u)$'s can be classified into subclasses as follows: (a) Count the number of $l$'s, $m$'s, and $n$'s in $t$ and $u$ together. (b) Define the signature of $L(t,u)$ by ascertaining whether the three numbers obtained in (a) are even or odd (thus one obtains eight types of signatures: *eee*, *eeo*, *eoe*, ..., *ooo*, where *e* denotes "even", and *o* denotes "odd"). (c) Observables with the same signature belong to a given subclass.

Similarly, bilinear combinations of amplitudes, that is, $a^{r_1}_{J_1}a^{r_2*}_{J_2}$'s, can be classified into product sets according to the evenness or oddness of the total numbers of $l$'s, $m$'s, and $n$'s in the two $T$'s together which go with the two $a$'s in Eq. (A4).

Finally, the notation for the symmetries is self-evident except for $B$, which denotes detailed balancing. Under $B$, initial and final particles are interchanged, $\hat{l}$ and $\hat{n}$ turn into minus themselves, $\hat{m}$ remains the same, and the spin vector $\vec{S}$ changes sign. $B$ is of course not a true symmetry, but rather a transformation under which for dynamical reasons, the reaction remains invariant under certain circumstances.

Under various symmetries, $\hat{l}$, $\hat{m}$, $\hat{n}$, and the spin vector $\vec{S}$ go either into themselves or into minus themselves. Whether a term in the $M$ matrix is invariant under a symmetry transformation or changes its sign will therefore depend on whether the number of those ingredients (i.e., $\hat{l}$'s, $m$'s, $n$'s, and $S$'s) which change their signs under that transformation is even or odd. This number of the crucial ingredients is called the characteristics quantity, and it can be composed of some sum of $\lambda_M$, $\mu_M$, and $\nu_M$, which are the numbers of $l$'s, the number of $m$'s, and the number of $n$'s, respectively.

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# Writing Hamiltonians in Terms of Local Currents*

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Several relativistic models are presented which are based on local currents and combine nontrivial internal-symmetry groups with $q$-number Schwinger terms. Each model is given by specifying a Lie algebra of equal-time current commutators together with a consistent expression for the Hamiltonian as a function of the currents.

## I. INTRODUCTION

Several authors have been pursuing the idea of writing nonrelativistic and relativistic models with local currents as the basic dynamical variables.[1] Sugawara proposed such a model, with internal symmetry and finite $c$-number Schwinger terms.[2] Its simplicity and internal consistency inspired considerable investigation; but Dashen and Frishman showed that it possessed "too much symmetry", leading to consequences not observed in the physical world.[3] Simultaneously, Dashen and Sharp proposed a quark model based on local currents, but for which they were unable to identify the Schwinger terms.[4]

In this paper we present some relativistic models, based on local currents, with nontrivial internal-symmetry groups and $q$-number Schwinger terms. The models we discuss resemble Sugawara's model in that the equal-time current algebras reduce to his if the Schwinger terms are replaced by $c$-numbers. However, the Hamiltonians we write in terms of the local currents are not generalizations of Sugawara's Hamiltonian, but resemble more closely the Hamiltonian in a model proposed by Sharp, which we discuss in Sec. II.[5]

For each model we present a Lie algebra of equal-time current commutators, together with an expression for the Hamiltonian as a function of the local currents. Concrete representations in Hilbert space have not yet been obtained for such an algebraic system, except for nonrelativistic models[6]; but some things can be said without making a commitment to a particular representation. For example, the generators of the internal-symmetry group $G$ (the "charges"), obtained formally by integrating the time components of the local currents over all space, cannot be represented as operators in the physical Hilbert space.[7] Nevertheless, the invariance of the Hamiltonian with respect to $G$ can be specified by requiring equations of continuity, such as Eq. (3.20) below, to follow from the current algebra. Relativistic