Renormalizable Massive Vector-Meson Theory – Perturbation Theory of the Higgs Phenomenon*

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We establish that the Abelian gauge theory first considered by Higgs in which the gauge vector boson acquires a finite mass due to the spontaneous breakdown of symmetry is renormalizable, in the sense that the Bogoliubov-Parasiuk-Hepp program can be executed in such a way that the Ward-Takahashi identities are satisfied. This paper contains the global study of the Ward-Takahashi identities and low-energy theorems of the model. We show that the Goldstone boson and the scalar excitation of zero mass associated with the Landau-gauge vector propagator are unphysical, and disappear from the S matrix.

I. INTRODUCTION

The Goldstone theorem[1,2] states that in a theory with a conserved current, but in which the vacuum (or the ground state) is not invariant under the symmetry usually associated with the current conservation, there must exist massless particles or excitations whose energy tends to zero in the long-wavelength limit, *provided* that the theory is manifestly relativistically invariant, or, in nonrelativistic cases, provided that forces in the theory have short ranges.[3] Nonrelativistically, the conversion of the phonon (Goldstone particle) into the plasmon (a massive particle) in the presence of a Coulomb interaction is a well-known evasion to the theorem.[4] Higgs[5] and Guralnik, Hagen, and Kibble[6] pointed out that in a gauge theory quantized in the radiation gauge, *manifest* covariance is lost (while the theory is still relativistically invariant), so that when the vacuum is not invariant under the gauge transformation there is no Goldstone boson. Higgs[7] and Kibble[8] have studied the spontaneous breakdown of Abelian and non-Abelian gauge symmetry, respectively, and have shown, in both cases, that the would-be Goldstone boson and the gauge vector particle, which would have only two transverse polarizations in the normal case, combine together to produce a massive vector boson. In a manifestly covariant formulation of gauge theories, for example in the Landau gauge, the Goldstone theorem can be proved, but due to the lack of positivity in the Hilbert-space structure, the Goldstone boson can and in fact does decouple from the physical states.

The author's recent interest in this phenomenon stems from Weinberg's model of leptons[9] in which electromagnetic and weak interactions are both mediated by Yang-Mills vector bosons, and in which the mass of the vector bosons mediating weak interactions is due to the Higgs-Kibble mechanism. Weinberg suggested that the theory might be renormalizable since the equations of motion of this theory are formally the same as those of a Yang-Mills theory.

Subsequently, 't Hooft[10] examined the question of renormalization of theories of this genre, and came to a conclusion affirming Weinberg's conjecture.

In this paper I propose to give a proof that the model originally studied by Higgs[7] – scalar electrodynamics in which the normal vacuum is unstable – is renormalizable. We choose to study an Abelian case first, because it is simpler and because all tools we need are available. We attempt to untangle the difficulties associated with the Higgs phenomenon *per se* from those that have to do with a non-Abelian gauge group. By "renormalizable," we mean that the theory can be rendered finite by specifying a finite number of values of primitively divergent vertices at subtraction points in such a way that the Ward-Takahashi identities are satisfied by the Green's functions of the theory. The Ward-Takahashi identities are the precise mathematical statement of gauge invariance in terms of the Green's functions, and allow us to explore the consequences of broken symmetry. They enable us to demonstrate both that there is a Goldstone boson in the theory and that it is unphysical. The part of the theory that has to do with the stability of an asymmetric vacuum is completely analogous to the U(1) version of the σ model,[11] whose renormalizability we have studied previously.[12,13] In fact the present discussion makes an essential use of the technique developed by Symanzik[13] in this connection. We shall consider a set of gauges characterized by a continuous parameter α, all of which gives a relativistically invariant theory, and all of which gives the same S matrix.

The reasons why theories of this type can be renormalized are, first, that because of gauge invariance the vector propagator may be taken, for example, as

$$-i(g_{\mu\nu}-k_\mu k_\nu/k^2)(k^2-m^2)^{-1}, \tag{1.1}$$

instead of

$$-i(g_{\mu\nu}-k_\mu k_\nu/m^2)(k^2-m^2)^{-1}, \tag{1.2}$$

and, second, that the ultraviolet behaviors of the Green's functions are the same as those of a theory with symmetric vacuum. This fact manifests itself in that the same regularization method makes divergent integrals finite in both theories. The negative-metric zero-mass pole in the propagator (1.1) is canceled exactly by the positive-metric pole due to the Goldstone boson in the S matrix, making both particles associated with these poles unphysical.

This paper is organized as follows: Section II is a brief exposition of the Higgs model. In Sec. III we quantize the model by the functional integration method of Feynman,[14] with the modification necessary for gauge theories as pointed out by Popov and Faddeev.[15] I found this method particularly appealing, since we can dispense with the technique (but perhaps not with the spirit) of the Gupta-Bleuler formalism.[16] The Feynman rules are derived and the necessity and the method of a gauge-invariant (Pauli-Villars)[17] regularization of divergent integrals are discussed. These sections contain nothing substantially new, but they are included here partly for pedagogical purpose and partly from my desire to have this paper reasonably self-contained.

Section IV is a discussion of the Ward-Takahashi (WT) identities. These are discussed globally in terms of the generating functional of the Green's functions. The identities for the irreducible vertices are then derived by constructing the generating functional of the irreducible vertices. Section V gives the prescription for determining the values of primitively divergent irreducible vertices at subtraction points. The prescription is derived from the WT identities. This prescription, together with the finiteness of the number of primitively divergent vertices, the existence of a gauge-invariant regularization method, and the Bogoliubov-Parasiuk-Hepp (BPH) theorem,[18,19] then establishes the renormalizability of the theory as we defined above. Section VI gives the proof, based, again, on the WT identities, that the Goldstone boson and the massless ghost associated with the longitudinal part of the vector propagator decouple from the physical states (i.e., disappear from the S matrix) and that the S matrix is independent of the particular gauge chosen.

II. MODEL

A model which exhibits the Higgs phenomenon, without the complications due to non-Abelian gauge invariance, is the Abelian gauge generalization of the model first discussed by Goldstone.[1] Following Higgs,[7] we consider the Lagrangian

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2+(\partial_\mu+ieA_\mu)\phi^*(\partial^\mu-ieA^\mu)\phi\\ &-\mu^2(\phi^*\phi)-\lambda(\phi^*\phi)^2 .\end{aligned} \tag{2.1}$$

If $\mu^2>0$, the Lagrangian describes charged scalar bosons interacting with the radiation field. If $\mu^2<0$, on the other hand, ϕ must develop a vacuum expectation value so as to make the physical masses of scalar bosons non-negative. The situation here is very similar to that of the σ model.[20] We can adjust the phase of the ϕ field so that the vacuum expectation value is real. Let

$$\phi(x)=(\tfrac{1}{2})^{1/2}[v+\psi(x)+i\chi(x)], \tag{2.2}$$

so that

$$\langle\phi(x)\rangle_0=(\tfrac{1}{2})^{1/2}v \tag{2.3}$$

and

$$\psi^*(x)=\psi(x),\quad \chi^*(x)=\chi(x).$$

The Lagrangian (2.1) may be rewritten as

$$\begin{aligned}\mathcal{L}[A_\mu,\psi,\chi]= &-\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2+\tfrac{1}{2}(ev)^2A_\mu{}^2\\ &+\tfrac{1}{2}(\partial_\mu\psi+eA_\mu\chi)^2+\tfrac{1}{2}(\partial_\mu\chi-eA_\mu\psi)^2\\ &-evA_\mu(\partial^\mu\chi-eA^\mu\psi)\\ &-\tfrac{1}{2}(\lambda v^2+\mu^2)\chi^2-\tfrac{1}{2}(3\lambda v^2+\mu^2)\psi^2\\ &-\lambda v\psi(\psi^2+\chi^2)-\tfrac{1}{4}\lambda(\psi^2+\chi^2)^2\\ &-v(\lambda v^2+\mu^2)\psi .\end{aligned} \tag{2.4}$$

In Eq. (2.4) the mass of the vector boson appears as $m^2=(ev)^2$. The vacuum expectation value v is determined by the condition (2.3). In the tree approximation, this requires the absence of the linear ψ term (the last term) in Eq. (2.4):

$$v^2=-\mu^2/\lambda . \tag{2.5}$$

[In higher orders, the term $-iv(\lambda v^2+\mu^2)$ should cancel the ψ-to-vacuum diagrams. This condition gives an eigenvalue equation for v. The similar problem in the σ model is discussed in Ref. 20.] Thus the mass of χ is zero. In this sense χ is the Goldstone boson. The mass of ψ is $2\lambda v^2$, which is positive.

In the case $\mu^2>0$, there are two scalar degrees of freedom and two polarizations associated with a massless vector boson. Equation (2.4), for $\mu^2 < 0$, appears to describe two scalar degrees of freedom and three polarizations associated with a massive vector boson. What really happens, however, is that one scalar degree of freedom, χ, decouples from the rest and disappears from the physical spectrum. To see this we shall parametrize the complex field ϕ somewhat differently from Eq. (2.2). We write, following Kibble,[8]

$$\phi(x) = (\tfrac{1}{2})^{1/2}[v + \rho(x)]\exp[i\zeta(x)/v], \tag{2.6}$$

and transform the vector field according to

$$B_\mu(x) = A_\mu(x) - \frac{1}{ev}\partial_\mu \zeta(x). \tag{2.7}$$

The Lagrangian is then transformed into

$$\begin{aligned}\mathcal{L}[B_\mu, \rho] = &-\tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \tfrac{1}{2}(ev)^2 B_\mu{}^2 \\ &+ \tfrac{1}{2}(\partial_\mu \rho)^2 - \tfrac{1}{2}(3\lambda v^2 + \mu^2)\rho^2 - \tfrac{1}{4}\lambda\rho^4 \\ &+ \tfrac{1}{2}e^2 B_\mu{}^2(2v\rho + \rho^2) - v(\lambda v^2 + \mu^2)\rho,\end{aligned} \tag{2.8}$$

from which the would-be Goldstone-boson field ζ has been completely transformed away. The transformation (2.7) shows that the massive vector boson with three polarization degrees of freedom is made up of the two transverse polarizations associated with A_μ and the would-be Goldstone boson χ as the longitudinal degree of freedom.

The Lagrangian in Eq. (2.8), while advantageous in not containing redundant fields, is not renormalizable, as a simple power-counting of Feynman integrals indicates, if we treat

$$\begin{aligned}\mathcal{L}_0[B_\mu, \rho] = &-\tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \tfrac{1}{2}(ev)^2 B_\mu{}^2 \\ &+ \tfrac{1}{2}(\partial_\mu \rho)^2 - \tfrac{1}{2}(3\lambda v^2 + \mu^2)\rho^2\end{aligned}$$

as the unperturbed Lagrangian density.

The renormalization program we shall develop is based on the Lagrangian (2.4), which makes reference to the redundant field χ. Our task is therefore to show that all divergences in the theory can be removed by the redefinition of parameters of the theory, and the T matrix so renormalized does not contain poles corresponding to the unphysical χ-field excitation. Both of these objectives are attained thanks to the invariance of the Lagrangian under the local gauge transformation,

$$\begin{aligned}&A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\theta, \\ &\phi_1 \to \phi_1\cos\theta - \phi_2\sin\theta, \\ &\phi_2 \to \phi_2\cos\theta + \phi_1\sin\theta,\end{aligned} \tag{2.9}$$

where

$$\phi(x) = (\tfrac{1}{2})^{1/2}[\phi_1(x) + i\phi_2(x)]. \tag{2.10}$$

III. QUANTIZATION

A. Quantization by Functional Integration

Following Feynman,[14] we express the vacuum-to-vacuum amplitude by the functional integral,

$$\langle \text{out}|\text{in}\rangle \sim \int [dA_\mu][d\phi_1][d\phi_2]\exp\{iS[A_\mu, \phi_1, \phi_2]\}, \tag{3.1}$$

where $S = \int d^4x\,\mathcal{L}(x)$ is the action, and the measure $[dA_\mu]$ of the functional integration is the usual one for the vector field,

$$[dA_\mu] = \prod_\mu \prod_x dA_\mu(x). \tag{3.2}$$

Popov and Faddeev[15] pointed out that in a gauge-invariant theory, the action is invariant under the substitution

$$\begin{aligned}&A_\mu \to A_\mu^\theta = A_\mu + \frac{1}{e}\partial_\mu\theta, \\ &\phi_1 \to \phi_1^\theta = \phi_1\cos\theta - \phi_2\sin\theta, \\ &\phi_2 \to \phi_2^\theta = \phi_2\cos\theta + \phi_1\sin\theta,\end{aligned} \tag{3.3}$$

where A_μ^θ, for example, is a result of performing an element θ of the gauge group to the field A_μ. In other words the action is constant over an orbit of the gauge group, which is formed by all $A_\mu^\theta(x)$, $\phi_1^\theta(x)$, $\phi_2^\theta(x)$ for fixed $A_\mu(x)$, $\phi_1(x)$, $\phi_2(x)$ and θ running all over the gauge group. Hence the integral (3.1) is proportional to the infinite factor $\int\Pi_x d\theta(x)$. In order that the formula (3.1) be not more singular than those of the usual theories without gauge degrees of freedom, this factor should be extracted before proceeding to quantization.

The extraction of this infinite factor can be effected by the following device. Let the gauge condition

$$f[A_\mu(x)] = 0 \tag{3.4}$$

be such that the equation $f[A_\mu^\theta(x)] = 0$ has a unique solution $\theta(x)$ for a given $A_\mu(x)$. We may write Eq. (3.1) as

$$\begin{aligned}\langle \text{out}|\text{in}\rangle \sim \int [d\theta]\int [dA_\mu][d\phi_1][d\phi_2]\,\Delta_f[A] \\ \times \prod_x \delta(f[A_\mu^\theta(x)])e^{iS},\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}\Delta_f[A] &= \det\{\partial f[A^\theta(x)]/\partial\theta(x')\} \\ &= \left(\int [d\theta]\prod_x \delta(f[A_\mu^\theta(x)])\right)^{-1}\end{aligned} \tag{3.6}$$

is independent of θ, i.e., $\Delta_f[A^\theta] = \Delta_f[A]$. The integrand of Eq. (3.5),

$$\int [dA_\mu][d\phi_1][d\phi_2]\Delta_f[A]\prod_x \delta(f[A_\mu^\theta(x)])e^{iS},$$

is independent of θ, due to the gauge invariance of S and the functional measure. Therefore, after removing the factor $\int[d\theta]$, we can write Eq. (3.5) as[15]

$$\langle \text{out}|\text{in}\rangle \sim \int [dA_\mu][d\phi_1][d\phi_2]\Delta_f[A]\prod_x \delta(f[A_\mu(x)])e^{iS}. \tag{3.7}$$

A convenient gauge condition is

$$f[A_\mu(x)] = \partial^\mu A_\mu(x) - c(x). \tag{3.8}$$

With this choice, $\Delta_f[A]$ is simply a factor independent of A and may be dropped from Eq. (3.7). Equation (3.7) is in fact independent of $c(x)$, so

$$\langle \text{out}|\text{in}\rangle \sim \int [dc]\exp\left(-i\frac{1}{2\alpha}\int d^4x\, c^2(x)\right)\int [dA_\mu][d\phi_1][d\phi_2]\prod_x \delta(\partial^\mu A_\mu(x) - c(x))\, e^{iS} = \int [dA_\mu][d\phi_1][d\phi_2]e^{iS_\alpha}, \tag{3.9}$$

where

$$S_\alpha = S[A_\mu, \phi_1, \phi_2] - \frac{1}{2\alpha}\int d^4x[\partial^\mu A_\mu(x)]^2 = \int d^4x\left\{\mathcal{L}(x) - \frac{1}{2\alpha}[\partial^\mu A_\mu(x)]^2\right\}. \tag{3.10}$$

Equation (3.10) forms the basis of a quantum theory of the Lagrangian $\mathcal{L}$. We define

$$\exp\{iZ_\alpha[\eta_\mu, J_1, J_2]\} \equiv \int [dA_\mu][d\phi_1][d\phi_2]\exp\left[i\left(S_\alpha + \int d^4x[\vec{J}(x)\cdot\vec{\phi}(x) - \eta_\mu(x)A^\mu(x)]\right)\right]. \tag{3.11}$$

The quantity $Z_\alpha[\eta_\mu, J_1, J_2]$ is the generating functional of the connected Green's functions. For $\mu^2 > 0$, we have

$$i\frac{\delta^{n+m+l}Z_\alpha}{\delta J_1(x_1)\cdots\delta J_1(x_n)\delta J_2(y_1)\cdots\delta J_2(y_m)\delta\eta^{\mu_1}(z_1)\cdots\delta\eta^{\mu_l}(z_l)}\bigg|_{J_1=J_2=\eta_\mu=0}$$
$$= (i)^{n+m-l}\langle(\phi_1(x_1)\cdots\phi_1(x_n)\phi_2(y_1)\cdots\phi_2(y_m)A_{\mu_1}(z_1)\cdots A_{\mu_l}(z_l))_+\rangle_0^{\text{connected}}. \tag{3.12}$$

In particular, the free vector-boson propagator is

$$-i[g_{\mu\nu} - k_\mu k_\nu(1-\alpha)/k^2]/k^2,$$

so that the choice $\alpha = 1$ corresponds to the Feynman gauge, while $\alpha = 0$ corresponds to the (transverse) Landau gauge.

For $\mu^2 < 0$,[21] we translate the variable ϕ_1 by a constant v:

$$\phi_1 = \psi + v, \quad \phi_2 = \chi, \tag{3.13}$$

where v is determined from the condition

$$\frac{\delta Z_\alpha}{\delta J_1(x)}\bigg|_{J_1=J_2=\eta_\mu=0} = v \tag{3.14}$$

[the left-hand side of Eq. (3.14) is independent of x due to translational invariance], which is the functional expression for Eq. (2.3). Equation (3.11) now takes the form

$$Z_\alpha[\eta_\mu, J_1, J_2] = -i\ln\int [dA_\mu][d\psi][d\chi]\exp\left(i\int d^4x\{\mathcal{L}[A_\mu, \psi, \chi] - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 + (v+\psi)J_1 + \chi J_2 - \eta^\mu A_\mu\}\right), \tag{3.15}$$

where $\mathcal{L}[A_\mu, \psi, \chi]$ is the expression in Eq. (2.4). Analogously to Eq. (3.12), we have

$$i\frac{\delta^{n+m+l}Z_\alpha}{\delta J_1(x_1)\cdots\delta J_1(x_n)\delta J_2(y_1)\cdots\delta J_2(y_m)\delta\eta^{\mu_1}(z_1)\cdots\delta\eta^{\mu_l}(z_l)}\bigg|_{J_1=J_2=\eta_\mu=0}$$
$$= (i)^{n+m-l}\langle(\psi(x_1)\cdots\psi(x_n)\chi(y_1)\cdots\chi(y_m)A_{\mu_1}(z_1)\cdots A_{\mu_l}(z_l))_+\rangle_0^{\text{connected}},$$
$$\text{except for } n=1, \quad m=l=0. \tag{3.16}$$

An important feature of Eq. (3.16) is that this formula gives the Green's functions of the field ψ whose vacuum expectation value is zero, rather than those of the field ϕ_1.[20]

B. Perturbation Theory

Perturbation theory of the Higgs phenomenon is based on the formula (3.15). We divide the gauge-dependent Lagrangian

$$\mathcal{L}_\alpha = \mathcal{L}[A_\mu, \psi, \chi] - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 \qquad (3.17)$$

into two parts, $\mathcal{L}_{\alpha 0}$ and $\mathcal{L}_I$:

$$\mathcal{L}_\alpha = \mathcal{L}_{\alpha 0} + \mathcal{L}_I, \qquad (3.18)$$

$$\mathcal{L}_{\alpha 0} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + \tfrac{1}{2}(ev)^2 A_\mu{}^2 + \tfrac{1}{2}(\partial_\mu \psi)^2 - \tfrac{1}{2}(2\lambda v^2)\psi^2 + \tfrac{1}{2}(\partial_\mu \chi)^2 - evA_\mu \partial^\mu \chi, \qquad (3.19)$$

$$\mathcal{L}_I = eA_\mu(\chi\partial^\mu\psi - \psi\partial^\mu\chi) + \tfrac{1}{2}e^2 A_\mu{}^2(\psi^2+\chi^2) + e^2 v A_\mu{}^2\psi + \lambda v\psi(\psi^2+\chi^2) - \tfrac{1}{4}\lambda(\psi^2+\chi^2)^2 - \delta m^2(\psi^2+\chi^2) - v\delta m^2 \psi, \qquad (3.20)$$

where $\delta m^2 = \lambda^2 v^2 + \mu^2$, and treat $\mathcal{L}_{\alpha 0}$ and $\mathcal{L}_I$, respectively, as the unperturbed and perturbing Lagrangian.

The terms proportional to δm^2 in Eq. (3.20) are to be treated as counterterms. Since μ^2 has been eliminated from the Lagrangian, v may be treated as a free parameter. Equation (3.14), the condition that the field ψ have no vacuum expectation value, then serves to fix the value of δm^2: If we denote by $ivF(v^2)$ the sum of all loop diagrams for the ψ-to-vacuum transition, Eq. (3.14) requires[20]

$$v[F(v^2) - \delta m^2] = 0$$

which fixes the size of δm^2. A successful renormalization program requires that the divergences in the ψ and χ self-energies be removed by the same δm^2.

We may expand the T matrix and various Green's functions in powers of e and $\lambda^{1/2}$, regarding ev and $\lambda^{1/2}v$ as fixed numbers. This is equivalent to ordering the perturbation series by the number of loops each term in the series contains.

The various bare propagators may be evaluated from Eq. (3.16) by keeping only $\mathcal{L}_{\alpha 0}$ in $\mathcal{L}_\alpha$. The results are

$$\int d^4x\, e^{ik\cdot x}\langle (A_\mu(x)A_\nu(0))_+\rangle_0^{\text{bare}} = -i(g_{\mu\nu} - k_\mu k_\nu/k^2)[k^2 - (ev)^2]^{-1} - i\alpha k_\mu k_\nu/(k^2)^2, \qquad (3.21)$$

$$\int d^4x\, e^{ik\cdot x}\langle (\psi(x)\psi(0))_+\rangle_0^{\text{bare}} = i(k^2 - 2\lambda v^2)^{-1}, \qquad (3.22)$$

$$\int d^4x\, e^{ik\cdot x}\langle (\chi(x)\chi(0))_+\rangle_0^{\text{bare}} = i(k^2)^{-1} - i\alpha(ev)(k^2)^{-2}, \qquad (3.23)$$

$$\int d^4x\, e^{ik\cdot x}\langle (A_\mu(x)\chi(0))_+\rangle_0^{\text{bare}} = \alpha(ev)k_\mu/k^2. \qquad (3.24)$$

The most convenient gauge is obtained by setting $\alpha = 0$ in which case the decoupling of A_μ and χ occurs and the $(k^2)^{-2}$ terms in the propagators disappear. [In the T matrix the $(k^2)^{-2}$ terms should cancel regardless of the value of α.[22] The actual Feynman-diagram calculation is greatly simplified by setting $\alpha = 0$.] The vertices implied by Eq. (3.20) are listed in Fig. 1.

A power-counting argument shows that irreducible two-point vertices (such as self-energies) are quadratically divergent; all irreducible three- and four-point vertices are logarithmically divergent, except the $A_\mu\chi\psi$ vertex which is linearly divergent. Thus, the theory is renormalizable, meaning that the introduction of a finite number of subtractions makes the theory finite according to the BPH the-

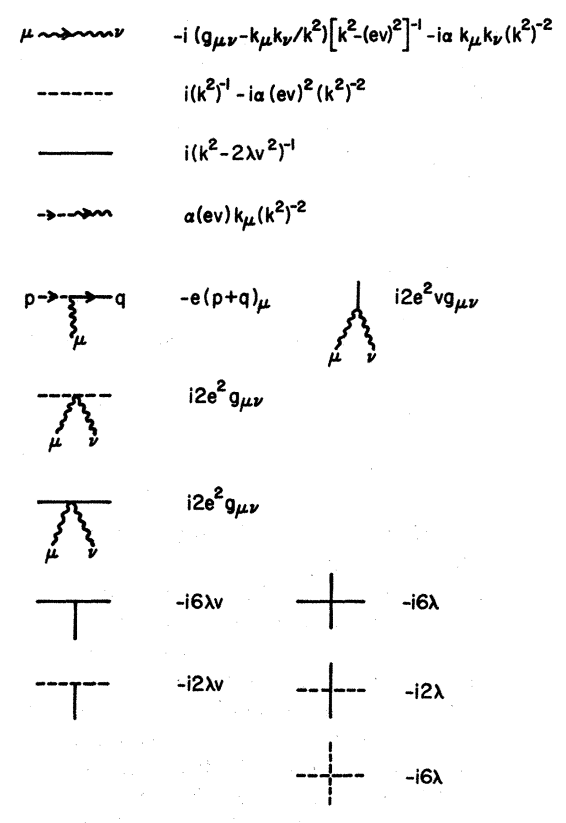

FIG. 1. The Feynman rules in the α gauge. The wavy, straight, and dashed lines stand for, respectively, the A_μ, ψ, and χ lines.

orem. Gauge invariance of the theory relates certain of the subtraction constants, and these relations assure us that the negative-metric scalar excitation implied by the vector-boson propagator and the χ-field excitation disappear from the S matrix.

C. Regularization

Divergent Feynman integrals can be regularized by the Pauli-Villars[17] technique in a gauge-invariant way. That such a procedure is possible can be seen best if we write down a Lagrangian which includes regulator fields[23]:

$$\mathcal{L}_\alpha = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + \sum_{i=1} \eta_i \left[-\tfrac{1}{4}(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu)^2 - \frac{1}{2\alpha}(\partial^\mu A^i_\mu)^2 + \tfrac{1}{2} m_i{}^2 (A^i_\mu)^2 \right]$$
$$+ \sum_{j=0} \xi_j \left[\left(\partial_\mu + ie \sum_{i=0} A^i_\mu \right) \phi_j^* \left(\partial^\mu - ie \sum_{i=0} A^{i\mu} \right) \phi_j - \mu_j{}^2 (\phi_j^* \phi_j) \right] - \tfrac{1}{4}\lambda \left| \sum_{j=0} \phi_j \right|^4$$
$$+ \sum_{k=0} \xi'_k \left[\left(\partial_\mu + ie \sum_{i=0} A^i_\mu \right) f_k^* \left(\partial^\mu - ie \sum_{i=0} A^{i\mu} \right) f_k - \mu'^2_k (f_k^* f_k) \right], \tag{3.25}$$

where $A^0_\mu = A_\mu$, $\phi_0 = \phi$, and η_i, ξ_j, and ξ'_k are the signature factors ± 1. The fields f_k and f_k^* are to be quantized according to the Fermi-Dirac statistics.[24] Equation (3.25) is invariant under the gauge transformation

$$A^i_\mu \to A^i_\mu + \delta^{i0} \frac{1}{e} \partial_\mu \theta ,$$
$$\phi_j \to e^{i\theta} \phi_j ,$$
$$f_k \to e^{i\theta} f_k .$$

If we choose the η_i and $m_i{}^2$ such that[25]

$$1 + \sum_{i=1} \eta_i = 0$$

and

$$\sum_{i=1} \eta_i m_i{}^2 = 0 ,$$

the A_μ propagator is regularized sufficiently to behave like k^{-6} for large k. If we further choose ξ_j and $\mu_j{}^2$ so that

$$1 + \sum_{j=1} \xi_j = 0 ,$$
$$\mu^2 + \sum_{j=1} \xi_j \mu_j{}^2 = 0 ,$$

all loop integrations are made finite except those loops with two or four external A_μ lines and no other external lines. The latter are made finite by the intervention of the fermion loops of the f_k, which have the relative negative signature.

With this regularization, the Feynman amplitude becomes finite, and because of the gauge invariance enjoyed by the regularized Lagrangian (4.1), the Ward-Takahashi (WT) identities hold for the regularized Green's functions. The subtracted amplitudes (i.e., the amplitudes one obtains by the R operation of Bogoliubov and Shirkov[18] and Hepp[19]) are then finite (i.e., independent of $m_i{}^2$, $\mu_j{}^2$, and μ'^2_k as they go to infinity) and satisfy the WT identities. Thus if we choose the subtraction constants in accordance with the WT identities, then the full amplitudes satisfy the WT identities.

IV. WARD-TAKAHASHI IDENTITIES

A. WT Identities for Green's Functions

All Ward-Takahashi identities can be studied globally if one considers the response of the generating functional Z_α to the gauge transformation (2.9). We perform the gauge transformation (2.9) on the variables of integration. Due to the invariance of the action this transformation will change only the source terms and the gauge term:

$$\delta\left(-\frac{1}{2\alpha}(\partial^\mu A_\mu)^2 + J_1\phi_1 + J_2\phi_2 - \eta^\mu A_\mu \right)$$
$$= \frac{1}{e}\left(-\frac{1}{2\alpha}\partial^2 \partial_\mu A^\mu - e(J_1\phi_2 - J_2\phi_1) + \partial^\mu \eta_\mu \right).$$

Since a transformation of integration variables does not change the value of an integral, we may put the variation of Z_α with respect to θ equal to zero. In this way we obtain[26]

$$\frac{1}{\alpha}\partial^2 \partial_\mu \frac{\delta Z_\alpha}{\delta \eta_\mu(x)} - e\left(J_1(x) \frac{\delta Z_\alpha}{\delta J_2(x)} - J_2(x) \frac{\delta Z_\alpha}{\delta J_1(x)} \right) + \partial^\mu \eta_\mu(x) = 0 . \tag{4.1}$$

Equation (4.1) summarizes all the WT identities which connect Green's functions of Eq. (3.16).

For example, if we differentiate Eq. (4.1) with respect to $\eta^\nu(y)$ and $J_2(y)$, respectively, and then let $J_1 = J_2 = \eta_\lambda = 0$, we obtain

$$\frac{1}{\alpha}\partial^\mu \partial^2 \frac{\delta^2 Z_\alpha}{\delta \eta^\mu(x) \delta \eta^\nu(y)}\bigg|_0 = -\partial^\nu \delta^4(x-y) , \tag{4.2}$$

$$\frac{1}{\alpha}\partial^\mu\partial^2\frac{\delta^2 Z_\alpha}{\delta\eta_\mu(x)\delta J_2(y)}\bigg|_0 = -e\frac{\delta Z_\alpha}{\delta J_1(x)}\bigg|_0 \delta^4(x-y) = -ev\delta^4(x-y)\,, \tag{4.3}$$

where the notation $|_0$ means the limit in which the external sources are turned off. Equations (4.2) and (4.3) specify uniquely the α-dependent parts of the propagators:

$$\int d^4x\, e^{ik\cdot x}\langle (A_\mu(x)A_\nu(0))_+\rangle_0 = -i\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)F(k^2) - i\alpha k_\mu k_\nu (k^2)^{-2}\,, \tag{4.4}$$

$$\int d^4x\, e^{ik\cdot x}\langle (A_\mu(x)\chi(0))_+\rangle_0 = \frac{\alpha(ev)k_\mu}{(k^2)^2}\,. \tag{4.5}$$

Thus, in the limit $\alpha\to 0$, the A_μ propagator is purely transverse and the fields A_μ and χ decouple. [Note also that Eq. (4.1) holds whether or not ϕ_1 has a nonvanishing vacuum expectation value. The difference arises solely from the value of $\delta Z_\alpha/\delta J_1(x)|_0$.]

Actually what we need for the determination of subtraction for primitively divergent vertices are the WT identities for (single-particle) irreducible vertices. We shall derive them by first constructing the generating functional for irreducible vertices, and then by deriving an equation satisfied by it.

B. Generating Functional for Irreducible Vertices

Let us define the "c fields" Φ_1, Φ_2, and $\mathfrak{A}_\mu$ by

$$\frac{\delta Z_\alpha}{\delta J_1(x)} = \Phi_1(x)\,,\quad \frac{\delta Z_\alpha}{\delta J_2(x)} = \Phi_2(x)\,,\quad -\frac{\delta Z_\alpha}{\delta\eta^\mu(x)} = \mathfrak{A}_\mu(x)\,. \tag{4.6}$$

We have $\Phi_1(x)|_0 = v$, $\Phi_2(x)|_0 = 0$, $\mathfrak{A}_\mu(x)|_0 = 0$.

After Schwinger[27] and Jona-Lasinio,[28] we define the functional Legendre transform of Z_α:

$$W_\alpha[\mathfrak{A}_\mu,\Phi_1,\Phi_2] = Z_\alpha[\eta_\mu,J_1,J_2] - \int d^4x[J_1(x)\Phi_1(x)+J_2(x)\Phi_2(x) - \eta^\mu(x)\mathfrak{A}_\mu(x)]\,. \tag{4.7}$$

We have the "Maxwell's equations" dual to Eq. (4.6):

$$\frac{\delta W_\alpha}{\delta\Phi_1(x)} = -J_1(x)\,,\quad \frac{\delta W_\alpha}{\delta\Phi_2(x)} = -J_2(x)\,,\quad \frac{\delta W_\alpha}{\delta\mathfrak{A}_\mu(x)} = +\eta^\mu(x)\,. \tag{4.8}$$

Also, we have the conditions

$$J_1(x)|_v = 0\,,\quad J_2(x)|_v = 0\,,\quad \mathfrak{A}_\mu(x)|_v = 0\,,$$

where the notation $|_v$ means the limit $\phi_1 = v$, $\phi_2 = \mathfrak{A}_\mu = 0$.

From the first of Eqs. (4.8) we obtain

$$\int d^4z\,\frac{\delta^2 W_\alpha}{\delta\Phi_1(x)\delta\Phi_1(z)}\,\frac{\delta\Phi_1(z)}{\delta J_1(y)} = -\delta^4(x-y)$$

or

$$\int d^4z\,\frac{\delta^2 W_\alpha}{\delta\Phi_1(x)\delta\Phi_1(z)}\bigg|_v \frac{\delta^2 Z_\alpha}{\delta J_1(z)\delta J_1(y)}\bigg|_0 = -\delta^4(x-y)\,. \tag{4.9}$$

Since

$$\frac{\delta^2 Z_\alpha}{\delta J_1(x)\delta J_1(y)}\bigg|_0 = i\langle(\psi(x)\psi(0))_+\rangle_0 = -\tilde{\Delta}_\psi(x-y)\,,$$

we see that $\delta^2 W_\alpha/\delta\Phi_1(x)\delta\Phi_2(y)$ is the inverse of the propagator for the ψ field:

$$\frac{\delta^2 W_\alpha}{\delta\Phi_1(x)\delta\Phi_1(y)}\bigg|_v = \tilde{\Delta}_\psi^{-1}(x-y)\,,$$

$$\int d^4z\,\tilde{\Delta}_\psi^{-1}(x-z)\tilde{\Delta}_\psi(z-y) = \delta^4(x-y)\,. \tag{4.10}$$

Likewise, from the last two of Eqs. (4.8), we obtain

$$\int d^4z \begin{pmatrix} \dfrac{\delta^2 W_\alpha}{\delta\Phi_2(x)\delta\Phi_2(z)}\bigg|_v & \dfrac{\delta^2 W_\alpha}{\delta\Phi_2(x)\delta\mathfrak{A}_\lambda(z)}\bigg|_v \\ \dfrac{\delta^2 W_\alpha}{\delta\mathfrak{A}_\mu(x)\delta\Phi_2(z)}\bigg|_v & \dfrac{\delta^2 W_\alpha}{\delta\mathfrak{A}_\mu(x)\delta\mathfrak{A}_\lambda(z)}\bigg|_v \end{pmatrix} \begin{pmatrix} 1 & \\ & -g_{\lambda\rho} \end{pmatrix} \begin{pmatrix} \dfrac{\delta^2 Z_\alpha}{\delta J_2(z)\delta J_2(y)}\bigg|_0 & \dfrac{\delta^2 Z_\alpha}{\delta J_2(z)\delta\eta_\nu(y)}\bigg|_0 \\ \dfrac{\delta^2 Z_\alpha}{\delta\eta_\rho(z)\delta J_2(y)}\bigg|_0 & \dfrac{\delta^2 Z_\alpha}{\delta\eta_\rho(z)\delta\eta_\nu(y)}\bigg|_0 \end{pmatrix} = \begin{pmatrix} -1 & \\ & g^{\mu\nu} \end{pmatrix}\,. \tag{4.11}$$

Equation (4.11) is somewhat simpler in the momentum space. Define

$$-\frac{\delta^2 Z_\alpha}{\delta J_2(x)\delta J_2(y)}\bigg|_0 = \int\frac{d^4k}{(2\pi)^4}\,\Delta(k^2)e^{ik\cdot(x-y)}\,,$$

$$-\left.\frac{\delta^2 Z_\alpha}{\delta\eta^\mu(x)\delta J_2(y)}\right|_0 = \int\frac{d^4k}{(2\pi)^4}\,\Delta_\mu(k)e^{ik\cdot(x-y)}\,,\qquad(4.12)$$

$$-\left.\frac{\delta^2 Z_\alpha}{\delta\eta^\mu(x)\delta\eta^\nu(y)}\right|_0 = \int\frac{d^4k}{(2\pi)^4}\,\Delta_{\mu\nu}(k)e^{ik\cdot(x-y)}$$

and

$$\left.\frac{\delta^2 W_\alpha}{\delta\Phi_2(x)\delta\Phi_2(y)}\right|_v = \int\frac{d^4k}{(2\pi)^4}\,\Gamma(k^2)e^{ik\cdot(x-y)}\,,$$

$$\left.\frac{\delta^2 W_\alpha}{\delta\mathcal{A}^\mu(x)\delta\Phi_2(y)}\right|_v = \int\frac{d^4k}{(2\pi)^4}\,\Gamma_\mu(k)e^{ik\cdot(x-y)}\,,\qquad(4.13)$$

$$\left.\frac{\delta^2 W_\alpha}{\delta\mathcal{A}^\mu(x)\delta\mathcal{A}^\nu(y)}\right|_v = \int\frac{d^4k}{(2\pi)^4}\,\Gamma_{\mu\nu}(k)e^{ik\cdot(x-y)}\,.$$

Then, Eq. (4.11) gives

$$\begin{pmatrix}\Gamma(k^2) & \Gamma_\lambda(-k)\\ \Gamma_\mu(k) & \Gamma_{\mu\lambda}(k)\end{pmatrix}\begin{pmatrix}1 & \\ & -g^{\lambda\rho}\end{pmatrix}\begin{pmatrix}\Delta(k^2) & \Delta_\nu(-k)\\ \Delta_\rho(k) & \Delta_{\rho\nu}(k)\end{pmatrix}=\begin{pmatrix}1 & \\ & -g_{\mu\nu}\end{pmatrix}.\qquad(4.14)$$

The utility of W_α lies in that it is the generating functional of irreducible vertices. Thus, if we define

$$\tilde\Gamma_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n;y_1,\ldots,y_m;z_1,\ldots,z_l)=\left.\frac{\delta^{n+m+l}Z_\alpha}{\delta\mathcal{A}^{\mu_1}(x_1)\cdots\delta\mathcal{A}^{\mu_n}(x_n)\delta\Phi_1(y_1)\cdots\delta\Phi_1(y_m)\delta\Phi_2(z_1)\cdots\delta\Phi_2(z_l)}\right|_v,\qquad(4.15)$$

$$\prod_{i=1}^n\int d^4x_ie^{ik_i\cdot x_i}\prod_{j=1}^m\int d^4y_je^{ip_j\cdot y_j}\prod_{h=1}^l\int d^4z_he^{iq_h\cdot z_h}\tilde\Gamma_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n;y_1,\ldots,y_m;z_1,\ldots,z_l)$$
$$=(2\pi)^4\delta^4(\textstyle\sum k+\sum p+\sum q)\Gamma_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n;p_1,\ldots,p_m;q_1,\ldots,q_l)\,,\qquad(4.16)$$

$\Gamma_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n;p_1,\ldots,p_m;q_1,\ldots,q_l)$ is the single-particle irreducible vertex for n A_μ's, m ψ's, and l χ's. The proof is rather simple, but clumsy in notation, and I refer the reader to Jona-Lasinio's elegant paper.[28]

C. WT Identities for Irreducible Vertices

The statement of gauge invariance, Eq. (4.1), may be translated into an equation for W_α. Making use of Eqs. (4.6) and (4.8), we write

$$-\frac{1}{\alpha}\partial^2\partial_\mu\mathcal{A}^\mu(x)-e\left(\Phi_1(x)\frac{\delta W_\alpha}{\delta\Phi_2(x)}-\Phi_2(x)\frac{\delta W_\alpha}{\delta\Phi_1(x)}\right)+\partial^\mu[\delta W_\alpha/\delta\mathcal{A}^\mu(x)]=0\,.\qquad(4.17)$$

Equation (4.17) stands for an infinite number of WT identities relating irreducible vertices of Eqs. (4.10), (4.13), and (4.15). For example, by differentiating Eq. (4.17) with respect to $\mathcal{A}^\nu(y)$ and $\Phi_2(y)$, respectively, and setting $\mathcal{A}_\nu=\Phi_2=0$, $\Phi_1=v$, we obtain

$$-\frac{1}{\alpha}\partial^\mu\partial^2\delta^4(x-y)+\partial_\nu\frac{\delta^2W_\alpha}{\delta\mathcal{A}_\nu(x)\delta\mathcal{A}_\mu(y)}-ev\frac{\delta^2W_\alpha}{\delta\Phi_2(x)\delta\mathcal{A}_\mu(y)}=0\,,\qquad(4.18)$$

$$-ev\frac{\delta^2W_\alpha}{\delta\Phi_2(x)\delta\Phi_2(y)}+\partial^\mu\frac{\delta^2W_\alpha}{\delta\mathcal{A}^\mu(x)\delta\Phi_2(y)}=0\,.\qquad(4.19)$$

Equation (4.17) can be cast in a more useful form. Define W by

$$W_\alpha=W-\frac{1}{2\alpha}\int d^4x[\partial^\mu\mathcal{A}_\mu(x)]^2\,.\qquad(4.20)$$

Then W satisfies the equation

$$\partial^\mu\left(\frac{\delta W_\alpha}{\delta\mathcal{A}^\mu(x)}\right)-e\left(\Phi_1(x)\frac{\delta W_\alpha}{\delta\Phi_2(x)}-\Phi_2(x)\frac{\delta W_\alpha}{\delta\Phi_1(x)}\right)=0\qquad(4.21)$$

which also shows the fact that W has no explicit dependence on α. Equation (4.21) is the statement that W is invariant under the local gauge transformation of the c fields:

$$\begin{aligned}&\mathcal{A}_\mu(x)\to\mathcal{A}_\mu(x)+\frac{1}{e}\partial_\mu\theta(x)\,,\\&\Phi_1(x)\to\Phi_1(x)\cos\theta-\Phi_2(x)\sin\theta\,,\\&\Phi_2(x)\to\Phi_2(x)\cos\theta+\Phi_1(x)\sin\theta\,,\end{aligned}\qquad(4.22)$$

so it can be written compactly as

$$\frac{\delta}{\delta\theta(x)} W[\mathfrak{A}_\mu, \Phi_1, \Phi_2] = 0 . \tag{4.23}$$

Thus, the totality of the WT identities is seen to imply the structure of W_α expressed in Eq. (4.20) wherein W is a functional of the c fields invariant under the local gauge transformation (4.22). Furthermore, this is all that the totality of the WT identities implies.

V. LOW-ENERGY THEOREMS AND RENORMALIZATION

Construction of a renormalized perturbation expansion according to the BPH program requires prescribing values of primitively divergent vertices (for which the overall superficial degrees of divergence are $\geqslant 0$) at subtraction points. Choosing as such the points of all external momenta equal to zero, we require the following 17 constants:

$$\begin{aligned}
&\Gamma_{\mu\nu}(0,0;\;;\;) \equiv g_{\mu\nu}\Gamma_{200} ,\\
&\frac{d}{dk^2}\Gamma^\mu_\mu(k,-k;\;;\;)\bigg|_{k=0} \equiv -3\Gamma_{200}' - \frac{1}{\alpha} ,\\
&\frac{d}{dk^2}k^\mu\Gamma_\mu(k;\;;-k)\bigg|_{k=0} \equiv i\Gamma_{101} ,\\
&\Gamma(\;;0,0;\;) \equiv \Gamma_{020} ,\\
&\frac{d}{dk^2}\Gamma(\;;k,-k;\;) \equiv \Gamma_{020}' ,\\
&\Gamma(\;;\;;0,0) \equiv \Gamma_{002} ,\\
&\frac{d}{dk^2}\Gamma(\;;\;;k,-k) = \Gamma_{002}' ,\\
&\frac{d}{dp_\mu}\Gamma_\nu(-p;p;0)\bigg|_{p=0} = -i\Gamma_{111}g^\mu_\nu ,\\
&\frac{d}{dq_\mu}\Gamma_\nu(-q;0;q)\bigg|_{q=0} = -i\Gamma_{111}'g^\mu_\nu ,\\
&\Gamma_{\mu\nu}(0,0;0;\;) = g_{\mu\nu}\Gamma_{210} ,\\
&\Gamma_{\mu\nu}(0,0;0,0;\;) = g_{\mu\nu}\Gamma_{220} ,\\
&\Gamma_{\mu\nu}(0,0;\;;0,0) = g_{\mu\nu}\Gamma_{202} ,\\
&\Gamma(\;;0,0,0;\;) = \Gamma_{030} ,\\
&\Gamma(\;;0;0,0) = \Gamma_{012} ,\\
&\Gamma(\;;0,0,0,0;\;) = \Gamma_{040} ,\\
&\Gamma(\;;0,0;0,0) = \Gamma_{022} ,\\
&\Gamma(\;;\;;0,0,0,0) = \Gamma_{004} .
\end{aligned} \tag{5.1}$$

Before proceeding further, a remark is in order. The BPH procedure involves expanding a regularized Feynman integral in a Taylor series of external momenta p_j about $p_j = 0$. Because of the singularities of the propagators $\Delta(k^2)$, $\Delta_\mu(k)$, and $\Delta_{\mu\nu}(k)$ of Eq. (4.12) at $k^2 = 0$, such a power-series expansion is not always possible. To circumvent this difficulty, we shall replace the factors k^2 in the denominators by $k^2 - a^2 + i\epsilon$. At the very end of the calculation, but not before, the limit $a^2 \to 0$ should be taken. While Green's functions depend nonanalytically on a^2 (through, for example, the factor $\ln a^2$), the T-matrix elements are independent of a^2 in this limit as we shall show (see Sec. VI).

The 17 constants specified in Eq. (5.1) are not all independent. To explore the consequences of the WT identities (4.17) on these constants, we shall write down the most general expression for W which is gauge-invariant and which contains no more than two derivatives of the c fields. The generating functional W_α so constructed will give exactly the first two terms of the power-series expansion in external momenta of the irreducible vertices (4.15). We have

$$W_\alpha[\mathfrak{A}_\mu, \Phi_1, \Phi_2] = \int d^4x\{-\tfrac{1}{4}(\partial_\rho\mathfrak{A}_\sigma - \partial_\sigma\mathfrak{A}_\rho)^2 A(\Phi_1{}^2+\Phi_2{}^2) + \tfrac{1}{2}[(\partial_\mu\Phi_1 + e\mathfrak{A}_\mu\Phi_2)^2 + (\partial_\mu\Phi_2 - e\mathfrak{A}_\mu\Phi_1)^2]B(\Phi_1{}^2+\Phi_2{}^2)$$
$$+ \tfrac{1}{2}(\Phi_1\partial_\mu\Phi_1 + \Phi_2\partial_\mu\Phi_2)^2 C(\Phi_1{}^2+\Phi_2{}^2) + D(\Phi_1{}^2+\Phi_2{}^2)\}(x) + O(\partial^3) , \tag{5.2}$$

where A, B, C, and D are nonsingular functions of their arguments. The condition for the spontaneous breakdown of symmetry,

$$\frac{\delta W}{\delta\Phi_1(x)}\bigg|_v = 0 , \tag{5.3}$$

implies that

$$D'(v^2) = 0 . \tag{5.4}$$

Since irreducible vertices are the expansion coefficients of W_α about $\phi_1 = v$, $\phi_2 = \mathfrak{A}_\mu = 0$, it is more convenient to express W_α in terms of $\psi = \phi_1 - v$, $\chi = \Phi_2$, and $\mathfrak{A}_\mu$. [We are using the symbols which we previously used as variables of integration.] Expanding A, B, C, and D in the form

$$A(\Phi_1{}^2+\Phi_2{}^2)=A(\psi^2+2v\psi+\chi^2+v^2)$$
$$=\sum_{n=0}^{\infty}\alpha_n(\psi^2+2v\psi+\chi^2)^n\,,\quad \text{etc.},$$

we have

$$\begin{aligned}W_\alpha[\mathcal{a}_\mu,\psi,\chi]=\int d^4x\{&-\tfrac{1}{4}(\partial_\rho\mathcal{a}_\sigma-\partial_\sigma\mathcal{a}_\rho)^2\sum_{n=0}\alpha_n(\psi^2+2v\psi+\chi^2)^n-(1/2\alpha)(\partial^\mu\mathcal{a}_\mu)^2\\&+\tfrac{1}{2}[(\partial_\mu\psi+e\mathcal{a}_\mu\chi)^2+(\partial_\mu\chi-e\mathcal{a}_\mu\psi)^2+2ev\mathcal{a}_\mu(\partial^\mu\chi-e\mathcal{a}^\mu\psi)+e^2v^2\mathcal{a}_\mu{}^2]\sum_{n=0}\beta_n(\psi^2+2v\psi+\chi^2)^n\\&+\tfrac{1}{2}(v\partial_\mu\psi+\psi\partial_\mu\psi+\chi\partial_\mu\chi)\sum_{n=0}\gamma_n(\psi^2+2v\psi+\chi^2)^n+\sum_{n=1}\delta_n(\psi^2+\chi^2+2v\psi)^n\}+O(\partial^3)\,.\end{aligned}\tag{5.5}$$

Equation (5.5) embodies all the low-energy theorems of the theory which follow from gauge invariance.[29]

We can prescribe the values of the constants (5.1) in accordance with the WT identities. We may set

$$\begin{aligned}&\Gamma_{200}=(ev)^2,\quad \Gamma_{200}'=1\,,\\&\Gamma_{020}=-2\lambda v^2,\quad \Gamma_{002}'=1\,,\\&\Gamma_{111}=e\,.\end{aligned}\tag{5.6}$$

Equation (5.2) defines three fundamental parameters e, λ, and v, and is equivalent to the statement $\alpha_0=\beta_0=1$ and $\delta_2=\frac{1}{4}\lambda$. Equation (5.4) implies $\delta_1=0$, so we have

$$\Gamma_{002}=0\tag{5.7}$$

which is the Goldstone theorem. The fact $\alpha_0=1$ leads to

$$\Gamma_{101}=ev\,.\tag{5.8}$$

The remaining ten constants are expressible in terms of e, λ, and v. From Eq. (5.5) we find that

$$\begin{aligned}&\Gamma_{020}'=1+v^2\gamma_0\,,\\&\Gamma_{111}'=-e(1-4v^2\beta_1)\,,\\&\Gamma_{210}=-2e^2v(1-2v^2\beta_1)\,,\\&\Gamma_{220}=e^2(1+2v^2\beta_1+4v^4\beta_2)\,,\\&\Gamma_{202}=e^2(1+2v^2\beta_1)\,,\\&\Gamma_{030}=-6\lambda v+48v^3\delta_3\,,\\&\Gamma_{012}=-2\lambda v\,,\\&\Gamma_{040}=-6\lambda+(12\times24)v^2\delta_3\,,\\&\Gamma_{022}=-2\lambda+(12\times4)v^2\delta_3\,,\\&\Gamma_{004}=-6\lambda\,.\end{aligned}\tag{5.9}$$

Equation (5.5) also tells us that

$$\begin{aligned}&e^2\beta_1=\frac{1}{4\times4\,!}(3\Gamma_{204}+\Gamma_{240})\,,\\&e^2v^2\beta_2=\frac{1}{4\times4\,!}(\Gamma_{204}-\Gamma_{240})\,,\\&\gamma_0=\tfrac{1}{2}\left(\frac{d}{dp^2}\Gamma(\;;\;;p,-p,0,0)\right)_{p=0}\,,\\&\delta_3=\tfrac{1}{6}\Gamma(\;;\;;0,0,0,0,0,0)\,,\end{aligned}\tag{5.10}$$

where $\Gamma_{\mu\nu}(0,0;\;;0,0,0,0)=g_{\mu\nu}\Gamma_{204}$ and similarly for Γ_{204}. Since the right-hand sides of Eqs. (5.10) are not primitively divergent, β_1, β_2, γ_0, and δ_3 can be computed in terms of e, λ, and v.

A perturbative construction of the constants defined in Eq. (5.9) may proceed in the following way. If we expand an irreducible vertex with E external lines in powers of e, with ev and $\lambda^{1/2}/e$ fixed, then the n-loop terms are of order e^{E-2+2L}. Therefore we see from Eq. (5.10) that

$$\begin{aligned}&\beta_1=e^4(\beta_{11}+\beta_{12}e^2+\cdots)\,,\\&\beta_2=e^6(\beta_{21}+\beta_{22}e^2+\cdots)\,,\\&\gamma_0=e^4(\gamma_{01}+\gamma_{11}e^2+\cdots)\,,\\&\delta_3=e^6(\delta_{31}+\delta_{32}e^2+\cdots)\,,\end{aligned}$$

where the second subscript of the expansion coefficient refers to the number of loops of the diagrams to which it is associated. The coefficient δ_{3n}, for example, is computed from the n-loop approximation to Γ_{006}, which is primitively convergent; the construction of Γ_{006} for up to n loops requires the knowledge of primitive vertices of Eq. (5.1) up to at most $n-1$ loops.

Finally we see from Eqs. (4.18), (4.19), and (5.7) that Γ, Γ_λ, and $\Gamma_{\mu\nu}$ of Eq. (4.13) are constrained to the following forms:

$$\begin{aligned}&\Gamma(k^2)=k^2X(k^2)\,,\\&\Gamma_\lambda(k)=ievX(k^2)\,,\\&\Gamma_{\lambda\mu}(k)=-(k^2g_{\lambda\mu}-k_\lambda k_\mu)Y(k^2)\\&\qquad+g_{\mu\nu}(ev)^2X(k^2)-\frac{1}{\alpha}k_\mu k_\nu\,.\end{aligned}\tag{5.11}$$

From Eqs. (5.5) and (5.6), we learn that the low-energy limits of Γ, Γ_λ, $\Gamma_{\mu\nu}$ are not renormalized:

$$X(0)=Y(0)=1\,.\tag{5.12}$$

We see therefore that Δ, Δ_μ, and $\Delta_{\mu\nu}$ are given by

$$\Delta(k^2)=\frac{1}{k^2X(k^2)}-\alpha\frac{(ev)^2}{(k^2)^2},$$

$$\Delta_\mu(k)=-i\alpha\frac{(ev)}{(k^2)^2}k_\mu, \tag{5.13}$$

$$\Delta_{\mu\nu}(k)=-(g_{\mu\nu}-k_\mu k_\nu/k^2)[k^2Y(k^2)-(ev)^2X(k^2)]^{-1}-\alpha k_\mu k_\nu/(k^2)^2.$$

The α-dependent parts of Δ, Δ_μ, and $\Delta_{\mu\nu}$ are not renormalized to all orders [see also Eqs. (4.4) and (4.5)], and $F(0)$ of Eq. (4.4) is given by

$$F(0)=-(ev)^{-2}. \tag{5.14}$$

VI. UNITARITY AND GAUGE INDEPENDENCE OF THE S MATRIX

By the regularization procedure of Sec. III C and the renormalization of Sec. V, finite irreducible vertices can be constructed which satisfy the WT identities (4.17). Green's functions, Eq. (3.16), can then be constructed from irreducible vertices by the rules of tree diagrams.[30] The Green's functions so constructed, of course, satisfy their own WT identities (4.1). We wish now to demonstrate, by means of Eq. (4.1), that the T-matrix elements between two physical states of the A_μ and ψ quanta are independent of the gauge α, and that the T matrix does not have a pole at $k^2=0$, the residues of the poles of the χ propagator and the longitudinal part of the vector propagator (of negative metric) canceling.

Since the ψ propagator has a branch point at $k^2=4a^2\to 0$, it is in practice inconvenient to locate the mass of the ψ particle from its propagator. The vector propagator $\Delta_{\mu\nu}$ does not suffer from the infrared divergence. From Eq. (5.13), we see that the mass of the A_μ particle, M^2, is given by

$$Y(M^2)M^2-X(M^2)(ev)^2=0. \tag{6.1}$$

We suggest that the most convenient way of getting at the ψ mass and the physical T matrix between states consisting of an arbitrary number of A_μ and ψ quanta is to consider first the T matrix for the process

$$A_\mu(k_1)+A_\nu(k_2)\to(n-2)\,A_\lambda\text{'s}, \tag{6.2}$$

and look for the pole in the variables $s=(k_1+k_2)^2$. The location of the pole, $s=m^2$, gives the mass of the ψ particle, and the residue gives the product of the $AA\psi$ coupling constant and the T matrix for the process

$$\psi(k_1+k_2)\to(n-2)\,A_\lambda\text{'s}.$$

By repeating this procedure, we can obtain all elements of the T matrix from those for the processes (6.2). We shall carry out our program by showing the α independence and the lack of the pole at $k^2=0$ of the Green's function $G_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)$:

$$G_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)(2\pi)^4\delta(\textstyle\sum k)=\prod_{i=1}^n\int d^4x_ie^{ik_i\cdot x_i}\left.\frac{\delta^nZ_\alpha}{\delta\eta^{\mu_1}(x_1)\cdots\delta\eta^{\mu_n}(x_n)}\right|_0. \tag{6.3}$$

We shall further show that $G_{\mu_1\cdots\mu_n}$ has no infrared divergence (i.e., is finite as $a^2\to 0$) and no branch point at any (subenergy)$^2=0$. Since the T matrix for the process (6.2) is obtained from the Green's function by the process of "amputation,"[31]

$$\lim_{k_1^2\to M^2}\left[\prod_i(k_i^2-M^2)\right]G_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n),$$

the above properties are transmitted to the entire T-matrix elements. In particular, the determination of the ψ mass does not suffer from the infrared difficulty. This might seem paradoxical. However, experiment with lower-order perturbation calculations suggest the following resolution: The ψ propagator has the form

$$\Delta_\psi(k^2)=F(k^2)G(k^2)F(k^2),$$

where $F(k^2)$ has a branch point at $k^2=0$ and no pole, and $G(k^2)$ has a pole at $k^2=m^2$ but no branch point at $k^2=0$. The factor $F(k^2)$ is canceled by the inverse factor from the vertex to which this propagator is attached.

From Eq. (4.1), we obtain

$$\frac{1}{\alpha}\partial^2\partial_{\mu_1}\left.\frac{\delta^nZ_\alpha}{\delta\eta_{\mu_1}(x_1)\cdots\delta\eta_{\mu_n}(x_n)}\right|_0=0. \tag{6.4}$$

If we write $G_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)$ as

$$G_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)=\Delta_{\mu_1}{}^{\nu}(k)H_{\nu;\mu_2\cdots\mu_n}(k;k_2,\ldots,k_n)+\Delta_{\mu_1}(k)J_{\mu_2\cdots\mu_n}(k;k_2,\ldots,k_n), \tag{6.5}$$

then Eq. (6.4) implies, by virtue of Eqs. (4.2) and (4.3),

$$ik^\mu H_{\mu;\lambda_1\cdots\lambda_n}(k;k_1,\ldots,k_n)+evJ_{\lambda_1\cdots\lambda_n}(k;k_1,\ldots,k_n)=0. \tag{6.6}$$

Let us consider a part of the G which is reducible with respect to an A_μ or χ propagator. Suppressing all inessential labels, such a term can be written as (see Fig. 2)

$$H_\mu^{(1)}(k;\cdots)\Delta^{\mu\nu}(k)H_\nu^{(2)}(-k;\cdots)+H_\mu^{(1)}(k;\cdots)\Delta^\mu(k)J^{(2)}(-k;\cdots)+J^{(1)}(k;\cdots)\Delta^\nu(-k)H_\nu^{(2)}(-k;\cdots)+J^{(1)}(k;\cdots)\Delta(k^2)J^{(2)}(-k;\cdots). \tag{6.7}$$

If we substitute Eqs. (5.13) in Eq. (6.7), the coefficient of α is

$$\frac{1}{(k^2)^2}[ik^\mu H_\mu^{(1)}(k;\cdots)+evJ^{(1)}(k;\cdots)]\times[-ik^\mu H_\mu^{(2)}(-k;\cdots)+evJ^{(2)}(-k;\cdots)],$$

which is identically zero due to Eq. (6.6). Now consider the residue of the pole at $k^2=0$ in Eq. (6.7). It is

$$\left[ik^\mu H_\mu^{(1)}(k;\cdots)F(k^2)(-ik^\nu)H_\nu^{(2)}(-k;\cdots)+J^{(1)}(k;\cdots)\frac{1}{X(k^2)}J^{(2)}(-k;\cdots)\right]_{k^2=0},$$

which is zero, due again to Eq. (6.6) and to Eqs. (5.12) and (5.14).

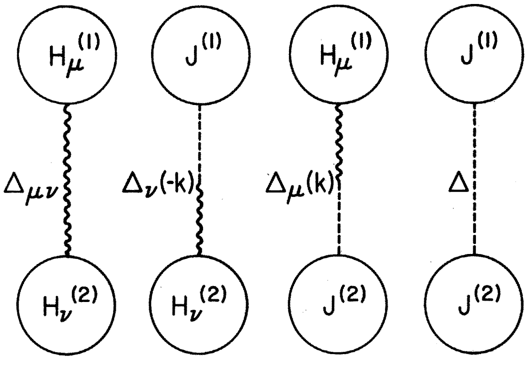

FIG. 2. Diagrammatic representation of Eq. (6.7).

Let us now consider the α independence in the general case. The generating functional $T_\alpha[A_\mu,\psi,\chi]$ of the Green's functions, from whose external lines the propagators of Eqs. (3.21)–(3.24) have been removed, may be written as

$$\exp(iT_\alpha[A_\mu,\psi,\chi])=\exp(i\alpha U[\delta/\delta A_\mu,\delta/\delta\chi])\times\exp\left[\tfrac{1}{2}i\int d^4x\,d^4y\left(\Delta_{\mu\nu}^{\rm tr}(x-y)\frac{\delta^2}{\delta A_\mu(x)\delta A_\nu(y)}+\Delta_F(x-y;2\lambda v^2)\frac{\delta^2}{\delta\psi(x)\delta\psi(y)}+D_F(x-y)\frac{\delta^2}{\delta\chi(x)\delta\chi(y)}\right)\right]\exp\left(i\int d^4z\,\mathcal{L}_I'(z)\right),\tag{6.8}$$

where $\Delta_F(x,\mu^2)$ and $D_F(x)$ are the usual Feynman propagators and

$$\Delta_{\mu\nu}^{\rm tr}(x)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2-(ev)^2+i\epsilon}.$$

The interaction Lagrangian $\mathcal{L}_I'$ is the sum of $\mathcal{L}_I$ in Eq. (3.20) and renormalization counterterms which ensure the renormalization conditions of Eq. (5.6). The operator U generates the gauge-dependent part of the off-shell T matrix; it is given by

$$U[\delta/\delta A_\mu,\delta/\delta\chi]=\tfrac{1}{2}\int d^4x\,d^4y\,d^4z\,D(x-z)D(z-y)V(x)V(y),\tag{6.9}$$

where V is the functional differential operator

$$V(x)=\partial_\mu\frac{\delta}{\delta A_\mu(x)}+ev\frac{\delta}{\delta\chi(x)}.\tag{6.10}$$

Equation (6.4) and its generalizations give

$$V(x_1)\cdots V(x_n)T_\alpha[A_\mu,\psi,\chi]|_{A_\mu=A_\mu^{\rm tr};\,\psi=\chi=0}=0,\tag{6.11}$$

where $A_\mu^{\rm tr}=(g_\mu^\nu-\partial_\mu\partial^\nu/\partial^2)A_\nu$ is the restriction of A_μ to the transverse components. We therefore have

$$\frac{d}{d\alpha}\exp(iT_\alpha[A_\mu,\psi,\chi])\Big|_{A_\mu=A_\mu^{\rm tr};\,\psi=\chi=0}=0,\tag{6.12}$$

from which follows the α independence of all on-shell physical T-matrix elements.

Let us now consider the unitarity in the general case. An absorptive part of a T-matrix element is given by the Landau-Cutkosky rule. We may write it abstractly as

$$\mathrm{Abs}\,T\sim\sum_n T_n^{(1)}\rho_n T_n^{(2)},\tag{6.13}$$

where n labels intermediate states, and ρ_n is the metric factor associated with the intermediate state n. Since we are concerned only with the cancellation of the contributions of the Goldstone bosons and the massless scalar excitations associated with the longitudinal part of the vector propagator, which goes as

$$\sim-\frac{k_\mu k_\nu}{k^2}\frac{1}{(ev)^2},\tag{6.14}$$

we may dispense with any reference to massive particles in the sum over intermediate states. Let us consider an intermediate state of N massless particles of both kinds. Let

$$T_{i_1\cdots i_N}(p_1,\ldots,p_N)\tag{6.15}$$

be the T-matrix element for N zero-mass bosons, where i denotes the kind of particles: $i=0$ is for the (positive metric) Goldstone boson, and $i=1$ for the (negative metric) scalar excitation associated with the expression (6.14). The metric ρ_n in Eq. (6.13) is given by

$$\rho_N = \exp\left(i\pi \sum_{j=1}^{N} i_j\right). \tag{6.16}$$

We may separate Eq. (6.4) into parts, which are either longitudinal or transverse with respect to each x_i, $i=2, \ldots, N$. In this way, we obtain altogether $2^N - 1$ equations, which are of the form

$$S T_{i_1 i_2 \cdots i_N} = 0, \tag{6.17}$$

where the operation S is defined as summing over the indices belonging to a nonempty subset of $\{i_1, \ldots, i_N\}$ and setting the rest of the indices equal to 1. In deriving Eq. (6.17) from Eq. (6.4) we used the fact that $p_i^2 = 0$. Equation (6.17) allows us to express the $2N$ components of Eq. (6.15) in terms of one function T. We can write

$$T_{i_1 \cdots i_N}(p_1, \ldots, p_N) = T \exp\left(i\pi \sum_{j=1}^{N} i_j\right).$$

Therefore we see that[32]

$$\sum_{i_1, \ldots, i_N} T^{(1)}_{i_1 \cdots i_N} \rho_N T^{(2)}_{i_1 \cdots i_N} = T^{(1)} T^{(2)} \sum_{i_1, \ldots, i_N} \exp\left(3i\pi \sum_{j=1}^{N} i_j\right) = 0,$$

which shows that an intermediate state with any number of massless particles of any kind does not contribute to the unitarity sum. We see further that the T-matrix element cannot have a branch point at any subenergy $=0$, and therefore there cannot be any singularity as $a^2 \to 0$.

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[1] J. Goldstone, Nuovo Cimento 19, 154 (1961).

[2] The first proof of the Goldstone theorem was given by J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962); S. Bludman and A. Klein, *ibid.* 131, 2363 (1963).

[3] An excellent survey in this subject is given in T. W. B. Kibble, in *Proceedings of the 1967 International Conference on Particles and Fields,* edited by C. R. Hagen *et al.* (Interscience, New York, 1967); G. Guralnik, C. R. Hagen, and T. W. B. Kibble, in *Advances in Particle Physics,* edited by R. L. Cool and R. E. Marshak (Interscience, New York, 1968), Vol. II.

[4] See, for example, Y. Nambu, Phys. Rev. 117, 648 (1960); P. W. Anderson, *ibid.* 130, 439 (1963); A. Klein and B. W. Lee, Phys. Rev. Letters 12, 266 (1964).

[5] P. Higgs, Phys. Letters 12, 132 (1964).

[6] G. Guralnik, C. R. Hagen, and T. W. B. Kibble, Phys. Rev. Letters 13, 585 (1964).

[7] P. Higgs, Phys. Rev. 145, 1156 (1966).

[8] T. W. B. Kibble, Phys. Rev. 155, 1554 (1967).

[9] S. Weinberg, Phys. Rev. Letters 19, 1264 (1967).

[10] G. 't Hooft, University of Utrecht report (unpublished).

[11] M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

[12] B. W. Lee, Nucl. Phys. B9, 649 (1969); J.-L. Gervais and B. W. Lee, *ibid.* B12, 627 (1969).

[13] K. Symanzik, Lett. Nuovo Cimento 2, 10 (1969); Commun. Math. Phys. 16, 48 (1970).

[14] R. P. Feynman, Rev. Mod. Phys. 20, 267 (1948).

[15] L. D. Faddeev and V. N. Popov, Phys. Letters 25B, 29 (1967); V. N. Popov and L. D. Faddeev, Kiev Report No. ITP 67-36, 1967 (unpublished): I regret that this excellent paper is not available to a wider audience.

[16] S. N. Gupta, Proc. Phys. Soc. (London) 63, 681 (1950); 64, 850 (1951); K. Bleuler, Helv. Phys. Acta 23, 567 (1950).

[17] W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949).

[18] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959).

[19] K. Hepp, Commun. Math. Phys. 1, 95 (1965); *Théorie de la Renormalisation* (Springer, Berlin, 1969).

[20] The σ model has been discussed in the spirit of this paper in the forthcoming book: B. W. Lee, *Chiral Dynamics* (Gordon and Breach, New York, to be published).

[21] The quantization of the σ model by the functional integration is discussed by A. Vassiliev, *Etudes des Symetries Brisées par des Méthodes Functionelles* (Gordon and Breach, New York, to be published).

[22] See Sec. VI.

[23] S. N. Gupta, Proc. Phys. Soc. (London) A66, 129 (1953).

[24] That is to say, the integration variables f_k and f_k^* are anticommuting c numbers.

[25] This paragraph summarizes the results of a rather lengthy but straightforward analysis.

[26] Similar expressions have been written down for electrodynamics and non-Abelian gauge theories by A. Slavonov, Kiev Report No. ITP 71-83E, 1971 (unpublished).

[27] J. Schwinger, Proc. Nat. Acad. Sci. (U.S.) 37, 452 (1951).

[28] G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964).

[29] Similar considerations for the case of chiral symmetry lead to the so-called chiral phenomenological Lagrangian. See Ref. 20 and B. Zumino in *Lectures on Elementary Particles and Quantum Field Theory,* 1970 Brandeis University Summer Institute in Theoretical Physics, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1971), Vol. I.

[30] This is obvious from the discussion of Ref. 28.

[31] Furthermore, a finite renormalization of the external wave functions is necessary, but this is trivial.

[32] For the proof of unitarity presented here, the author owes a lot to J. Zinn-Justin.