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Bound Geodesics in the Kerr Metric*

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The bound geodesics (orbits) of a particle in the Kerr metric are examined. (By "bound" we signify that the particle ranges over a finite interval of radius, neither being captured by the black hole nor escaping to infinity.) All orbits either remain in the equatorial plane or cross it repeatedly. A point where a nonequatorial orbit intersects the equatorial plane is called a node. The nodes of a spherical (i.e., constant radius) orbit are dragged in the sense of the spin of the black hole. A spherical orbit near the one-way membrane traces out a helix-like path lying on a sphere enclosing the black hole.

I. INTRODUCTION

Do effects of general relativity proper play a central role in such celestial phenomena as the pulsars and quasars? This question merits serious consideration if only for the reason that there is no (macroscopic) device as effective as the gravitational field of a collapsed star for the release of energy.[1]

At present, two sorts of energy-releasing processes are known. In one, a particle falling into a black hole emits electromagnetic or gravitational waves; the hole does not participate actively. In the other, a possible mechanism for which has been proposed by Penrose, the black hole itself provides a source of energy. A particle near the one-way membrane breaks up into two particles. One can arrange that one of the fragments will escape to infinity with an energy larger than that possessed by the original particle provided the other fragment is captured by the black hole. It may be that this mechanism, more contrived than the radiation process, will have less direct importance for astrophysics.

The Penrose process does not violate conservation of energy. As shown by Christodoulou,[2] the extra energy of the escaping particle is taken from the rotational energy of the black hole. By repeating the Penrose process many times one may deplete a black hole of its entire rotational energy, which can amount for a charged black hole to as much as 50% of its rest mass.[3]

The energy release possible through radiation processes is equally remarkable. Christodoulou and Ruffini[3] have shown that a charged particle falling into a charged black hole can, under appropriate conditions, emit its entire rest-mass energy.

Preliminary to investigating the problem of radiation, one must understand the kinematics of test particles for which the effect of radiation is neglected. Darwin[4,5] has already considered the geodesics of a particle in the spherically symmetric Schwarzschild field. The Kerr-Newman field, which describes a body having charge, mass, and spin angular momentum, is richer in structure than the Schwarzschild field. In this paper, we study the bound geodesics or "orbits" in the charge-free Kerr metric.[6] We have restricted ourselves to the charge-free case both because this case can be a very relevant one for astrophysics, and because we would like to distinguish the purely relativistic effects of the gravitational field from the electrodynamic ones.

In Sec. II, we deduce a necessary condition for binding to occur. Subsequently, we specialize to spherical orbits, that is, orbits of constant radius. In Newtonian mechanics, one can understand much about the general orbit by studying the circular-orbit case. Here, too, one believes that most features of interest are present in the case of spherical orbits.

In the limit of large radius, a spherical orbit goes asymptotically to a Keplerian circle. Considering a sequence of orbits of ever smaller radius, we find that as the radius decreases, the line of the (ascending or descending) node is increasingly "dragged" in the sense of the spin of the black hole. (The line of the ascending node is a line of constant azimuth in the equatorial plane passing through that point at which the orbit, going from negative to positive latitudes, intersects the equatorial plane.)

At a certain radius, a horizon occurs. This horizon, known as the one-way membrane, has the property that anything penetrating the surface from outside will be unable to escape again. The most rapidly rotating Kerr particle for which causality does not break down[7] has an angular momentum equal to its mass squared (in geometrical units). Stable orbits are possible down to its one-way membrane. As the orbital radius approaches that of the horizon, the dragging of the nodes increases without limit. During the time that a particle makes one oscillation in latitude, it will be swept through many complete azimuthal revolutions. Consequently, an orbit near the horizon will have a helix-like shape with axis parallel to the spin of the black hole.

II. CONDITIONS FOR BINDING

Carter[7] has given the first integrals of the equations of motion of a particle in the Kerr-Newman field. For the sake of a clearer physical interpretation, we use his Eq. (9) to transform to Boyer-Lindquist coordinates; in these coordinates, the metric is symmetric under simultaneous inversion of the axial and stationary Killing vectors. The metric is

$$ds^2 = \rho^2\Delta^{-1}dr^2 + \rho^2 d^2\theta + \rho^{-2}\sin^2\theta[a\,dt - (r^2+a^2)d\phi]^2 - \rho^{-2}\Delta(dt - a\sin^2\theta\, d\phi)^2,$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2mr + a^2 + e^2, \tag{1}$$

a, m, and e are, respectively, the specific angular momentum, mass, and charge of the black hole. Specializing to the charge-free case, the equations of motion take the form[8]

$$\rho^2\dot{r} = \pm\sqrt{R}, \tag{2a}$$

$$\rho^2\dot{\theta} = \pm\sqrt{\Theta}, \tag{2b}$$

$$\rho^2\dot{\phi} = (\Phi\sin^{-2}\theta - aE) + a\Delta^{-1}P, \tag{2c}$$

$$\rho^2\dot{t} = a(\Phi - aE\sin^2\theta) + (r^2+a^2)\Delta^{-1}P, \tag{2d}$$

with

$$\Theta = Q - \cos^2\theta[a^2(\mu^2 - E^2) + \Phi^2\sin^{-2}\theta],$$
$$P = E(r^2+a^2) - \Phi a, \tag{3}$$
$$R = P^2 - \Delta[\mu^2 r^2 + Q + (\Phi - aE)^2].$$

The dot denotes differentiation with respect to a parameter λ, defined in terms of the proper time by

$$\tau = \mu\lambda. \tag{4}$$

The signs in (2a) and (2b) can be chosen independently. The constant a denotes the angular momentum per unit mass of the central body. E, Φ, and Q are three constants of the particle's motion. E and Φ refer, respectively, to the energy and to the z component of angular momentum; Q is related to the θ velocity, $\dot{\theta}$.

We rewrite $R(r)$ in a form independent of the masses. Writing $R(r)$ out in detail,

$$R(r) = (E^2-\mu^2)r^4 + 2\mu^2 m r^3 + [a^2(E^2-\mu^2) - \Phi^2 - Q]r^2 + 2m[(aE-\Phi)^2 + Q]r - a^2 Q.$$

Dividing through by $\mu^2 m^4$,

$$R(r)/\mu^2 m^4 = (\hat{E}^2 - 1)\hat{r}^4 + 2\hat{r}^3 + [\hat{a}^2(\hat{E}^2-1) - \hat{\Phi}^2 - \hat{Q}]\hat{r}^2 + 2[(\hat{a}\hat{E} - \hat{\Phi})^2 + \hat{Q}]\hat{r} - \hat{a}^2\hat{Q}, \tag{5}$$

where

$$\hat{E} = E/\mu, \quad \hat{\Phi} = \Phi/m\mu, \quad Q = Q/m^2\mu^2, \quad \hat{r} = r/m, \quad \hat{a} = a/m. \tag{6}$$

We shall henceforth take

$$\mu = m = 1,$$

which is equivalent to using the caret variables. Since $\mu^2 = 1$ is greater than zero, the geodesics described by (1), (2), and (3) are timelike.

One can determine the values of E, Φ, and Q for which binding occurs by using the effective potentials.

We first discuss the effective radial potential $V(r)$. $V(r)$ is defined as that value of E such that $\dot{r}=0$ at radius r; by Eq. (2a) it follows that $R(r)=0$. Rewriting $R(r)$ in a more convenient form,

$$R(r) = [r^4 + a^2(r^2+2r)]E^2 - 4a\Phi rE + a^2\Phi^2 - (r^2 + Q + \Phi^2)(r^2 - 2r + a^2), \tag{7}$$

a quadratic in E. $R(r)$ has not one but two roots:

$$V_\pm(r, \Phi, Q) = \frac{4a\Phi r \pm \sqrt{D}}{2[r^4 + a^2(r^2+2r)]}. \tag{8}$$

The discriminant D depends on Φ only through Φ^2. In general, the reality of the radial velocity leads by (2a) to the requirement that $R(r)$ be non-negative. If D is negative, this requirement is fulfilled

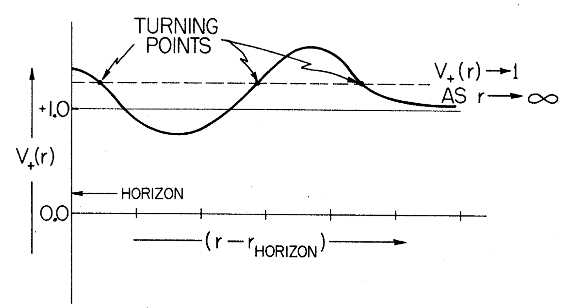

FIG. 1. Sketch of an effective radial potential which would bind a particle of energy greater than unity (dashed line). There must be at least three turning points. We show that such a potential is impossible by proving that there may be at most two turning points for $E > 1$.

for all values of the energy. If, however, D is non-negative, the energy must satisfy either of the following conditions:

$$E \geq V_+(r, \Phi, Q) \tag{9a}$$

or

$$E \leq V_-(r, \Phi, Q). \tag{9b}$$

Whether or not the effective potential is real, we may always make the one-to-one correspondence,

$$E, \Phi, Q \leftrightarrow -E, -\Phi, Q. \tag{10}$$

Inspection of the equations of motion, (2) and (3), shows that with proper choice of sign, the 4-velocities of two such corresponding motions will differ only in sign.

We now prove that if $E^2 \geq 1$, and if the specific angular momentum of the black hole lies in the causality-preserving range $0 \leq a \leq 1$, then the motion is unbound.

The proof depends on setting an upper limit on the number of turning points. Consider a conceivable $V_+(r)$ sketched in Fig. 1. Notice that by (7) and (8), $V_+(r)$ goes asymptotically to unity at large radii. As indicated by the figure a bound state with $E \geq 1$ is only possible if there is a range of energies for which three or more turning points occur. We will prove in fact, that there are at most two. $E \leq -1$ also satisfies $E^2 \geq 1$. By the correspondence (10), however, if our theorem is proven for $E \geq 1$, it will necessarily also hold for $E \leq -1$.

Let us reexpress $R(r)$ in terms of the new coordinate x, defined by

$$r = x + 1.$$

Substituting this into (5) yields

$$R(x) = (E^2 - 1)x^4 + (4E^2 - 2)x^3 + [(6 + a^2)E^2 - a^2 - \Phi^2 - Q]x^2 + [(4 + 2a^2)(E^2 - 1) + 6 + 2a^2E^2 - 4a\Phi E]x + [(2aE - \Phi)^2 + (E^2 + 1 + Q)(1 - a^2)]. \tag{11}$$

In the language of classical algebra, whenever a term in the polynomial is followed by one of the opposite sign, that is described as a "variation of sign."[9] A polynomial (with real coefficients) cannot have more positive roots than there are variations of sign.[10] If $E \geq 1$, the first two terms are non-negative. Write the last three terms as

$$c_2x^2,\ c_1x,\ c_0.$$

As many as three variations of sign can occur if

$$c_2, c_0 < 0, \quad c_1 > 0. \tag{12}$$

From (12),

$$c_1 - (c_0 + c_2) > 0,$$

or by (11),

$$Qa^2 > 3E^2 - 1 > 0. \tag{13}$$

If $a = 0$ (Schwarzschild), (13) is impossible. If $0 < a \leq 1$, (13) implies a positive Q, whence c_0 is non-negative; but c_0 non-negative contradicts (12). Since (12) cannot be satisfied, there may be at most two variations of sign and hence two positive roots. We have shown this for $r \geq 1$. It must be true *a fortiori* outside the horizon, since the latter occurs at

$$r = 1 + (1 - a^2)^{1/2} \geq 1.$$

When $E^2 < 1$, the first term is negative. Four variations would be possible only if the succeeding terms had the signs +, −, +, −. But if the second term is positive, then

$$E^2 > \tfrac{1}{2}.$$

Equation (13) continues to hold, showing that the last three terms cannot have the supposed signs. Hence, not more than three variations of sign are possible. One sees then that for given Q, Φ, and $|E| < 1$, there may be at most one region of binding.

By considering the motion in latitude, and using the preceding theorem, we will prove that Q must be ≥ 0 for binding.

By analogy with $V(r)$ the effective θ potential, $V^2(\theta)$, is defined as that value of E^2 which makes $\dot{\theta} = 0$ when the polar angle $= \theta$. By (2b) and (3),

$$0 = \Theta(\theta) = Q - \cos^2\theta\{a^2[1 - V^2(\theta)] + \Phi^2 \sin^{-2}\theta\}$$

or

$$V^2(\theta) = 1 + a^{-2}(\Phi^2 \sin^{-2}\theta - Q\cos^{-2}\theta). \tag{14}$$

The significance of $V^2(\theta)$ can be seen by writing

$$\Theta(\theta) = a^2\cos^2\theta[E^2 - V^2(\theta)]. \tag{15}$$

A particle at polar angle θ can only satisfy

$$E^2 < V^2(\theta) \tag{16}$$

if it is confined to the equatorial plane and has $Q = 0$. That Q is not negative follows from (14) according to which $Q > 0$ implies

$$E^2 > V^2(\pi/2) = -\infty.$$

For all cases other than $Q = 0$,

$$E^2 \geq V^2(\theta). \tag{17}$$

But (14) shows that if $Q<0$, then

$$V^2(\theta)>1, \tag{18}$$

By our earlier theorem, (17) and (18) together imply that the particle is unbound. Thus, for binding we require

$$Q\geqslant 0. \tag{19}$$

It follows from (2b), (3), and the theorem that an orbit has $Q=0$ if and only if it is confined to the equatorial plane.

Typical examples of $V^2(\theta)$ with $Q>0$ are shown in Fig. 2.[11] A particle can only reach the axis ($\theta=0, \pi$) if $\Phi=0$. With the help of the figure we draw the conclusion: Every orbit either remains in the equatorial plane ($Q=0$) or crosses it repeatedly ($Q>0$).

III. SPHERICAL ORBITS

From here on we will treat the most interesting case, the extreme Kerr case:

$$a=1.$$

In addition, we restrict ourselves to spherical orbits. The particle's radial coordinate will be stable at some value r_0 if $R(r)$ vanishes at $r=r_0$

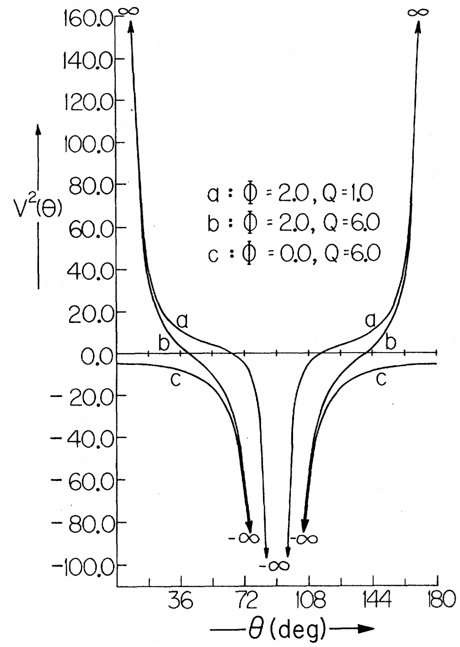

FIG. 2. Examples of the effective θ potential with $Q>0$. Only when $\Phi=0$ can particles of finite energy reach the axis ($\theta=0°$, 180°). In this and subsequent figures, the spin parameter, a, equals unity.

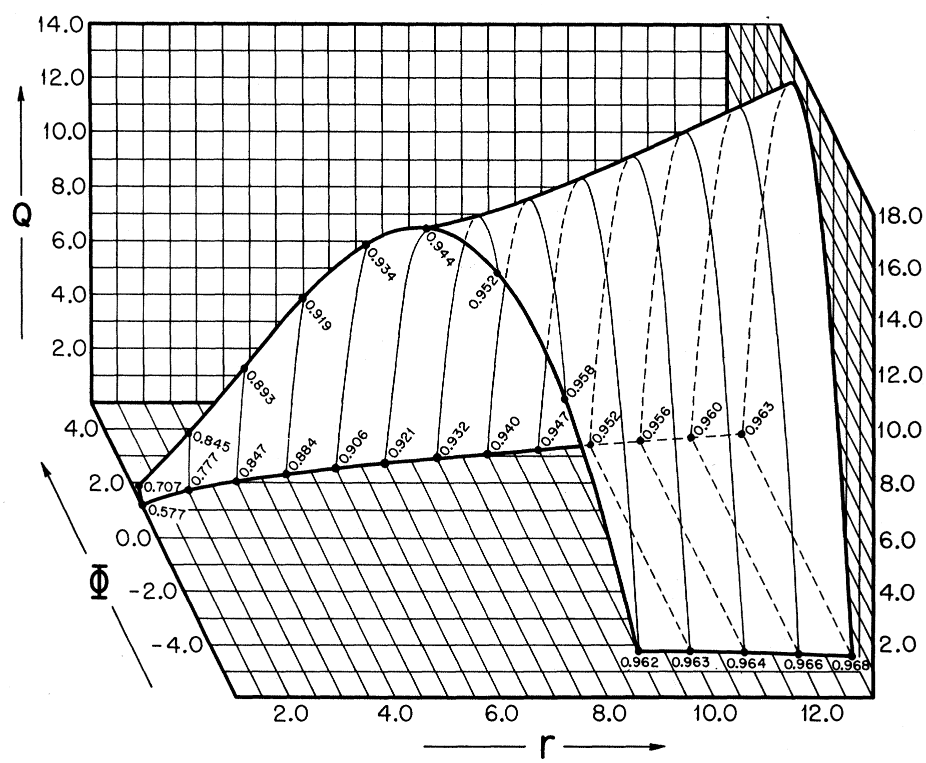

FIG. 3. Portion of the surface of stable spherical orbits. The energies of the orbits of least and greatest binding are given for integral values of the radius. For small radii, only orbits revolving in the sense of the black hole (those with $\Phi>0$) are stable.

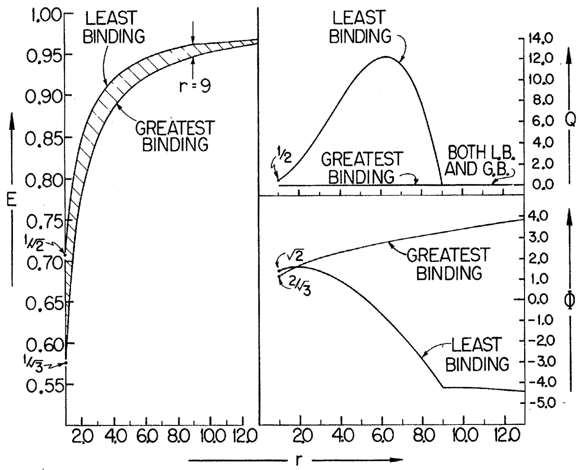

FIG. 4. Constants of motion of spherical orbits of least and greatest binding. As seen in the plot of Φ versus r and in Fig. 3, all orbits with radius $\lesssim 5.3$ are co-revolving ($\Phi > 0$). The discontinuity of slope at $r = 9$ for the orbits of least binding is not puzzling in view of the same discontinuity evident in Fig. 3.

and goes negative nearby. This will be the case if

$$R(r_0) = 0, \tag{20a}$$

$$\left.\frac{\partial R}{\partial r}\right|_{r=r_0} = 0, \tag{20b}$$

$$\left.\frac{\partial^2 R}{\partial r^2}\right|_{r=r_0} < 0. \tag{20c}$$

Replacing (20c) by

$$\left.\frac{\partial^2 R}{\partial r^2}\right|_{r=r_0} > 0 \tag{20c$'$}$$

yields instead the conditions for an unstable spherical orbit.

In the following we shall only be interested in the plus-root solutions, that is, those satisfying (9a). The behavior of the minus-root solutions is easily determined with the help of (10).

Solving Eqs. (20a) and (20b) simultaneously eliminates two of the four unknowns, E, Q, Φ, and r. Imposing the stability requirement (20c) and the boundedness requirement (19) determines a subset of this two-parameter set of trajectories.[12] In Fig. 3, the set of stable spherical orbits of small radius is represented as a two-dimensional surface in (r, Q, Φ) space. The intersections of the surface with the $Q = 0$ plane are the equatorial orbits. For $r \geq 9$, there are two such lines, standing for the co-revolving[13] ($\Phi > 0$) and counterrevolving ($\Phi < 0$) equatorial orbits.

For a fixed radius the energy varies monotonically along the surface. The co-revolving equatorial orbits have the largest binding energy, $1 - E$. For a given radius ≥ 9, the counterrevolving equatorial orbit has the smallest binding energy. For a radius < 9, such an orbit is unstable (and thus not shown in Fig. 3). Instead, the orbit situated at an inflection point of the effective radial potential, that is, with

$$\frac{\partial^2 R}{\partial r^2} = 0,$$

is the least tightly bound (for a given radius < 9); the set of these orbits constitutes the curved edge of the opening in the surface. (See Fig. 4.)

For large radii, the surface goes to the Schwarzschild limit:

$$Q(r) = \frac{r^2}{r-3} - \Phi^2 \geq 0, \qquad E^2(r) = 1 - \frac{r-4}{r(r-3)}. \tag{21}$$

Stable orbits occur all the way to the horizon, at $r = 1$. The one-parameter family of horizon-skimming orbits is

$$2/\sqrt{3} \leq \Phi \leq \sqrt{2}, \tag{22a}$$

$$Q = \tfrac{3}{4}\Phi^2 - 1, \tag{22b}$$

$$E = \tfrac{1}{2}\Phi. \tag{22c}$$

The lower limit on Φ results from the restriction to $Q \geq 0$. To understand the upper limit one must consider an orbit just outside the horizon. Setting $r = 1 + \lambda$, with $0 < \lambda \ll 1$, one can show that

$$\left.\frac{\partial^2 R}{\partial r^2}\right|_{r=1+\lambda} = 2(\Phi^2 - 2)\lambda + O(\lambda^2). \tag{23}$$

Applying (20c) yields the upper limit, $\Phi = \sqrt{2}$.

Equations (22b) and (23) do not of themselves rule out the alternative range

$$-\sqrt{2} \leq \Phi \leq -2/\sqrt{3}. \tag{22a$'$}$$

Consideration of $V_+(r)$, however, shows that (22a) describes a solution of (9a), and (22a$'$) a solution of (9b).

From (21), observe that in the Schwarzschild case, the squared angular momentum, $\Phi^2 + Q$, depends only on the radius. By contrast, it follows from Eqs. (22) that, for orbits at the horizon, $\Phi^2 + Q$ varies almost by a factor of 2 in the extreme Kerr case.

IV. DRAGGING OF THE NODES

Lense and Thirring[14] have shown that in the weak-field limit, the nodes of a circular orbit are dragged in the sense of the spin by an angle

$$\Delta\Omega \cong 2(a/m)(m/r)^{3/2} \tag{24}$$

per revolution. An exact expression for $\Delta\Omega$, correct for all distances, can be given in the Kerr case.

Assume, say, that θ is decreasing. Dividing (2c) by (2b) then gives

$$\frac{d\phi}{d\theta}=-\frac{\Phi\sin^{-2}\theta-E+P\Delta^{-1}}{[Q-\cos^2\theta(1-E^2+\Phi^2\sin^{-2}\theta)]^{1/2}}. \qquad (25)$$

Setting

$$z=\cos^2\theta \qquad (26)$$

and assuming $\theta\leqslant\frac{1}{2}\pi$, we integrate to obtain

$$\phi=\tfrac{1}{2}\Phi\int\frac{dz}{(1-z)Y(z)}+\frac{P\Delta^{-1}-E}{2}\int\frac{dz}{Y(z)}, \qquad (27)$$

where

$$Y(z)=[\beta z^3-(\alpha+\beta)z^2+Qz]^{1/2}, \qquad (28)$$

with $\alpha=\Phi^2+Q$ and $\beta=1-E^2$. Turning points in θ occur when the denominator of (25) vanishes or, equivalently, when

$$\beta z^2-(\alpha+\beta)z+Q=0. \qquad (29)$$

Since α, β, and Q are all non-negative and since $\alpha\geqslant Q$, the roots, $z_\pm$, of (29) are real and non-negative. The range of z for which the motion takes place includes the equatorial value, $z=0$:

$$0\leqslant z\leqslant z_-. \qquad (30)$$

Equation (30) corresponds to one-quarter of a complete oscillation in latitude. From (2c) it is clear that the azimuth changes by the same amount in each quarter oscillation; that is, one gets the same change whatever the signs of $\dot\theta$ and $\theta-\frac{1}{2}\pi$.

Using a standard table of integrals,[15] the change of azimuth during one-quarter oscillation of latitude is cast into a more intelligible form:

$$\Delta\phi=\frac{1}{(\beta z_+)^{1/2}}[\Phi\Pi(-z_-,k)+(P\Delta^{-1}-E)K(k)], \qquad (31)$$

where

$$k^2=z_-/z_+$$

and

$$K(k)=\int_0^{\pi/2}\frac{dx}{(1-k^2\sin^2x)^{1/2}},$$

$$\Pi(n,k)=\int_0^{\pi/2}\frac{dx}{(1+n\sin^2x)(1-k^2\sin^2x)^{1/2}}$$

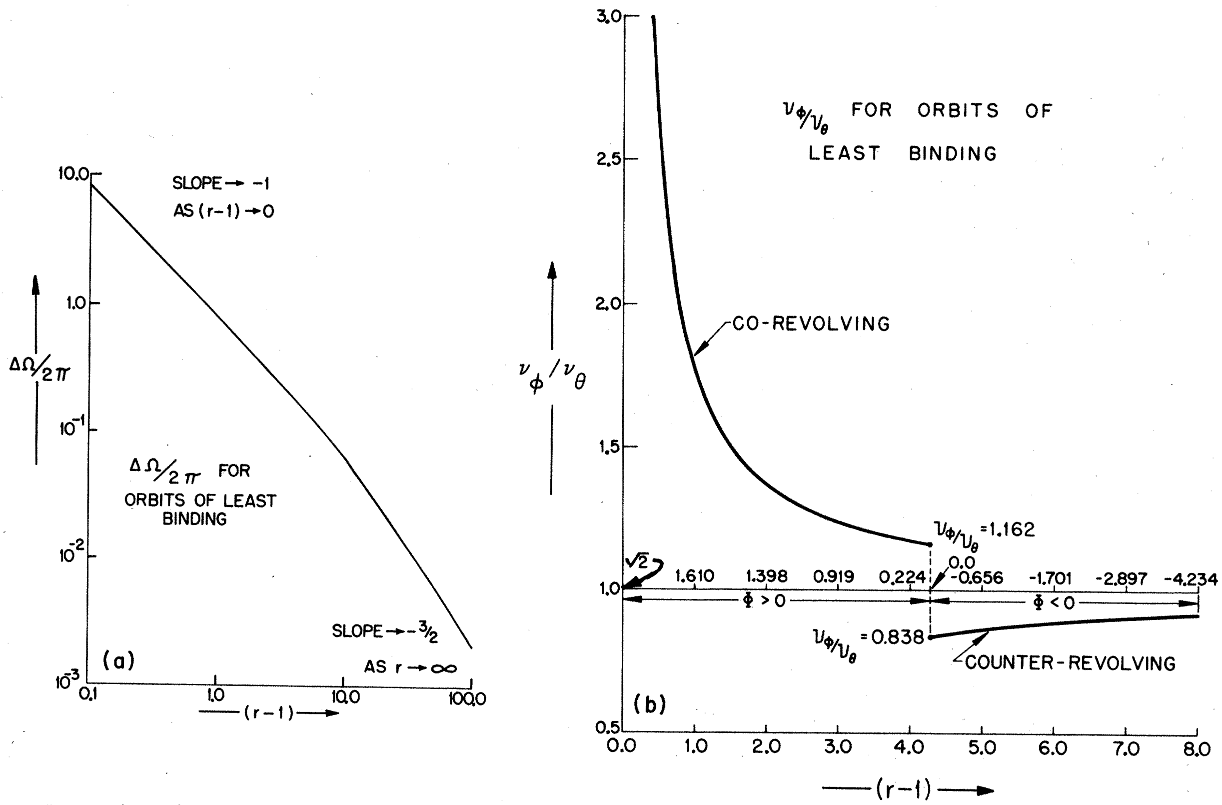

FIG. 5. (a) Angle of dragging of nodes per revolution versus distance from the one-way membrane. $\Delta\Omega$ increases continuously as the orbital radius decreases. (b) Ratio of the ϕ and θ frequencies versus distance from the one-way membrane. This changes discontinuously from a value less than unity to a value greater than unity, in going from counterrevolving to co-revolving orbits.

are elliptic integrals of the first and third kinds, respectively.

An orbit is called co-revolving if $\Delta\phi$ is positive. The first term on the right-hand side of (31) has the sign of Φ; one can show that the second term is always positive for a spherical orbit satisfying (9a). When Φ is negative the first term dominates the second; even when Φ approaches zero, the integral by which it is multiplied blows up so that the first term remains dominant. It follows that the sign of Φ determines that of $\Delta\phi$ and hence whether an orbit is co-revolving.

If the θ and ϕ frequencies were equal, $\Delta\phi$ would equal $\frac{1}{2}\pi$. Thus the ratio of the frequencies is given in general by

$$\nu_\phi/\nu_\theta = |\Delta\phi|/\tfrac{1}{2}\pi, \tag{32}$$

Substituting values of E, Φ, and Q for various orbits, one finds that

$$\begin{aligned}\nu_\phi/\nu_\theta &< 1 \quad \text{for} \quad \Phi < 0 \\ &> 1 \quad \text{for} \quad \Phi > 0.\end{aligned} \tag{33}$$

Equation (33) signifies that the nodes are always dragged in the sense of the spin.

The angle of advance of the nodes per nodal period is

$$\Delta\Omega = 2\pi\,|\nu_\phi/\nu_\theta - 1|. \tag{34}$$

One finds, as one would expect, that $\Delta\Omega$ always varies continuously when the constants of motion change continuously. ν_ϕ/ν_θ, however, undergoes a finite discontinuity when Φ passes through zero. Figure 5 displays the contrasting behavior of $\Delta\Omega$ and ν_ϕ/ν_θ for the least tightly bound orbits. The plot of ν_ϕ/ν_θ for the most tightly bound orbits does not show the discontinuity since such orbits are all co-revolving.

For large radii, one may replace E, Φ, and Q by their Schwarzschild values. From (21), (31), (32), and (34), the leading term of $\Delta\Omega$ is just the Lense-Thirring result, (24), with $a = m = 1$.

Contrast (24) with the effect in Newtonian theory of a mass quadrupole moment:

$$\Delta\Omega \propto \frac{\cos i}{r^2}, \tag{35}$$

where i is the angle of inclination. The $\Delta\Omega$ of (35) differs from that of (24) in these respects: (a) higher-order dependence on the radius; (b) dependence on the inclination of the orbit, e.g., the nodes do not regress for a polar orbit; and (c) the sense of rotation of the nodes depends on the motion of the orbiting particle – the nodes move contrary to the azimuthal velocity.

For small radii, only the term Δ^{-1} in (31) diverges. Using Eq. (8), the effective potential at $r = 1 + \lambda$ (expanded about $r = 1$) is

$$V(r) \simeq \tfrac{1}{2}\Phi + \frac{\partial V}{\partial r}\lambda + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\lambda^2.$$

Since the slope of $V(r)$ vanishes at $r = 1 + \lambda$,

$$\lambda \simeq -\frac{\partial V}{\partial r}\Big/\frac{\partial^2 V}{\partial r^2}.$$

Thus

$$E = V(r) \simeq \tfrac{1}{2}\Phi - \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\lambda^2 = \tfrac{1}{2}\Phi + O(\lambda^2).$$

Using this to evaluate P to order λ, one finds

$$P\Delta^{-1} = \Phi\lambda^{-1} + O(1). \tag{36}$$

There results the asymptotic formula

$$\frac{\nu_\phi}{\nu_\theta} \simeq \frac{1}{r-1}\,\frac{2K(k)}{\pi(\beta z_+)^{1/2}}. \tag{37}$$

Substitution of the values (22) for orbits at the horizon reveals that the coefficient of $(r-1)^{-1}$ varies from about 0.817 for $\Phi = 2/\sqrt{3}$ to about 0.835 for $\Phi = \sqrt{2}$.

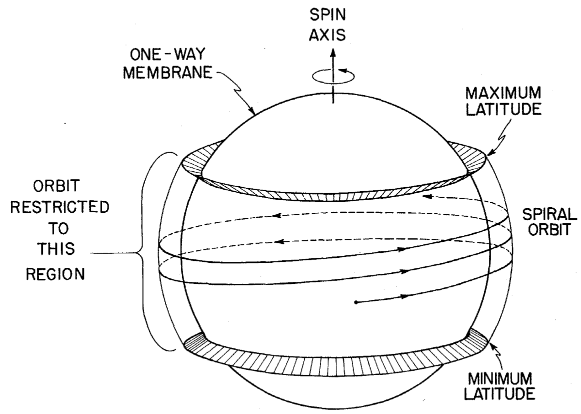

FIG. 6. Sketch of path of spherical orbit near one-way membrane (not drawn to scale.)

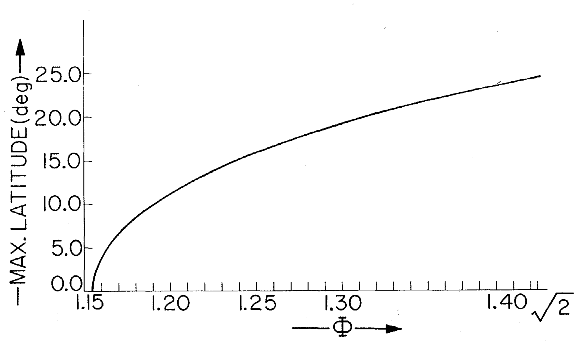

FIG. 7. Maximum latitude (= – minimum latitude) of orbits at the one-way membrane.

An orbital path close to the horizon is sketched in Fig. 6. The particle traces out a kind of helix lying on a sphere. As the particle approaches the maximum latitude, the angular separation between successive loops of the helix decreases. Reaching the maximum latitude, the particle begins the winding descent to the minimum latitude, located below the equatorial plane symmetrically to the maximum. Using Eqs. (22), (26), (29), and (30), we have plotted in Fig. 7 the maximum latitudes of orbits at the horizon.

It is known that there is a close resemblance between the linearized gravitational theory and classical electromagnetism. This prompts one to ask whether there occur orbits analogous to the spirals of electrons in the earth's magnetic field. The best place to look for such exotic behavior is near the horizon. From Eqs. (3), (9a), and (8) it follows that $P\Delta^{-1}$ always diverges at the horizon. Divide Eq. (2c) by Eq. (2d) to obtain

$$\frac{d\phi}{dt} \approx \tfrac{1}{2} \quad \text{for} \quad r \approx 1 .$$

One can see from this that the retrograde motion required for any kind of looping is impossible: Dragging forces a particle near the horizon to revolve always in the same sense as the black hole.

Unlike the weak-field region it is not true in general that $\Delta\Omega$ depends only on the radius. For example, for $r = 9$, $\Delta\Omega/2\pi$ decreases from 0.0814 for the counterrevolving equatorial orbit to 0.0607 for the co-revolving equatorial orbit.[16]

V. PERIODS

The proper θ period is obtained by integration of (2b). Squaring (2b), multiplying through by $\cos^2\theta \sin^2\theta$, and making the change of variables (26), one obtains

$$\dot{z}^2 = \frac{4}{r^2 + z^2} Y(z)$$

with $Y(z)$ as in (27). Hence, apart from a sign,

$$d\tau = \frac{(r^2 + z)dz}{2Y(z)} .$$

Integrating,

$$\tfrac{1}{4}\tau_{\theta,p} = \int_0^{z_-} \frac{(r^2+z)dz}{2Y(z)}$$

or

$$\tau_{\theta,p} = \frac{4}{(\beta z_+)^{1/2}}(r^2 + z_+)K(k) - 4\left(\frac{z_+}{\beta}\right)^{1/2} E(k), \qquad (38)$$

where

$$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 x)^{1/2} dx$$

is the complete elliptic integral of the second kind.

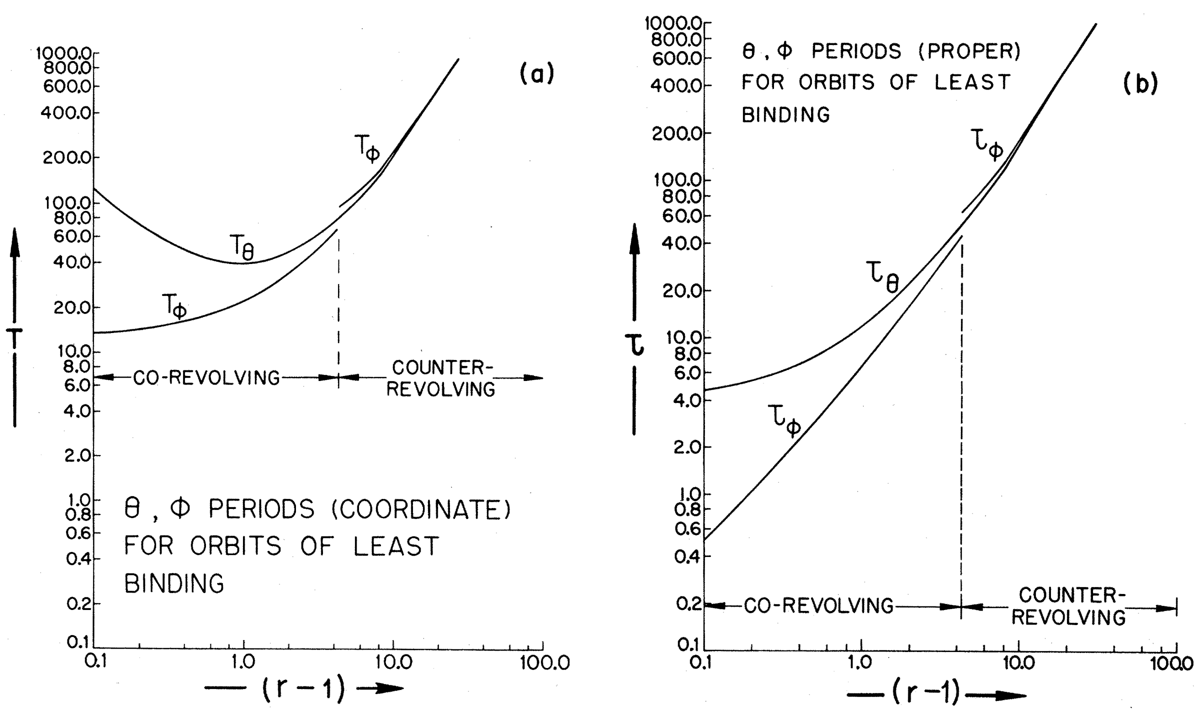

FIG. 8. The θ, ϕ periods (coordinate and proper) for orbits of least binding. The ϕ period is greater (less) than the θ period for counterrevolving (co-revolving) orbits. For the most tightly bound orbits (not shown), which are all co-revolving and equatorial, the ϕ period is everywhere less than the θ period.

To obtain the coordinate period, divide Eq. (2d) by Eq. (2b). Integration then yields

$$\tau_{\theta,c}=4\left[E\left(\frac{z_+}{\beta}\right)^{1/2}+\frac{1}{(\beta z_+)^{1/2}}[\Phi+P\Delta^{-1}(r^2+1)-E]\right]K(k)$$

$$-4E\left(\frac{z_+}{\beta}\right)^{1/2}E(k). \qquad (39)$$

To find the proper azimuthal period divide $\tau_{\theta,p}$ by

$$\nu_\phi/\nu_\theta=\tau_\theta/\tau_\phi,$$

which is given by (31) and (32); likewise for the coordinate period.

The periods determined above refer to $\mu=m=1$. The scaling law for times is obtained in similar manner to Eqs. (6). Equation (2b) is to be cast into the mass-independent form

$$\hat{\rho}^2\frac{d\hat{\theta}}{d\hat{\lambda}}=(\hat{\Theta})^{1/2}, \qquad (40)$$

where $\hat{\Theta}$ is the function Θ expressed in terms of caret variables. The polar angle is already scale free. From (1), (3), and (6) one sees that

$$\hat{\rho}^2=m^{-2}\rho^2,$$

$$\hat{\Theta}=(m\mu)^{-2}\Theta.$$

It follows that (2b) can be cast into the form (40) by setting

$$d\hat{\lambda}=\mu m^{-1}d\lambda=m^{-1}d\tau.$$

The second equality results from (4). Putting $m=1$,

$$d\hat{\lambda}=d\hat{\tau}$$

or

$$d\hat{\tau}=m^{-1}d\tau. \qquad (41)$$

Similarly, from (2d), (3), and (6),

$$d\hat{t}=m^{-1}dt. \qquad (42)$$

Equations (41) and (42) together with Eq. (6) constitute a general rule applicable to geodesics with $\mu^2>0$.

Figure 8 shows the periods for the least-bound orbits with radii ≤30. Here we see that the discontinuity in ν_ϕ/ν_θ mentioned earlier is due entirel to a discontinuity in ν_ϕ.

Note added in proof. One can show from (7) that

$$D=4r\Delta\{\Phi^2r^3+(r^2+Q)[r^3+a^2(r+2)]\}.$$

Equations (8) and (19) then imply that for an orbit the effective radial potential is everywhere real outside the horizon.

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[2]D. Christodoulou, Phys. Rev. Letters 25, 1596 (1970).

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[4]C. Darwin, Proc. Roy. Soc. (London) A249, 180 (1958).

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[6]By "bound" we mean that the particle remains within a finite radial distance of the black hole but is never captured.

[7]B. Carter, Phys. Rev. 174, 1559 (1968).

[8]We have corrected the signs of the terms containing P in Eqs. (2c) and (2d).

[9]The polynomial must be written in the order of decreasing (or increasing) powers. Terms with zero coefficients are not included.

[10]R. Descartes, *The Geometry of René Descartes*, translated by D. E. Smith and M. L. Latham (Open Court, Chicago, 1925). Descartes merely states the rule. For a proof see, e.g., H. E. Buchanan and L. C. Emmons, *A Brief Course in Advanced Algebra* (Houghton-Mifflin, Boston, 1937).

[11]The symbol $V^2(\theta)$ does not stand for the square of a real-valued function; rather it is meant to suggest a function having the dimensions of squared energy. That it can take on negative values is consequently not problematical.

[12]The artificiality of considering the θ and r motions separately here reveals itself in the curious circumstance that a trajectory may be "stable" and yet unbound.

[13]By co-revolving we mean that during one complete oscillation in latitude, the particle's azimuth changes in the sense of the spin of the black hole, that is to say, it increases. In Sec. IV we explain why an orbit is co-revolving if and only if $\Phi>0$.

[14]J. Lense and H. Thirring, Physik. Z. 19, 156 (1918).

[15]W. Grobner and N. Hofreiter, *Integraltafel* (Springer, Vienna, 1949), Vol. 1.

[16]To give a meaning to $\Delta\Omega$ for equatorial orbits, one imagines the motion to deviate infinitesimally from the equatorial plane.