and L. M. Matarrese, p. 67.
[20]Z. Bay, paper presented at the Fourth International Conference on Atomic Masses and Fundamental Constants, Teddington, England, 1971 (to be published).

PHYSICAL REVIEW D VOLUME 5, NUMBER 4 15 FEBRUARY 1972

Momentum and Angular Momentum in Relativistic Classical Particle Mechanics*

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(Received 19 October 1971)

For a classical-mechanical system of any fixed number of particles it is observed that space-translation invariance and conservation of angular momentum imply conservation of momentum. For three particles it is shown, as previously for two, that Poincaré invariance implies that the total kinematic momentum cannot be a constant of motion unless the accelerations are zero. The equations involved make it appear most likely that this is true for any number of particles.

We have learned only recently how relativistically invariant classical mechanics can describe interactions of a fixed number of particles, without fields, as in ordinary Newtonian equations of motion.[1-8] As yet, not very much is known about these interactions. For two particles it has been shown that their constants of motion do not include the total kinematic particle momentum or angular momentum.[4,9] (These quantities could have the same values before and after a collision by being asymptotic limits of constants of the motion which would depend on the interaction and could correspond to translation and rotation invariance. From the field-theory point of view there is momentum in the fields that propagate the interaction; Newton's third law does not hold because the fields do not propagate the interaction instantaneously.)

Here we observe that for any number of particles the impossibility of kinematic momentum being a constant of motion implies the same for angular momentum. We prove the statement about momentum for three particles. The equations involved make it appear most likely that it is true for any number of particles.

The idea is very simple. Suppose the kinematic momentum is a constant of motion. It is also space-translation invariant. We assume the dynamics is Poincaré invariant. It follows that every Lorentz transform of the kinematic momentum is a constant of motion, that is, the sum of the kinematic momenta of the particles taken at the same time in the transformed frame. It seems the only way every one of these can be a constant of motion is for the individual particle momenta to be constants of motion, which means there is no interaction.[10]

The same idea for space-translation invariance shows that conservation of total kinematic momentum follows from that of angular momentum, as we shall see as soon as we introduce some notation.

Let $\vec{x}^n$ and $\vec{v}^n$ be the position and velocity of the nth particle, m_n its mass, and

$$\vec{u}^n = m_n\vec{v}^n[1-(\vec{v}^n)^2]^{-1/2}$$

its (kinematic) momentum.

Suppose that the angular momentum

$$\sum_n \vec{x}^n \times \vec{u}^n$$

is a constant of motion. If the dynamics is space-translation invariant, it follows that the translated angular momentum

$$\sum_n (\vec{x}^n + \vec{\epsilon}) \times \vec{u}^n,$$

that is, the angular momentum in a frame translated a distance $\vec{\epsilon}$, is a constant of motion. For this to be true for every $\vec{\epsilon}$ the momentum

$$\sum_n \vec{u}^n$$

must be a constant of motion.

For a Lorentz transformation with velocity $\tanh\epsilon$ in the kth direction, the jth component of the transformed position of the nth particle, that is, the position at time zero in the transformed frame, is

$$x_j^n + \epsilon x_k^n v_j^n$$

to first order in ϵ, where $\vec{x}^n$ and $\vec{v}^n$ are the position and velocity at time zero in the original frame.[5,11,12] We can use a bracket-generator symbol $[\ ,\vec{K}]$ for Lorentz transformations[6] and write

$$[x_j^n, K_k] = x_k^n v_j^n$$

for $j, k = 1, 2, 3$. The first-order part of the similarly transformed velocity is[5,11,12]

$$[v_j^n, K_k] = x_k^n \dot{v}_j^n + v_j^n v_k^n - \delta_{jk}\,.$$

Here and in the following a dot means a time derivative. From these transformations we find that

$$[u_l^n, K_k] = x_k^n \dot{u}_l^n - [(\vec{u}^n)^2 + m_n^{\,2}]^{1/2}\delta_{kl}$$

for $k, l = 1, 2, 3$.

Suppose that $\sum_n \vec{u}^n$ is a constant of motion. It is also space-translation invariant, because space translation does not change velocities. We can use a bracket-generator symbol $[\ ,H]$ for time derivatives and $[\ ,\vec{P}]$ for space translations.[6] Then using the bracket relations of the Poincaré group,[6] we get

$$[[\sum_n u_l^n, K_k], H] = [[\sum_n u_l^n, H], K_k] + [\sum_n u_l^n, P_k] = 0$$

and

$$[[\sum_n u_l^n, K_k], P_j] = [[\sum_n u_l^n, P_j], K_k] + [\sum_n u_l^n, \delta_{jk}H] = 0\,;$$

so $[\sum_n u_l^n, K_k]$ is a constant of motion and space-translation invariant for $k, l = 1, 2, 3$. Repeating this with $[\sum_n u_l^n, K_k]$ and then $[[\sum_n u_l^n, K_k], K_j]$, etc., in place of $\sum_n u_l^n$, we see that $[[\sum_n u_l^n, K_k], K_j]$ and then $[[[\sum_n u_l^n, K_k], K_j], K_i]$, etc., are constants of motion.

To calculate $[[\sum_n u_l^n, K_k], K_j]$, for example, we use a bracket relation of the Poincaré group again to get

$$\begin{aligned}[\dot{u}_l^n, K_j] &= [[u_l^n, H], K_j] \\ &= [[u_l^n, K_j], H] - [u_l^n, P_j] \\ &= \frac{d}{dt}\{x_j^n \dot{u}_l^n - [(\vec{u}^n)^2 + m_n^{\,2}]^{1/2}\delta_{jl}\}\,.\end{aligned}$$

To find $[\dot{u}_l^n, K_i]$ we can use another bracket relation, viz., that space translations commute with time derivatives, to see that $\dot{u}_l^n$ is space-translation invariant:

$$[\dot{u}_l^n, P_i] = [[u_l^n, H], P_i] = [[u_l^n, P_i], H] = 0\,.$$

In this way we find that if $\sum_n u_l^n$ is a constant of motion, then

$$\sum_n \left(x_k^n \dot{u}_l^n - \delta_{kl}[(\vec{u}^n)^2 + m_n^{\,2}]^{1/2}\right), \tag{1}$$

$$\sum_n \left(\frac{d}{dt}(x_j^n x_k^n \dot{u}_l^n) - (\delta_{kl}x_j^n + \delta_{jl}x_k^n) \times \frac{d}{dt}[(\vec{u}^n)^2 + m_n^{\,2}]^{1/2}\right), \tag{2}$$

$$\sum_n \left(\frac{d^2}{dt^2}(x_i^n x_j^n x_k^n \dot{u}_l^n) - \frac{d}{dt}\{(\delta_{kl}x_i^n x_j^n + \delta_{jl}x_i^n x_k^n + \delta_{il}x_j^n x_k^n) \times \frac{d}{dt}[(\vec{u}^n)^2 + m_n^{\,2}]^{1/2}\}\right), \tag{3}$$

etc., also are constants of motion.

For three particles this implies that the individual particle momenta $\vec{u}^n$ are constants of motion.[10] The rest of the paper is a proof of this statement. For three particles we do not need the last constant of motion (3). You are invited to use it to try to construct a proof for four particles. For five particles you probably need one more constant of motion, etc.

The constants of motion (1) and (2) are space-translation invariant. [You can easily see that (1) is, because $\sum_n \dot{u}_l^n = 0$, and that (2) is, because (1) is a constant of motion.] Therefore in working with these constants of motion or their time derivatives (since time derivatives commute with space translations) we can choose the origin to be at $\vec{x}^1$ and write $\vec{x}^n - \vec{x}^1$ in place of $\vec{x}^n$.

Let $\vec{e}$ be a vector perpendicular to $\vec{x}^2 - \vec{x}^1$ and $\vec{x}^3 - \vec{x}^1$. We take the time derivative of (1), change $\vec{x}^n$ to $\vec{x}^n - \vec{x}^1$, multiply by e_k, sum over $k = 1, 2, 3$, and get

$$\sum_{n=1}^{3} \vec{v}^n \times (\dot{\vec{u}}^n \times \vec{e}) = 0\,.$$

Similarly, taking the time derivative of (2), changing $\vec{x}^n$ to $\vec{x}^n - \vec{x}^1$, multiplying by $e_j e_k$, and summing over $j, k = 1, 2, 3$, we get

$$\sum_{n=1}^{3} (\vec{e}\cdot\vec{v}^n)\vec{v}^n \times (\dot{\vec{u}}^n \times \vec{e}) = 0\,,$$

and, using the previous equation to eliminate $\vec{v}^3 \times (\dot{\vec{u}}^3 \times \vec{e})$, we get

$$\vec{e}\cdot(\vec{v}^1 - \vec{v}^3)\vec{v}^1 \times (\dot{\vec{u}}^1 \times \vec{e}) + \vec{e}\cdot(\vec{v}^2 - \vec{v}^3)\vec{v}^2 \times (\dot{\vec{u}}^2 \times \vec{e}) = 0\,.$$

Then taking a dot product with $\vec{v}^1$ or $\vec{v}^2$ and interchanging dot and cross products, we find that

$$\vec{e}\cdot(\vec{v}^1 - \vec{v}^3)\vec{v}^1 \times \vec{v}^2 \cdot \dot{\vec{u}}^1 \times \vec{e} = 0$$

and

$$\vec{e}\cdot(\vec{v}^2 - \vec{v}^3)\vec{v}^1 \times \vec{v}^2 \cdot \dot{\vec{u}}^2 \times \vec{e} = 0\,.$$

We see[10] that $\dot{\vec{u}}^1 \times \vec{e}$, $\dot{\vec{u}}^2 \times \vec{e}$, and $\dot{\vec{u}}^3 \times \vec{e}$ (since $\dot{\vec{u}}^3 = -\dot{\vec{u}}^1 - \dot{\vec{u}}^2$) are perpendicular to $\vec{v}^1 \times \vec{v}^2$ and (permuting indices) to $\vec{v}^2 \times \vec{v}^3$ and $\vec{v}^3 \times \vec{v}^1$. We conclude[10] that $\dot{\vec{u}}^1 \times \vec{e}$, $\dot{\vec{u}}^2 \times \vec{e}$, and $\dot{\vec{u}}^3 \times \vec{e}$ are zero, which means that $\dot{\vec{u}}^1$, $\dot{\vec{u}}^2$, and $\dot{\vec{u}}^3$ are perpendicular to $\vec{x}^2-\vec{x}^1$ and $\vec{x}^3-\vec{x}^1$.

If in (1) we change $\vec{x}^n$ to $\vec{x}^n-\vec{x}^1$ and sum over $k=l=1, 2, 3$, we find that the total kinetic energy

$$\sum_{n=1}^{3}[(\vec{u}^n)^2+m_n^2]^{1/2}$$

is a constant of motion. Taking its time derivative, we have

$$\sum_{n=1}^{3}\vec{v}^n\cdot\dot{\vec{u}}^n=0\,. \tag{4}$$

Again let $\vec{e}$ be a vector perpendicular to $\vec{x}^2-\vec{x}^1$ and $\vec{x}^3-\vec{x}^1$. We take the time derivative of (2), change $\vec{x}^n$ to $\vec{x}^n-\vec{x}^1$, multiply by e_j, sum over $j=1, 2, 3$ and $k=l=1, 2, 3$, and find that

$$\sum_{n=1}^{3}(\vec{e}\cdot\vec{v}^n)(\vec{v}^n\cdot\dot{\vec{u}}^n)=0\,.$$

Using Eq. (4) to eliminate $\vec{v}^3\cdot\dot{\vec{u}}^3$, we get

$$\vec{e}\cdot(\vec{v}^1-\vec{v}^3)(\vec{v}^1\cdot\dot{\vec{u}}^1)+\vec{e}\cdot(\vec{v}^2-\vec{v}^3)(\vec{v}^2\cdot\dot{\vec{u}}^2)=0\,. \tag{5}$$

We use $\dot{\vec{u}}^3=-\dot{\vec{u}}^1-\dot{\vec{u}}^2$ to eliminate $\dot{\vec{u}}^3$ from (4), multiply (4) by $\vec{e}\cdot\vec{v}^1$ and subtract it from (5), and find that

$$\vec{e}\cdot\dot{\vec{u}}^1\times[(\vec{v}^1-\vec{v}^3)\times\vec{v}^1]+\vec{e}\cdot(\vec{v}^2-\vec{v}^3)(\vec{v}^2\cdot\dot{\vec{u}}^2)-(\vec{e}\cdot\vec{v}^1)(\vec{v}^2-\vec{v}^3)\cdot\dot{\vec{u}}^2=0\,.$$

If we interchange the dot and cross products, we see that the first term is zero because $\vec{e}\times\dot{\vec{u}}^1$ is zero. This eliminates $\dot{\vec{u}}^1$; so we are left with

$$[\vec{e}\cdot(\vec{v}^2-\vec{v}^3-\vec{v}^1)\vec{v}^2+(\vec{e}\cdot\vec{v}^1)\vec{v}^3]\cdot\dot{\vec{u}}^2=0\,.$$

But $\dot{\vec{u}}^2$ is also perpendicular to the two other vectors $\vec{x}^2-\vec{x}^1$ and $\vec{x}^3-\vec{x}^1$. Therefore $\dot{\vec{u}}^2$ must be zero.[10] We conclude that $\dot{\vec{u}}^1$, $\dot{\vec{u}}^2$, and $\dot{\vec{u}}^3$ are all zero.

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*Work supported in part by the U. S. Atomic Energy Commission, under Contract No. AT-(30-1)-3829.

[1]D. G. Currie, Phys. Rev. 142, 817 (1966).

[2]D. G. Currie and T. F. Jordan, Phys. Rev. Letters 16, 1210 (1966).

[3]R. N. Hill, J. Math. Phys. 8, 201 (1967).

[4]D. G. Currie and T. F. Jordan, Phys. Rev. 167, 1178 (1968).

[5]D. G. Currie and T. F. Jordan, in *Lectures in Theoretical Physics,* edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. X A, p. 91.

[6]T. F. Jordan, Phys. Rev. 166, 1308 (1968).

[7]J. G. Wray, Phys. Rev. D 1, 2212 (1970).

[8]A different approach using non-Newtonian equations of motion is taken by H. Van Dam and E. P. Wigner, Phys. Rev. 138, B1576 (1965); A. Katz, J. Math. Phys. 10, 1929 (1969).

[9]Similar properties for both two and three particles were found by H. Van Dam and E. P. Wigner, Phys. Rev. 142, 838 (1966).

[10]We do not consider singular accelerations which are zero for almost all values of the positions and velocities.

[11]D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. 35, 350 (1963).

[12]D. G. Currie, J. Math. Phys. 4, 1470 (1963).